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A Riemannian metric $g = \sum g_{ij} dx^i dx^j$ gives rise to measurement of angles between vectors, and a conformally equivalent metric $\tilde{g} = \rho g$ for some positive function $\rho$ gives the same angle measurements. Two spaces $X, X'$ are said to be conformally equivalent if there is a map $T : X \to X'$ which preserves the angle measurements.

- We are interested to study conformal invariants, i.e. terms which are invariant under conformal change of metrics. This includes

- local or pointwise conformal invariants. Examples are curvature tensors, e.g. Weyl tensor $W_g$, which satisfies $W_{\tilde{g}} = \rho^{-1} W_g$ and measures the deviation of the metric from a conformally flat metric.
• Global or integral conformal invariants. Examples are integral of curvature invariants, e.g. integral of Gaussian curvature over a Riemann surface.

• We are also interested in conformal covariant operators, i.e. operators which transform by simple rules under conformal change of metrics; such operators are usually closely associated with local conformal invariants. e.g.

• $\Delta_g$ on compact surface,

• The conformal Laplace operator $L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ on $(M^n, g)$ for $n \geq 3$ where $R_g$ is the scalar curvature.
Important aspects of the theory includes:

- Existence and construction of local conformal invariants:
  E. Cartan’s theory of differential invariants.
  T. Thomas’s theory of tractor calculus.
  C. Fefferman and R. Graham introduced the ambient metric construction.
Construction and properties of conformally covariant operators and their associated $Q$ curvatures:

- **Paneitz** introduced 4-th order operators.
- **Graham-Jenne-Mason-Sparling** introduce the n-th order operator on n-manifolds (n even).
- **Branson** relates the operators to $Q$-curvature.
- **Fefferman-Graham, Zworski** relates the n-th order operators on n-manifolds to the scattering theory of conformally compact Einstein spaces.
- **Alexakis’** result on the structure of $Q$ curvatures
• Nonlinear PDE’s associated with the conformally covariant operators:
  Work on the Gauss curvature equation.
  Work on the Yamabe equation
  Work on the Q-curvature equation and the related fully nonlinear PDE’s.

• Connection to spectral theory.

• Applications to 4-dimensional conformal geometry and higher dimensional Kleinian groups.
  An existence theorem for conformal metrics of positive Ricci curvature. A conformal sphere theorem.

• Conformally compact Einstein structures.
Review of the Riemann curvature tensor:

- The Riemann tensor $Rm = R_{ijkl}$ is defined in terms of a nonlinear expression involving up to two derivatives of the metric.

- The sectional curvature of the plane $v \wedge w$ is given by $K(v \wedge w) = \sum R_{ijkl} v^i v^k w^j w^l$ when $v, w$ are orthonormal.

- Ricci curvature in the direction $v = v_1$ is given as a trace $Ric(v, v) = \sum_{i=2}^{n} K(v, v^i)$.

- The scalar curvature $R = \sum_{i=1}^{n} Ric(v_i, v_i)$.
Decomposition of the curvature tensor:

The Riemann curvature tensor has the decomposition

\[ Rm = W \bigoplus A \bigotimes g \]

where

\[ A = \frac{1}{n - 2} \left[ R_{i,j} - \frac{R}{2(n - 1)} g_{i,j} \right] \]

is called the Schouten tensor which is determined by the Ricci tensor; and \( \bigotimes \) is the Nomizu-Kulkarni product of the symmetric two tensors.

- The Weyl tensor satisfies \( W_{\bar{g}} = \rho^{-1} W_g \).

- The Ricci tensor controls the growth of volume of balls, and hence the topology of the underlying space.
§ Analytic aspects: A blow up sequence of functions

Sobolev Embedding Theorem:
For all $v \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$

\[
\begin{align*}
\lambda \left( \int_{\mathbb{R}^n} |v|^p dx \right)^{\frac{2}{p}} &\leq \int_{\mathbb{R}^n} |\nabla v|^2 dx.
\end{align*}
\]

- We say that $W_0^{1,2}(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n)$.

- By a dilation of $v(x)$ to $v(\lambda x)$, we see $p$ in $(\ast)$ is $p = \frac{2n}{n-2}$.

