Q-curvature and Conformal Covariant operators

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Question: What are the general conformal invariants? What are the conformal covariant operators and their related curvature invariants?
• Second order invariants:
  1. \((\Delta_g, K_g)\) on \((M^2, g)\) satisfying
     \[
     \Delta_{gw} = e^{-2w} \Delta_g
     \]
     and
     \[
     -\Delta_{gw} + K_g = K_{gw}e^{2w}.
     \]
  2. \((L_g, R_g)\) on \((M^n, g), n \geq 3,\) where
     \[
     L_g = -c_n \Delta_g + R_g
     \]
     where \(R_g\) the scalar curvature, satisfying
     \[
     L_{gw} = e^{-\frac{n+2}{2}w} L_g(e^{\frac{n-2}{2}w}).
     \]
     Yamabe equation
     \[
     L_g(e^{\frac{n-2}{2}w}) = R_{gw}e^{\frac{n+2}{2}w}.
     \]
4th order invariants:
When $n = 4$, Paneitz operator: (1983)

$$P_\varphi \equiv \Delta^2 \varphi + \delta \left[ \left( \frac{2}{3} R g - 2 \text{Ric} \right) d\varphi \right]$$

Satisfying:

$$P_{gw} = e^{-4w} P_g$$

$$P_{gw} + 2Qg = 2Q_{gw} e^{4w}$$

$$Q = \frac{1}{12} (-\Delta R + R^2 - 3 |\text{Ric}|^2)$$
3. On $(M^n, g)$, $n \neq 4$.

Existence of 4-th order conformal Paneitz operator $P^4_n$,

$$P^4_n = \Delta^2 + \delta (a_n Rg + b_n \text{Ric}) + \frac{n-4}{2} Q^4_n.$$ 

For $\tilde{g} = u^{n-4} g$: $P^4_n u = \tilde{Q}^n_{4} u^{\frac{n+4}{n-4}}.$

- $P^4_n$ is conformal covariant of bidegree $(\frac{n-4}{2}, \frac{n+4}{2}).$

- $Q^4_n$ is a fourth order curvature invariants. i.e. under dilation $\delta_t g = t^{-2} g,$

$$(Q^4_n)(\delta_t g) = t^4 (Q^4_n)(g).$$
Fefferman-Graham (1985) systematically construct (pointwise) conformal invariants:

**Example:** The Riemann curvature tensor has the decomposition

\[ R_{ijkl} = W_{ijkl} + [A_{jk}g_{il} + A_{il}g_{jk} - A_{jl}g_{ik} - A_{ik}g_{jl}] \]

where

\[ A = \frac{1}{n-2}[R_{ij} - \frac{R}{2(n-1)}g_{ij}] \]

is called the Schouten tensor. The Weyl tensor satisfies \( W_{gw} = e^{-2w}W_g \).

Graham-Jenne-Mason-Sparling (1992) applied method of construction to existence results of general conformal covariant operators \( P_{2k}^n \) for \( n \) even.
§2. Conformally compact Einstein manifold

Given \((M^n, g)\), denote \([g]\) class of conformal metrics \(g_w = e^{2w} g\) for \(w \in C^\infty(M^n)\).

**Definition:** Given \((X^{n+1}, M^n, g^+)\) with smooth boundary \(\partial X = M^n\). Let \(r\) be a defining function for \(M^n\) in \(X^{n+1}\) as follows:

- \(r > 0\) in \(X\);
- \(r = 0\) on \(M\);
- \(dr \neq 0\) on \(M\).

- We say \(g^+\) is **conformally compact**, if there exists some \(r\) so that \((X^{n+1}, r^2 g^+)\) is a compact manifold.
- \((X^{n+1}, M^n, g^+)\) is **conformally compact Einstein** if \(g^+\) is Einstein (i.e. \(Ric_{g^+} = cg^+\)).
- We call \(g^+\) a **Poincare metric** if \(Ric_{g^+} = -ng^+\).
Example:

On \((H^{n+1}, S^n, g_H)\)

\[
(H^{n+1}, (\frac{2}{1-|y|^2})^2|dy|^2).
\]

We can then view \((S^n, [g])\) as the compactification of \(H^{n+1}\) using the defining function

\[
r = 2 \frac{1 - |y|}{1 + |y|}
\]

\[
g_H = g^+ = r^{-2} \left( dr^2 + \left(1 - \frac{r^2}{4}\right) g \right).
\]
Given \((M^n, g)\), consider \(M^+ = M^n \times [0, 1]\) and metric \(g^+\) with

(i) \(g^+\) has \([g]\) as conformal infinity,
(ii) \(\text{Ric}(g^+) = -ng^+\).

