ON A FOURTH ORDER CURVATURE INVARIANT

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§0. Introduction

This article is a continuation of the survey article [C3] on the fourth order equation first introduced by Paneitz ([Pa]). In particular we describe the geometric aspect of the equation, as well as further analytic development since that survey was written.

On a Riemannian manifold (M^n, g) of dimension n, the Laplace Beltrami operator is the natural geometric operator. Under conformal change of metric $\tilde{g} = e^{2w} g$, when the dimension is two, Δ_{g_w} is related to Δ_g by the simple formula:

$$\Delta_{g_w}(\varphi) = e^{-2\omega} \Delta_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^2)\,. \tag{0.1}$$

In dimension greater than two, similar tranformation property continue to hold for a modification of the Laplacian operator called the conformal Laplacian operator $L \equiv -\frac{4(n-1)}{n-2}\Delta + R$ where R is the scalar curvature of the metric. We have

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}\omega} L_g\left(e^{\frac{n-2}{2}\omega}\varphi\right) \tag{0.2}$$

for all $\varphi \in C^\infty(M)$.

In general, we call a metrically defined operator A conformally covariant of bidegree (a, b), if under the conformal change of metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n)\,. \tag{0.3}$$

A particularly interesting such operator is a fourth order operator on 4-manifolds discovered by Paneitz [Pa] in 1983:

$$P\varphi \equiv \Delta^2\varphi + \delta\left(\frac{2}{3}RI - 2\text{Ric}\right)d\varphi \tag{0.4}$$

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where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P (which we will later denote by P_4) is conformal covariant of bidegree $(0,4)$ on 4-manifolds, i.e.

$$P_{g_\omega}(\varphi) = e^{-4\omega} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4)\,. \tag{0.5}$$

For manifold of general dimension n, when n is even, the existence of a n-th order operator P_n conformal covariant of bidegree $(0,n)$ was verified in [GJMS]. However it is only explicitly known on the standard Euclidean space $\mathbb{R}^n$ and hence on the standard sphere S^n. The explicit formula for P_n on standard sphere S^n has appeared in Branson [Br-1] and independently in Beckner [B].

In dimension four, the Paneitz equation has close connection with the Chern-Gauss-Bonnet formula. The Q curvature invariant defined by the Paneitz operator integrates to a numerical conformal invariant which is essentially bounded by the Euler number. We have obtained ([CY3]) general existence and uniqueness criteria for the Paneitz and related equations. The appearance of the traceless Ricci tensor in the equation can be exploited to characterize certain special conformal classes in the work of Gursky ([Gu1]). He has also shown that the criteria for existence in ([CY3])is satisfied by a large class of conformal structures. To further relate the role of the quantity Q curvature, it is natural to understand the contribution of the boundary term in the Chern-Gauss-Bonnet integral when we incorporate the Q curvature. Previously Chang-Qing ([CQ1], [CQ2]) had considered the boundary term and defined a third order boundary operator which we shall call P_3 operator which defines boundary curvature invariant. Recently we have derived in collaboration with Qing ([CQY1, CQY2]) Cohn-Vossen type inequality which express the difference of the Q curvature integral from the Euler number by an isoperimetric ratio analogous to Finn's result in dimension two. Such an integral formula gives a criteria for conformal compactification of complete conformally flat manifold of positive scalar curvature and finite Q curvature integral. As a consequence we obtain a classification of such manifolds with zero Q curvature.

One cannot discuss the Paneitz operator without mentioning its connection with the zeta function formulation of the determinant of the conformal Laplacian. Such consideration in fact lead to the relevance of the Paneitz operator to four dimensional conformal geometry, as well as the discovery of the boundary P_3 operator. It is appropriate to acknowledge on this occasion the pioneering work of Osgood-Phillips-Sarnak ([OPS1], [OPS2]) in two dimension.