The best constant $\lambda$ and the extremal functions $v$ for $(\ast)$: Assume $v(x) = v(|x|) = v(r)$,

\[
\begin{align*}
\left\{ \begin{array}{l}
 v'' + \frac{n-1}{r}v' + \lambda \frac{n+2}{v^{n-2}} = 0, \\
v(0) = a, \quad v'(0) = 0.
\end{array} \right.
\end{align*}
\]
One solution is
\[
\begin{align*}
\left\{ 
    v(x) &= \left( \frac{2}{1 + |x|^2} \right)^{\frac{n-2}{2}} \\
    \Lambda &= \frac{n(n-2)}{4} \omega_n^{2/n},
\end{align*}
\]
where \( \omega_n \) is the surface area of the unit sphere \( S^n \). We then observe that the inequality is invariant under:
\[
v \to v_\epsilon(x) = \epsilon \frac{2-n}{2} \frac{x - x_0}{\epsilon},
\]
where \( \epsilon > 0 \) and \( x_0 \) is any point in \( \mathbb{R}^n \). In other words, we have
\[
v_\epsilon(x) = \left( \frac{2\epsilon}{\epsilon^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}
\]
are all extremals for the Sobolev embedding \((*)\), we have the following remarkable theorem.
Theorem: (Bliss, Talenti, T. Aubin)

The best constant in the Sobolev inequality

\[ (*) \quad \Lambda \left( \int_{\mathbb{R}^n} |v|^p \, dx \right)^{2/p} \leq \int_{\mathbb{R}^n} |\nabla v|^2 \, dx. \]

for \( p = \frac{2n}{n-2} \) is \( \Lambda = \frac{n(n-2)}{4} \omega_n^{2/n} \). It is only realized by the functions \( v_\epsilon \).

Properties of \( v_\epsilon \): (fix \( x_0 = 0, \epsilon > 0, \))

\[ v_\epsilon(x) = \left( \frac{2\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} \]

(i) \( v_\epsilon(0) = (\frac{2}{\epsilon})^{\frac{n-2}{2}} \to \infty \) as \( \epsilon \to 0 \),

(ii) \( v_\epsilon(x) \to 0 \), for all \( x \neq 0 \), as \( \epsilon \to 0 \),

(iii) \( \int_{\mathbb{R}^n} |v_\epsilon(x)|^{\frac{2n}{n-2}} \, dx = \int_{\mathbb{R}^n} |v_1(x)|^{\frac{2n}{n-2}} \, dx \),

(iv) \( \int_{\mathbb{R}^n} |\nabla v_\epsilon(x)|^2 \, dx = \int_{\mathbb{R}^n} |\nabla v_1(x)|^2 \, dx \).
Thus $v_\epsilon$ is a sequence of functions

- bounded in $W^{1,2}(\mathbb{R}^n)$,

- The weak limit as $\epsilon \to 0$ is the zero function;

Hence it does not have a convergent subsequence in $L^{n-2}$. 

- The embedding of the Sobolev space $W^{1,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2}}$ is not compact. This lack of compactness due to the non-compact group of translations and dilations of $\mathbb{R}^n$ is the heart of the problem.
The Euler Lagrange equation for the extremal function satisfies:

\[-\Delta v = \frac{n(n - 2)}{4} v^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n.\]

Thus functions \( v_\epsilon \) above are solutions.

**Theorem:** (Caffarelli-Gidas-Spruck)

\( v_\epsilon \) are the only positive solutions of above equation.

We conclude:

- All critical points of the Sobolev embedding are minimal points.

- The positive solutions are unique up to dilations and translations.
Blow up sequence on the unit sphere $S^n$
Consider stereographic projection.

$$\pi : (S^n - \text{north pole}) \to \mathbb{R}^n$$

$$\xi \xrightarrow{\pi} x(\xi)$$

Sending the north pole on $S^n$ to $\infty$;
$\xi = (\xi_1, \xi_2, \ldots, \xi_{n+1})$ is a point in $S^n \subset \mathbb{R}^{n+1}$,
note{$x = (x_1, x_2, \ldots, x_n)$, then $\xi_i = \frac{2x_i}{1+|x|^2}$ for $1 \leq i \leq n$; $\xi_{n+1} = \frac{|x|^2-1}{|x|^2+1}$.

Suppose $u$ is a smooth function defined on $S^n$, note that the Jacobian of $\pi^{-1}$ as

$$J_{\pi^{-1}} = \left(\frac{2}{1+|x|^2}\right) I$$

$$v(x) = u(\xi(x)) \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}},$$
Sobolev inequality on $S^n$:

$$\Lambda \left( \int_{S^n} |u(\xi)|^{\frac{2n}{n-2}} d\sigma(\xi) \right)^{\frac{n-2}{2}} \leq \int_{S^n} |\nabla u(\xi)|^2 d\sigma(\xi)$$

$$+ \frac{n(n-2)}{4} \int_{S^n} |u(\xi)|^2 d\sigma(\xi),$$

where $d\sigma(\xi) = \left( \frac{2}{1+|x|^2} \right)^n$ is the standard area form on the unit sphere $S^n$.