In an appropriate coordinate system \((\xi, r)\), where \(\xi \in M\) with

(iii) \(g^+ = r^{-2} \left( dr^2 + \sum_{i,j=1}^{n} g^+_{ij}(\xi, r) d\xi_i d\xi_j \right)\),

and \(g^+_{ij}\) is even in \(r\).

**Theorem:** (C. Fefferman- R. Graham, ’85)

(a) In case \(n\) is odd, up to a diffeomorphism fixing \(M\), there is a unique formal power series solution of \(g^+\) to (i)–(iii).

(b) In case \(n\) is even, there exists a formal power series solution for \(g^+\) for which the components of \(\text{Ric}(g^+) + ng^+\) vanish to order \(n - 2\) in power series of \(r\).
Remarks:

- Conformally compact Einstein manifold is of current interest in the physics literature. The Ads/CFT correspondence proposed by Malda-cena involves string theory and super-gravity on such $X$.
- The construction of the Poincare metric is actually accomplished via the construction of a Ricci flat metric, called the ambient metric on the manifold $\tilde{G}$, where $\tilde{G} = G \times (-1,1)$ of dimension $n+2$ and $G$ is the metric bundle

$$G = \left\{ (\xi, t^2 g(\xi)) : \xi \in M^n, t > 0 \right\}$$

of the bundle of symmetric 2 tensors $S^2T^*M$ on $M$. The conformal invariants are then contractions of $(\tilde{\nabla}^{k_1} \tilde{R} \otimes \tilde{\nabla}^{k_2} \tilde{R} \otimes \ldots \tilde{\nabla}^{k_l} \tilde{R})$ restricted to $TM$ where $\tilde{R}$ denotes the curvature tensor of the ambient metric.
A model example is given by the standard sphere $(S^n, g)$. Denote $S^n = \{ \sum_{1}^{n+1} \xi_k^2 = 1 \}$.

$$G = \left\{ \sum_{1}^{n+1} p_k^2 - p_0^2 = 0 \right\}$$

under $\xi_k = p_k/p_0$ ($1 \leq k \leq n + 1$). Then the ambient space $\tilde{G}$ is Minkowski space $R^{n+1,1} = \{(p,p_0), |p| \in R^{n+1}, p_0 \in R \}$ with the Lorentz metric

$$\tilde{g} = |dp|^2 - dp_0^2,$$

The standard hyperbolic space is realized as the quadric $H^{n+1} = \{|p|^2 - p_0^2 = -1\} \subset R^{n+1,1}$. 

PICTURE
Graham, Jenne, Mason and Sparling (1992)
The existence of conformal covariant operator $P_{2k}^n$ on $(M^n, g)$ with:

- Order $2k$ with leading symbol $(-\Delta)^k$

- Conformal covariant of bi-degree $(\frac{n-2k}{2}, \frac{n+2k}{2})$; where $k \in \mathbb{N}$ when $n$ is odd, but $2k \leq n$ when $n$ is even.

- In general, the operators $P_{2k}^n$ is not unique, e.g. add $|W|^k$ to $P_{2k}^n$, where $W$ is the Weyl tensor, when $k$ is even.

- On $\mathbb{R}^n$, the operator is unique and is equal to $(-\Delta)^k$. Hence the formula for $P_{2k}^n$ on the standard sphere and on Einstein metric.
$Q$ curvature associated with $P_{2k}^n$.

- When $2k \neq n$, then $P_{2k}^n(1) = c(n, k)Q_{2k}^n$, e.g. when $k = 1, 2 < n$, $P_2^n = -c_n \Delta + R = L$, and $Q_2^n = R = P_2^n(1)$.

- When $2k = n$, $n$ even Branson ('93) justified the existence of $Q_n^n$ by a dimension continuation (in $n$) argument from $Q_{2k}^n$. e.g. When $k = 1$ and $n = 2k = 2$, $Q_2^2 = K$ the Gaussian curvature. When $k = 2$ and $n = 4$, $Q_4^4 = 2Q_4$.

- Graham and Zworski ('02) Existence of $Q_n^n$ when $n$ even, the analytic continuation of a spectral parameter in scattering theory.
Spectral Theory on \((X^{n+1}, M^n, g^+)\), with \(g^+\) Poincare metric and \((M^n, [g])\) as conformal infinity.