This paper is organized as follows. In section one, we discuss basic properties of the Laplace operator in dimension two and compared it, from the point of view of conformal geometry, to analogous properties of the Paneitz operator P_4. In section two, we discuss uniqueness results of the Paneitz equation on spheres. In section three, we consider the variational functional for the Paneitz operator P_4 on general compact 4-manifolds. In section four, we discuss another natural geometric functional-namely the zeta functional determinant for the conformal Laplacian operator-where Paneitz operator plays an important role. We survey some existence and regularity results of the extremal metrics of the zeta functional determinant, and indicate some recent geometric applications by M. Gursky of the extremal metrics to characterize some compact 4-dimensional conformal structures. In section five, we discuss the P_3 operator, which is conformally covariant of bidegree $(0, 3)$; and its associated curvatures T, operating on functions defined on the boundary of compact 4-manifolds. We mention some existence results for extremal metrics in this setting. In section six, we discuss the Chern-Gauss-Bonnet formula that relates the Q curvature integral to the Euler number. We discuss the generalization of the Cohn-Vossen inequality for surfaces to 4-dimensional conformally flat manifolds as well as some application to obtain further uniqueness result for the Paneitz equation on such manifolds. In section seven, we consider a variational problem in which the P_4 and P_3 operators both play a role. We define a numerical conformal invariant of the pair $(M^4, \partial M)$ and prove an existence theorem when this numerical invariant is suitably restricted. In order to avoid technical complications caused by the boundary geometry, we consider boundaries which are umbilic. We show that the argument of Gursky ([Gu2]) can be modified to determine the conformally flat pairs $(M, \partial M)$ that satisfies the simplest equations $Q = 0$ and $T = 0$.

We are indebted to many friends and collaborators among whom we shall mention Branson, Gilkey, Gursky, C.S. Lin, Qing, Sarnak and Xu.

§1. Properties of the Paneitz operator

On a compact Riemannian manifold (M^n, g) without boundary, when the dimension of the manifold is two, we denote by $P_2 \equiv -\Delta = -\Delta_g$, the Laplacian operator. When the dimension is four, we denote by $P = P_4$ the Paneitz operator as defined on (0.4). Thus both operators satisfy conformal covariance property $(P_n)_\omega = e^{-n\omega} P_n$, where $(P_n)_\omega$ denote the operator with respect to (M^n, g_ω), $g_\omega = e^{2\omega} g$. Here we list several such properties for comparison.

(i) On a compact surface, a natural curvature invariant associated with the Laplace operator is the Gauss curvature K. Under the conformal change of metric $g_\omega = e^{2\omega} g$, we have

$$\Delta\omega + K_\omega e^{2\omega} = K \quad \text{on} \quad M^2 \tag{1.1$_a$}$$

where K_ω denotes the Gaussian curvature of (M^2, g_ω). While on 4-manifold, we have

$$-P_4\omega + 2\tilde{Q}_w e^{4\omega} = 2\tilde{Q} \quad \text{on} \quad M^4 \tag{1.1$_b$}$$

where $\tilde{Q}$ is the curvature invariant

$$12\tilde{Q} = -\Delta R + R^2 - 3|\mathrm{Ric}|^2 \tag{1.2}$$

(ii) The analogy between K and $\tilde{Q}$ becomes more apparent if one considers the Gauss-Bonnet formulae:

$$2\pi\chi(M) = \int K dv \qquad \text{on} \quad M = M^2 \tag{1.3$_a$}$$

$$4\pi^2\chi(M) = \int \left(\tilde{Q} + \frac{|C|^2}{8}\right) dv \quad \text{on} \quad M = M^4 \tag{1.3$_b$}$$

where $\chi(M)$ denotes the Euler-characteristic of the manifold M, and $|C|^2$= norm squared of the Weyl tensor. Since $|C|^2 dv$ is a pointwise invariant under conformal change of metric, $\tilde{Q}dv$ is the term which measures the conformal change in formula $(1,3)_b$.

(iii) When $n \geq 3$, another natural analogue of $-\Delta$ on M^2 is the conformal Laplacian operator L as defined on (0.2). In this case, if we denote the conformal change of metric as $g_u = u^{\frac{4}{n-2}} g$ for some positive function u, then we may rewrite the conformal covariant property (0.2) for L as

$$L_u(\varphi) = u^{-\frac{n+2}{n-2}} L(u\varphi) \quad \text{on} \quad M^n, n \geq 3 \tag{1.4$_a$}$$

for all $\varphi \in C^\infty(M^n)$.