The transformed function $u(\xi)$ satisfies:

$$-\Delta_g u + \frac{n(n-2)}{4} u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \text{ on } S^n,$$

where $\Delta_g = \left( \frac{2}{1+|x|^2} \right)^2 \Delta x$.

• **Uniqueness** Functions $u_\epsilon$ obtained from $v_\epsilon$ are the only positive solutions.
On \((M^n, g)\), the **conformal Laplacian** \(L_g\)

\[ L_g = -\Delta_g + c_n R_g \]

where \(c_n = \frac{n-2}{4(n-1)}\), and \(R_g\) denotes the scalar curvature of the metric \(g\).

**Euler equation for Sobolev inequality on** \((M^n, g)\): **Yamabe equation**:

\[ L_g u = c_n R_{\bar{g}} u^{\frac{n+2}{n-2}}. \]

where the conformal metric \(\bar{g} = u^{\frac{4}{n-2}} g\) for some positive function \(u\).

- **Yamabe problem**:
  
  Given \((M^n, g)\), find positive function \(u\) so that \(R_{\bar{g}}\) a constant.
  
  (Yamabe, Trudinger, Aubin, Schoen).
Variational method:
Find the extremals for the inequality:

$$\Lambda_g \left( \int_M |u|^{2n/(n-2)} dv_g \right)^{n-2} \leq \int_M |\nabla_g u|^2 dv_g + c_n \int_M R_g |u|^2 dv_g,$$

for some constant $\Lambda_g \leq \Lambda$.

- This constant $\Lambda_g$ is called the Yamabe constant, and is conformally invariant.

A crucial ingredient in the proof: to establish some criteria for compactness of the minimizing sequence. That is to distinguish the manifold from the standard sphere by establishing $\Lambda_g < \Lambda_{gc}$.

- In the solution by Aubin, the non-vanishing of the Weyl tensor in high dimensions plays this crucial role.
Schoen uses the positive mass theorem to differentiate the conformal structure from the standard n-sphere.

Mass associated to a point $p$ is defined as the finite part $A$ in the asymptotic expansion of the Green’s function of the conformal Laplacian with pole at $p$: in a geodesic coordinate system $x$ whose origin is the given pole $p$, the Green’s function $G$ is the solution of the equation

$$L_gG = (n - 2)\omega_{n-1}\delta_p.$$ 

Near the point $p$ there is an expansion:

$$G(x) = |x|^{2-n} + A + O(|x|).$$

$A \geq 0$, $A = 0$ if and only if $(M^n, g) = (S^n, g_c)$. 
Moser-Trudinger Inequality

Sobolev embedding Theorem:

$$W^{1,q}_0(D) \hookrightarrow L^q$$ with $$\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$$.

When $$q = 2$$, $$p = \frac{2n}{n-2}$$ for $$n \geq 3$$.

When $$q = 2$$, $$n = 2$$ $0 < p < \infty$, but $$p \neq \infty$$.

Example: Take $$D$$ to be the unit ball $$B$$ in $$\mathbb{R}^2$$, $$w(x) = \log |\log(e - 1 + \frac{1}{|x|})|$$.

Theorem: (Moser, Trudinger)

Suppose $$D$$ is a smooth domain in $$\mathbb{R}^2$$, then there is a constant $$C$$, for all functions $$w \in W^{1,2}_0(D)$$ with $$\int_D |\nabla w(x)|^2 dx \leq 1$$, we have

$$\int_D e^{\alpha w^2}(x) dx \leq C|D|$$,

for any $$\alpha \leq 4\pi$$, with $$4\pi$$ being the best constant.
- Existence of extremal functions for Moser's inequality. (Carleson-Chang)

- Linearized form of the inequality is useful:

$$\log \frac{1}{|D|} \int_D e^{2w} \, dx \leq \frac{1}{4\pi} \int_D |\nabla w|^2 \, dx.$$ 

- (W. Chen and C. Li)
Suppose $w$ is in $C^2(\mathbb{R}^2)$, with $e^{2w} \in L^1(\mathbb{R}^2)$, and satisfies the equation

$$-\Delta w = e^{2w} \text{ on } \mathbb{R}^2.$$ 

Then

$$w(x) = \log \frac{2\epsilon}{\epsilon^2 + |x - x_0|^2}$$

for some $\epsilon > 0$ and some $x_0 \in \mathbb{R}^2$. 