- A basic fact is (Mazzeo, Melrose-Mazzeo)
  \[
  \sigma(-\Delta_{g^+}) = \left[\left(\frac{n}{2}\right)^2, \infty\right) \cup \sigma_{pp}(-\Delta_{g^+})
  \]
  the pure point spectrum \(\sigma_{pp}(-\Delta_{g^+})\) (\(L^2\) eigenvalues), is finite.
- For \(s(n-s) \notin \sigma_{pp}\), consider
  \[
  (-\Delta_{g^+} - s(n-s))u = 0.
  \]
  Given \(f \in C^\infty(M)\), then there is a meromorphic family of solutions \(u(s) = \varphi(s) f\)
  \[
  \varphi(s)f = Fr^{n-s} + Gr^s \quad \text{if} \quad s \notin n/2 + \mathbb{N}
  \]
  with \(F|_M = f\)

Define Scattering matrix to be
  \[
  S(s)f = G|_M
  \]
The relation of $f$ to $S(s)f$ is like that of the Dirichlet to Neumann data.

**Theorem:** (Graham-Zworski 2002)

Let $(X^{n+1}, M^n, g^+)$ be a Poincare metric with $(M^n, [g])$ as conformal infinity. Suppose $n$ is even, and $k \in \mathbb{N}$, $k \leq \frac{n}{2}$ and $s(n - s)$ not in $\sigma_{pp}(-\Delta_{g^+})$. Then the scattering matrix $S(s)$ has a simple pole at $s = \frac{n}{2} + k$ and

$$c_k P^n_{2k} = -\text{Res}_{s=n/2+k} S(s)$$

When $2k \neq n$, $P^n_{2k}(1) = c(n, k)Q^n_{2k}$

When $2k = n$, $c_{n/2} Q_n = S(n)1$. 
§3. Known facts for $Q_n$, n even:

- $Q_n$ is a conformal density of weight $-n$; i.e. with respect to the dilation $\delta_t$ of metric $g$ given by $\delta_t(g) = t^2 g$, we have $(Q_n)_{\delta tg} = t^{-n}(Q_n)_g$.

- $\int_{M^n}(Q_n)_g dv_g$ is conformally invariant.

- For $g_w = e^{2w} g$, we have $(P_n)_{g_w} + (Q_n)_g = (Q_n)_{g_w} e^{nw}$.

- When $(M^n, g)$ is locally conformally flat, then $(Q_n)_g = c_n \sigma_{\frac{n}{2}}(A_g) + \text{divergence terms}$, e.g. $Q_4 = \sigma_2(A_g) - \frac{1}{6} \Delta_g R$.

- Alexakis

$$Q_n = c_n \text{Pfaffian} + J + \text{div}(T_n).$$

where Pfaffian is the Euler class density, which is the integrand in the Gauss-Bonnet formula, $J$ is a pointwise conformal invariant, and $\text{div}(T_n)$ is a divergence term.
• Alexakis (also Fefferman-Hirachi) has extended the existence of conformal covariant operator to conformal densities of weight $\gamma$, where $\gamma \neq \left(-\frac{n}{2}\right) + k$ where $k$ is a positive integer and $\gamma$ not a nonnegative integer. An example of such operator is:

$$2P(f) = \nabla^i (||W||^2 \nabla^i f) + \frac{n-6}{n-2} ||W||^2 \Delta f.$$ 

with corresponding Q-curvature explicit.

• Fefferman and Hirachi have also extended the construction of conformal covariant operator and $Q$ curvature to $CR$ manifolds.

• Branson, Eastwood-Gover survey articles, AIM meeting August 2003.
§4. Renormalized Volume (Witten, Gubser-Klebanov-Polyakov, Henningson-Skenderis, Graham)

On conformal compact \((X^{n+1}, M^n, g^+)\) with defining function \(r\), For \(n\) odd,

\[
\text{Vol}_{g^+}(\{r > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \cdots + c_{n-1} \epsilon^{-1} + V + o(1)
\]

For \(n\) even,

\[
\text{Vol}_{g^+}(\{r > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \cdots + c_{n-2} \epsilon^{-2} + L \log \frac{1}{\epsilon} + V + o(1)
\]

• For \(n\) odd, \(V\) is independent of \(g \in [g]\), and for \(n\) even, \(L\) is independent of \(g \in [g]\), and hence are conformal invariants.
**Theorem:** (Graham-Zworski) When $n$ is even,

$$ L = -2 \int_M S(n)1 = 2c_n \frac{1}{2} \int_M Q_n dv_g. $$

**Theorem:** (Fefferman-Graham '02) Consider $v = \frac{d}{ds}|_{s=n} S(s)1$ then $v$ is a smooth function defined on $X$ solving

$$ -\Delta_{g^+}(v) = n $$

and with the asymptotic

$$ v = \begin{cases} 
\log x + A + Bx^n \log x & \text{for } n \text{ even} \\
\log x + A + Bx^n & \text{for } n \text{ odd} 
\end{cases} $$

where $A, B \in C^\infty(X)$ are even mod $O(x^\infty)$ and $A|_M = 0$. Moreover

(i) If $n$ is even, then

$$ B|_M = -2S(n)1 = -2c_n Q_n $$

hence $L = 2c_n \frac{1}{2} \int_M Q_n$. 