A differential equation associated with the operator L is the scalar curvature equation:

$$Lu = R_u u^{\frac{n+2}{n-2}} \qquad \text{on} \quad M^n, \quad n \geq 3\,. \tag{1.5$_a$}$$

Equation $(1.5)_a$ has been intensively studied in the recent decade. For example the famous Yamabe problem in differential geometry is the study of the equation $(1.5)_a$ for solutions $R_u \equiv$ constant; the problem has been completely solved by Yamabe [Y], Trudinger [T-1], Aubin [Au] and Schoen [Sc].

(iv) There is also a natural fourth order Paneitz operator P_4^n in all dimension $n \neq 4$, which enjoys the conformal covariance property with respect to conformal changes in metrics also. The relation of this operator to the Paneitz operator in dimension four is completely analogous to the relation of the conformal Laplacian to the Laplacian in dimension two. On (M^n, g) when $n \neq 4$, define

$$P_4^n = (-\Delta)^2 + \delta(a_n R + b_n Ric) d + \frac{n-4}{2} Q_4^n;$$

where

$$Q_4^n = c_n |Ric|^2 + d_n R^2 - \frac{1}{2(n-1)} \Delta R,$$

and $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$, $c_n = -\frac{2}{(n-2)^2}$, $d_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$ are dimensional constants. Thus $P_4^4 = P_4$, $Q_4^4 = \tilde{Q}$. Then (Branson [Br1]), we have for $g_u = u^{\frac{4}{n-4}} g$, $n \neq 4$

$$\left(P_4^n\right)_u (\varphi) = u^{-\frac{n+4}{n-4}} \left(P_4^n\right)(u\varphi) \tag{1.4$_b$}$$

for all $\varphi \in C^\infty(M^n)$. We also have the analogue for the Yamabe equation:

$$P_4^n u = Q_4^n u^{\frac{n+4}{n-4}} \qquad \text{on} \quad M^n, \quad n \neq 4\,. \tag{1.5$_b$}$$

We remark that on $\mathbb{R}^n$ with Euclidean metric, P_4^n reduces to the bi-Laplacian operator. Equation $(1.5)_b$ takes the form $(-\Delta)^2 u = c_n u^{\frac{n+4}{n-4}}$, an equation which has been studied in literature e.g. [PuS].

§2. Uniqueness result on S^n

In this section we will consider the behavior of the Paneitz operator on the standard spheres (S^n, g). First we recall the situation when $n = 2$. On (S^2, g), when one makes a conformal change of metric $g_\omega = e^{2\omega} g$, the Gaussian curvature $K_\omega = K(g_\omega)$ satisfies the differential equation

$$\Delta\omega + K_\omega e^{2\omega} = 1 \tag{2.1}$$

on S^2, where Δ denotes the Laplacian operator with respect to the metric g on S^2.

When $K_w \equiv 1$ on (2.1), the Cartan-Hadamard theorem asserts that $e^{2w} g$ is isometric to the standard metric g by a diffeomorphism φ; and the conformality requirements says φ is a conformal transformation of S^2. In particular, $w = \frac{1}{2}\log|J_\varphi|$, where J_φ denotes the Jacobian of the transformation φ.

In [CL], Chen and Li studied the corresponding equation of (2.1) on $\mathbb{R}^2$ with $K_w \equiv 1$, and they proved, using the method of moving plane, that when u is a smooth function defined on $\mathbb{R}^2$ satisfying

$$-\Delta u = e^{2u} \qquad \text{on} \quad \mathbb{R}^2 \tag{2.2}$$

with $\int_{\mathbb{R}^2} e^{2u} dx < \infty$, then $u(x)$ is of the form $u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$ for some $x_0 \in \mathbb{R}^2$ and some $\lambda > 0$. There is an alternative argument by Chanillo-Kiessling ([ChK]) for this uniqueness result using the isoperimetric inequality.