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§ Gaussian curvature on compact surface

- Recall on \((M^2, g)\) a compact surface, we have \(\Delta = \Delta_g\) and the Gaussian curvature \(K = K_g\).

- Under the conformal change \(g_w = e^{2w}g\),

\[
(1) \quad - \Delta g_w + K_g = K_w e^{2w} \text{ on } M
\]

\(K_w\) denotes the Gaussian curvature of \((M, g_w)\).

- The Gauss-Bonnet Theorem:

\[
2\pi \chi(M) = \int_M K_w dv_{g_w}
\]

where \(\chi(M)\) is the Euler characteristic of \(M\).

- Uniformization Theorem to classify compact closed surfaces can be viewed as finding solutions with \(K_w \equiv -1, 0, \text{ or } 1\) according to the sign of \(\int K dv_g\).
(1) $- \Delta_g w + K_g = K_w e^{2w}$ on $M$

Variational Functional:

$$J[w] = \int_M |\nabla w|^2 dv_g + 2 \int_M K_g w dv_g$$

$$- (\int_M K_g dv_g) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}.$$ 

Nirenberg problem: Which functions can be the Gaussian curvature function $K_w$, in particular on $(S^2, g_c)$.

• Kazdan-Warner

On $S^2 = \{(\xi_1, \xi_2, \xi_3)| \sum_{i=1}^3 \xi_i^2 = 1\}$, there is an obstruction for the problem:

$$\int_{S^2} \nabla K_w \cdot \nabla \xi \ e^{2w} dv_{g_c} = 0.$$
Theorem: (Moser)
Any positive $C^2$ even function $f$ (i.e. $f(\xi) = f(-\xi)$ for all $\xi \in S^2$) can be a Gaussian curvature function on $(S^2, g_c)$.

Theorem: (Onofri; T. Aubin) $J[w] \geq 0$
and $J[w] = 0$ precisely for conformal factors $w$ of the form $e^{2w}g = T^*g$ where $T$ is a Mobius transformation of the 2-sphere.

Leray-Schauder degree theory for (1):
(Chang-Yang, Chang-Gursky-Yang)
(C.C. Chen and C.S. Lin)

Assume $f$ is a Morse function satisfying the (non-degenerate condition) $\Delta f(\xi) \neq 0$ at the critical points $\xi$ of $f$,

$$\text{degree} = \sum_{\nabla f(q) = 0, \Delta f(q) < 0} (-1)^{\text{ind}(q)} - 1.$$
Geometric content of the functional $J[w]$

Polyakov-Ray-Singer Formula

On $(M^2, g)$

$$J[w] = 12\pi \log \left( \frac{\det(-\Delta_g)}{\det(-\Delta_{gw})} \right)$$

where the determinant of the Laplacian $\det(-\Delta_g)$ is defined by Ray-Singer as:

$$\log \det(-\Delta_g) := -\zeta'(0).$$

Definition

On compact Riemannian manifold $(M^n, g)$, consider eigenvalue of $-\Delta_g$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$$

and the zeta function

$$\zeta(s) := \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$
Formal differentiation leads to
\[ \zeta'(s) = \sum_{\lambda_k \neq 0} - (\log \lambda_k) \lambda_k^{-s}, \text{ i.e.} \]
\[ \zeta'(0) = - \sum_{\lambda_k \neq 0} \log \lambda_k = - \log \prod_{k=1}^{\infty} \lambda_k. \]

Apply Mellin transform for all \( x > 0 \),
\[ x^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-xt} t^{s-1} \, dt. \]

We can rewrite \( \zeta(s) \) in terms of the Gamma function:
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \sum_{j=1}^{\infty} e^{-\lambda_j t} t^{s-1} \, dt \]
\[ = \frac{1}{\Gamma(s)} \int_{0}^{\infty} (Z(t) - 1) t^{s-1} \, dt, \]

where \( Z(t) \) denotes the Heat kernel. The existence of \( \zeta'(0) \) can be justified via Weyl's asymptotic formula of the heat kernel.
• Onofri’s inequality is equivalent to the statement $\det(-\Delta_{g_c})$ is maximal among all metrics $g$ on $S^2$.

• Osgood-Phillips-Sarnak independently derived Onofri’s inequality and established the $C^\infty$ compactness of isospectral metrics on compact surfaces.

• Chang-Yang, Brooks-Perry-Peterson: Partial results for isospectral compactness for 3-manifolds.

• Okikiolu: Among all metrics with the same volume as the standard metric on the 3-sphere, the standard canonical metric is a local maximum for the functional $\det(-\Delta_g)$. 