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(ii) If \( n \) is odd, then

\[
B|_M = -\frac{d}{ds}|_{s=nS(s)}1,
\]

and if one defines \( Q_n(g^+, [g]) \) to be

\[
Q_n(g^+, [g]) = k_nB|_M
\]

then

\[
k_nV = \int_M Q_n(g^+, [g])dv_g.
\]

**Remark:** when \( n \) is odd, the \( Q \) curvature thus defined is not intrinsic, it depends not only on the boundary metric \( g \) on \( M \) but also on the extension of \( g^+ \) on \( X \).
On compact Riemannian 4-manifold \((X^4, M^3, g^+)\) with boundary, Chang-Qing introduced

\[
(P_b)_{gw} = e^{-3w}(P_b)_g, \quad \text{on } M \quad \text{and} \quad (P_b)_{gw} + T_g = T_g w e^{3w} \text{on } M.
\]

\[8\pi^2 \chi(X) = \int_{X^4} \left( \frac{1}{4} |W|^2 + Q_4 \right) dv + 2 \int_{M^3} (\mathcal{L} + T) d\sigma,\]

where \(\mathcal{L}\) is a point-wise conformal invariant term on the boundary of the manifold.

On conformally compact Einstein \((X^4, M^3, g^+)\):

\[
(P_b)_g = -\frac{1}{2} \frac{\partial}{\partial n} \Delta g^+ \big|_M, \quad T_g = \frac{1}{12} \frac{\partial R}{\partial n} \big|_M,
\]

and in this case \(\mathcal{L}\) vanishes.
When $n = 3$, on $(X^4, M^3, g^+)$, conformally compact Einstein

**Theorem:** (Chang-Qing-Yang)

On $(X^4, M^3, g^+)$

(i) $(Q_4)_{e^2v g^+} = 0,$

**Proof:** Recall

$$Q_4 = \frac{1}{6}(-\Delta R + R^2 - 3|Ric|^2).$$

Thus for $g^+$ a Poincare metric with $Ric g^+ = -3g^+$, we have $(Q_4)_{g^+} = 6$ and

$$(P_4)_{g^+} = (\Delta)g^+ + 2\Delta_{g^+}.$$ 

We then use the equations $-\Delta_{g^+}(v) = n = 3$ and

$$(P_4)_{g^+}(v) + (Q_4)_{g^+} = (Q_4)_{e^2v g^+}$$

to conclude the proof.
(i) \((Q_4)_{e^{2v}g^+} = 0\),
(ii) \(Q_3(e^{2v}g^+, [e^{2v}g]) = 3B|_{x=0} = T_{e^{2v}g}\).

As a consequence we have

\[
6V = \int_{X^4} (Q_4)_{e^{2v}g^+} + 2 \int_{M^3} T_{e^{2v}g} \\
= \int_{X^4} \sigma_2(A_{e^{2v}g^+}).
\]

Hence (M. Anderson)

\[
8\pi^2 \chi(X^4) = \frac{1}{4} \int_{X^4} |W|^2 dv_\bar{g} + \int_{X^4} \sigma_2(A_\bar{g}) \\
= \frac{1}{4} \int_{X^4} |W|^2 dv_\bar{g} + 6V,
\]

for \(\bar{g} = e^{2v}g^+\) or any conformal compact \(\bar{g}\).
Conformal Sphere Theorem:
(Chang-Gursky-Yang)
On \((M^4, g)\) with \(Y(M^4, g) > 0\). If
\[
\int_{M^4} |W_g|^2 dv_g < 16\pi^2 \chi(M^4),
\]
or equivalently
\[
\int_{M^4} \sigma_2(A_g) dv_g > 4\pi^2 \chi(M^4)
\]
then \(M^4\) is diffeomorphic to \(S^4\) or \(\mathbb{R}^4\).