On (S^n, g), denote $g_u = u^{\frac{4}{n-2}} g$ the conformal change of metric of g, where u is a positive function, then the scalar curvature $R_u = R(g_u)$ of the metric is determined by the following differential equation

$$-\frac{4(n-1)}{n-2}\Delta u + n(n-1)u = R_u u^{\frac{n+2}{n-2}}. \tag{2.3}$$

When $R_u = n(n-1)$, a uniqueness result established by Obata [Ob] states that this happens only if the metric g_u is isometric to g or equivalently $u = |J_\varphi|^{\frac{n-2}{2n}}$ for some conformal transformation φ of S^n. In [CGS] Caffarelli-Gidas-Spruck studied the corresponding equation on $\mathbb{R}^n$:

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \qquad u > 0 \quad \text{on} \quad \mathbb{R}^n\,. \tag{2.4}$$

They classified all solutions of (2.4), via the method of moving plane, as $u(x) = (\frac{2\lambda}{\lambda^2 + |x - x_0|^2})^{\frac{n-2}{2}}$ for some $x_0 \in \mathbb{R}^n$, $\lambda > 0$.

For all n, on (S^n, g), there also exists a n-th order (pseudo) differential operator $\mathbb{P}_n$ which is the pull back via sterographic projection of the operator $(-\Delta)^{n/2}$ from

$\mathbb{R}^n$ with Euclidean metric to (S^n, g). $\mathbb{P}_n$ is conformal covariant of bi-degree (0, n), i.e. $(\mathbb{P}_n)_w = e^{-nw}\mathbb{P}_n$. According to Branson [Br1] and Beckner [B]:

$$(2.5)\quad \begin{cases} \text{For} \quad n \quad \text{even} \quad \mathbb{P}_n & = \prod_{k=0}^{\frac{n-2}{2}}(-\Delta + k(n-k-1)), \\ \text{For} \quad n \quad \text{odd} \quad \mathbb{P}_n & = \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=0}^{\frac{n-3}{2}}(-\Delta + k(n-k-1)). \end{cases}$$

On general compact manifolds in the cases when the dimension of the manifold is two or four, there exist natural curvature invariants Q_n of order n which, under conformal change of metric $g_w = e^{2w}g$, is related to $P_n w$ through the following differential equation:

$$(2.6)\qquad -P_n w + (Q_n)_w e^{nw} = Q_n \quad \text{on} \quad M\,.$$

In the case when $n = 2$, P_2 is the negative of the Laplacian operator, $Q_2 = K$, the Gaussian curvature. When $n = 4$, P_4 is the Paneitz operator, $Q_4 = 2\tilde{Q}_4$ as defined in (1.2). In the special case of (S^2, g), $P_2 = \mathbb{P}_2$, similarly on (S^4, g), $P_4 = \mathbb{P}_4$. In section 5 below, we will also discuss the existence of P_1, P_3 operators and corresponding curvature invariants Q_1 and Q_3 defined on boundaries of general compact manifolds of dimension 2 and 4 respectively.

On (S^n, g), when the metric g_w is isometric to the standard metric, then $(Q_n)_w = Q_n = (n-1)!$. In this case, equation (2.6) becomes

$$(2.7)\qquad -P_n w + (n-1)! e^{nw} = (n-1)! \quad \text{on} \quad S^n$$

One can establish the following uniqueness result for solutions of equation (1.7).

Theorem 2.1. *[CY4] On (S^n, g), all smooth solutions of the equation (2.7) are of the form $e^{2w}g = \varphi^*(g)$ for some conformal transformation φ of S^n; i.e. $w = \frac{1}{n}\log|J_\varphi|$ for the transformation φ.*

Appealing to the conformal covariance of the equation (2.7) one can rewrite the equation on R^n:

$$(2.8)\qquad (-\Delta)^{n/2} u = (n-1)! e^{nu} \quad \text{on} \quad \mathbb{R}^n\,.$$

For the fourth order equation C.S. Lin ([L]) and Xu ([X1]) have obtained a classification of solutions of the equation (2.8) in $\mathbb{R}^4$. Then combining the arguments of Wei-Xu ([WX]) in even dimensions and that of Zhu ([Z]) in odd dimensions, solutions of equation (2.8) can be classified:

Theorem 2.2. *([L], [X1], [WX], [Z]) On $\mathbb{R}^n$, suppose u is a smooth solution of the equation (2.8) such that:*

$$(2.9)\qquad u(x) = \log\frac{2}{1+|x|^2} + o(|x|^2),$$

then $u(x)$ is of the form

$$u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \quad \text{for some} \quad x_0 \in \mathbb{R}^n \ \lambda > 0. \tag{2.10}$$

Remarks

1. In the case when w is a minimal solution of the functional with Euler-Lagrange equation (1.7), the result in Theorem 2.2 is a consequence of the sharp Sobolev type inequality of Milin-Lebedev when $n = 1$, that of Moser [Mo] and Onofri [On] when $n = 2$ and Beckner [B] for general n.

