Homework 5. Due on October 23

1. Decide if the following series converges, give your reasoning:
   \[
   (a) \sum \frac{n^2 - 5n + 3}{10n^2 + 5} \quad (b) \sum \frac{1}{n^2 - 3n + 6} \quad (c) \sum \frac{\sin n}{n^2} \quad (d) \sum \frac{1}{n \ln n}
   \]

2. If \( \sum x_n \) and \( \sum y_n \) are convergent, prove that \( \sum (x_n + y_n) \) is convergent.

3. Can you give an example of a convergent series \( \sum x_n \) and a divergent series \( \sum y_n \) such that \( \sum (x_n + y_n) \) is convergent? Explain.

4. If \( \sum a_n \) with \( a_n > 0 \) is convergent, then is \( \sum a_n^2 \) always convergent? Is \( \sum \sqrt{a_n} \) always convergent? Either prove or give counter an example in each case.

5. Suppose \( \{a_n\} \) is a decreasing sequence of strictly positive numbers. Prove that \( \sum a_n \) converges if and only if \( \sum 2^n a_{2^n} \) converges.

6. Use problem 5 to discuss the convergence or divergence of the \( p \)-series \( \sum \frac{1}{n^p} \).

7. Problem 9 on page 79 of Rudin. (You may treat \( z \) as a real number).

8. We say a sequence \( \{x_n\} \) of real numbers is contractive if there exists a constant \( C, 0 < C < 1 \) such that \( |x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \) for all \( n \in \mathbb{N} \). Prove that a contractive sequence is a Cauchy sequence.

9*. Decide the convergent or divergent of the series \( \sum \frac{\cos n}{n} \) and \( \sum \frac{\sin n}{n} \).