Note that on \((M^4, g)\), with \(Y(M^4, g) > 0\).
\[
\int_{M^4} \sigma_2(A_g) dv_g \leq 16\pi^2
\]
with equality if and only if \(M^4\) is diffeomorphic to \(S^4\).
**Theorem**: (Chang-Qing-Yang )

Suppose \((X^4, M^3, g^+)\) is a conformal compact Einstein manifold, and \((M^3, [g])\) has positive Yamabe constant, then

(i) \(V \leq \frac{4\pi^2}{3}\), with equality holds if and only if \((X^4, g^+)\) is the hyperbolic space \((H^4, g_H)\), and therefore \((M^3, g)\) is the standard 3-sphere.

(ii) If

\[ V > \frac{1}{3}(\frac{4\pi^2}{3}\chi(X)), \]

then \(X\) is homeomorphic to the 4-ball \(B^4\) up to a finite cover.

(iii) If

\[ V > \frac{1}{2}(\frac{4\pi^2}{3}\chi(X)), \]

then \(X\) is diffeomorphic to \(B^4\) and \(M\) is diffeomorphic to \(S^3\).
A crucial step in the proof of the theorem above is an earlier result:

**Theorem:** (Qing ’02)
Suppose \((X^{n+1}, M^n, g^+)\) is a conformal compact Einstein manifold, with \(Y(M^n, [g])\) positive, then there is a positive eigenfunction \(u\) satisfying

\[-\Delta_{g^+} u = (n + 1)u \text{ on } X^{n+1},\]

so that \((X^{n+1}, u^{-2}g^+)\) is a compact manifold with totally geodesic boundary and the scalar curvature is greater than or equal to \(\frac{n+1}{n-1} R_g\), where \(g \in [g]\) is the Yamabe metric.

**PICTURE**
Theorem:
(Chang-Qing-Yang, Epstein)
On conformally compact Einstein \((X^{n+1}, M^n, g^+)\), when \(n\) is odd,
\[
\int_{X^{n+1}} W_{n+1} \, dv_g + c_n V(X^{n+1}, g) = \chi(X^{n+1})
\]
for some curvature invariant \(W_{n+1}\), which is a sum of contractions of Weyl curvatures and/or its covariant derivatives in an Einstein metric.

Proof:
Use structure equation of \(Q_n\); in particular, the result of Alexakis that
\[
Q_n = a_n \text{Pfaffian} + J + \text{div}(T_n).
\]
§5. Renormalized volume when $n$ is even.

The renormalized volume can also be defined via the scattering matrix:

$$V(X^3, [g], g^+) = -\int_{M^2} \frac{d}{ds}|_{s=2} S(s) 1 dv_g, \text{ for } n = 2$$

$$V(X^5, [g], g^+) = -\int_{M^4} \frac{d}{ds}|_{s=4} S(s) 1 dv_g$$

$$- \frac{1}{32 \cdot 36} \int_{M^4} R^2[g] dv_g, \text{ for } n = 4$$

$$V(X^{n+1}, [g], g^+) = -\int_{M^n} \frac{d}{ds}|_{s=n} S(s) 1 dv_g$$

$+ \text{ correction terms, for } n \text{ even}$
**Definition:** We call a functional $\mathcal{F}$ defined on $(M^n, g)$ a conformal primitive of a curvature tensor $\mathcal{A}$ if

$$\frac{d}{d\alpha}|_{\alpha=0}\mathcal{F}[e^{2\alpha}w g] = -2c_n \int_M w \mathcal{A} dv_g.$$

**Theorem:** On $(X^{n+1}, M^n, g^+)$, $n$ even, the scattering term $S(g, g^+) = \frac{d}{ds}|_{s=n} S(s) 1(g, g^+)$ is the conformal primitive of $(Q_n)_g$.

**Corollary:** (Henningson-Skenderis, Graham)

On $(X^3, M^2, g^+)$, $V$ is the conformal primitive of $K$, the Gaussian curvature.

On $(X^5, M^4, g^+)$, $V$ is the conformal primitive of $\frac{1}{16} \sigma_2$, where $\sigma_2 = \frac{1}{6}(R^2 - 3|Ric|^2)$. 

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• Qing established the rigidity result that any conformal compact Einstein manifold with conformal infinity the standard n-sphere is the hyperbolic $n + 1$ space extending prior results of L. Andersson.

• X. Wang proved that on $(X^{n+1}, M^n, g^+)$ with $\lambda_0(g^+) > n - 1$, then $H_n(X, \mathbb{Z}) = 0$. In particular, the conformal infinity $M$ is connected; thus extending an earlier result of Witten-Yau.

Given $(M^n, [g])$ in general, both the existence and uniqueness problem of a conformal compact Einstein manifold with $(M^n, [g])$ as conformal infinity remain open.