2. The condition (2.9) is necessary as one can show using an argument of McOwen ([Mc]) that for any choice of $a_i > 0$, $1 \le i \le 4$ there exists solution v of the equation

$$(-\Delta)^2 v = exp(-\sum_{i=1}^{4} a_i x_i^2) e^{4v}, \text{ on } \mathbb{R}^4$$

which has the asymptotic behavior $v(x) \equiv -2\log(|x|)$. Then setting

$$u(x) = -\frac{1}{4}\sum_{i=1}^{4} a_i x_i^2 + v(x),$$

we obtain an extraneous solution of the equation which does not satisfy the condition (2.9). Geometrically such solutions correspond to metrics which are incomplete and has very large negative scalar curvature at infinity.

Concerning the analogue of the Yamabe equation $(1.5)_b$, Lin and Xu have also obtained uniquess result on R^n.

Theorem 2.3. *([L], [X1]) In $\mathbb{R}^n, n \ge 5$, the positive solutions of the equation $\Delta^2 u = u^{(n+4)/(n-4)}$ are of the form $u(x) = (\frac{\lambda}{\lambda^2 + |x-x_0|^2})^{\frac{n-4}{2}}$.*

In general dimensions $n \ge 5$ there is additional analytic difficulty associated with the lack of a good maximum principle for fourth order equations. In particular it is not a simple matter to verify that minimizing solutions to $(1.5)_b$ are positive.

§3. Existence and regularity of the Paneitz equation on 4-manifold

On (M^2, g) with Gaussian curvature $K = K_g$, consider the functional

$$J[w] = \int |\nabla w|^2 dv + 2\int K w dv - (\int K dv) \log ⨍ e^{2w} dv \tag{3.1}$$

where the gradient, the volume form are taken with respect to the metric g, and $⨍ \varphi dv = \int \varphi dv / \text{volume}$ for all φ.

The Euler-Lagrange equation for J is:

$$\Delta\omega + ce^{2\omega} = K \qquad \text{on} \quad M^2 \tag{3.2}$$

where c is a constant. Notice that (3.2) is a special case of equation $(1.1)_a$ with $K_\omega \equiv c$. For the special manifold (S^2, g), $K \equiv 1$, (3.2) is a special case of equation (2.1).

On a compact 4-manifold (M^4, g), denote by $k_p = \int Q dv$, and define

$$(3.3) \qquad II[\omega] = \int (P_4\omega)\omega + 4\int Q w dv - (\int Q dv) \log(⨍ e^{4\omega} dv)$$

Theorem 3.1. *([CY-3]) Suppose $k_p < 8\pi^2$, and suppose P_4 is a positive operator with* ker $P = \{\text{constants}\}$*; then* $\inf II_{w\in W^{2,2}}[w]$ *is attained. Denote the infimum by w_p, then the metric $g_p = e^{2w_p} g$ satisfies $Q_p \equiv \text{constant} = k_p / \int dv$.*

Remarks

(1) In general, the positivity of P_4 is a necessary condition for the functional II to be bounded from below. Recent work of Gursky [Gu-2] indicates that under the additional assumption that $k_p > 0$ and that g is of positive scalar class, P_4 is always positive. Furthermore, under the same assumption, $k_p < 8\pi^2$ is always satisfied unless (M^4, g) is conformally equivalent to (S^4, g); in the latter case then $k_p = 8\pi^2$ and the extremal metric for $II[w]$ has been studied in [BCY].

(2) Notice that the extremal function w_p in $W^{2,2}$ for II satisfies the equation

$$(3.4) \qquad -P_4 w_p + Q_p e^{4\omega_p} = Q$$

with $Q_p \equiv$ constant. Thus standard elliptic theory can be applied to establish the smoothness of w_p. This is in contrast with the smoothness property of the extremal function w_d of the log-determinant functional $F[w]$, in which $II[w]$ is one of the term. We will discuss regularity property of w_d in section 4.

(3) A key analytic fact used in establishing Theorem 3.1 above is the generalized Moser inequality established by Adams [A], which in the special case of domains Ω in $\mathbb{R}^4$ states that $W_0^{2,2}(\Omega) \hookrightarrow \exp L^2$.

§4 Zeta functional determinant

On compact surface (M^2, g), let $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ be the spectrum of the (negative of) Laplacian -Δ_g. Let $\zeta(s) = \sum \lambda_i^{-s}$ defined for $Res > \frac{1}{2}$, then ζ has a meromorphic continuation to the whole plane and is regular at the origin using the heat kernel expansion of Δ_g. $-\zeta'(0)$ is well-defined, and one may define $\log\det \Delta_g$ to be $-\zeta'(0)$ (as in Ray-Singer [RS]). In [Po], Polyakov further computed the logarithm of the ratio of determinant of two conformally related metrics $g_w = e^{2w} g$ on a compact surface without boundary.

$$(4.1) \qquad F[w] = \log \frac{det\Delta_w}{det\Delta} = \frac{1}{3}\int_M (|\nabla w|^2 + 2Kw) dv_g$$

under the normalization that $\text{vol}(g_w) = \text{vol}(g)$. Notice that $F[w]$ is essentially the same as the functional $J[w]$ in (3.1). In a series of papers, Osgood-Phillips-Sarnak ([OPS1], [OPS2])have shown among other things that $F[w]$ enjoys a certain compactness property on account of the Moser-Trudinger inequality, and proved that in each conformal class, the functional $F[w]$ attains its extrema at the constant curvature metrics.

When the dimension of a closed manifold is odd, it was shown in Branson [Br2] that $\log \det L_g$ is a conformal invariant. Thus the next natural dimension to study the generalized Polyakov formula (4.1) is four.

Suppose (M, g) is a compact, closed 4-manifold, and suppose A is a conformally covariant operator satisfying (0.3). In [BO] Branson-Orsted gave an explicit computation of the normalized form of $\log \frac{\det A_w}{\det A}$ which may be expressed as:

$$F[w] = \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w] \tag{4.2}$$

where $\gamma_1, \gamma_2, \gamma_3$ are constants depending only on A and

$$\begin{aligned} I[w] &= 4 \int |C|^2 w dv - \left(\int |C|^2 dv \right) \log \fint e^{4w} dv \\ II[w] &= \langle Pw, w \rangle + 4 \int Qw dv - \left(\int Q dv \right) \log \fint e^{4w} dv, \\ III[w] &= 12 \left(Y(w) - \frac{1}{3} \int (\triangle R)\, w dv \right), \end{aligned}$$

where C is the Weyl tensor, and $Y(w) = \int \left(\frac{\Delta(e^w)}{e^w} \right)^2 - \frac{1}{3} \int R |\nabla w|^2$. We also remark that the functional $III[w]$ [BCY] may be written as

$$III[w] = \frac{1}{3} \left[\int R_w^2\, dv_w - \int R^2\, dv \right]$$

so that when the background metric is assumed to be the Yamabe metric in a positive conformal class, the functional III is non-negative.

Let us define the conformal invariant:

$$\begin{aligned} k_d &= -\gamma_1 \int |C|^2 dv - \gamma_2 \int Q dv \\ &= (-\gamma_2)\, 4\pi^2 \chi(M) + \left(\frac{\gamma_2}{8} - \gamma_1 \right) \int |C|^2 dv \end{aligned} \tag{4.3}$$

Theorem 4.1. *([CY3]) If the functional F satisfies $\gamma_2 < 0$, $\gamma_3 < 0$, and $k_d < (-\gamma_2) 8\pi^2$, then $\sup_{w \in W^{2,2}} F[w]$ is attained by some function w_d and the metric $g_d = e^{2w_d} g_0$ satisfies the equation*

$$\gamma_1 |C_d|^2 + \gamma_2 Q_d - \gamma_3 \triangle_d R_d = -k_d \cdot Vol(g_d)^{-1}. \tag{4.4}$$

Further, all functions $\varphi \in W^{2,2}$ satisfy the inequality:

$$k_d \log \fint e^{4(\varphi - \tilde{\varphi})} dv_d \le (-\gamma_2) \langle P\varphi, \varphi \rangle - 12 \gamma_3 Y_d(\varphi). \tag{4.5}$$

where $\tilde{\varphi}$ denotes the mean value of φ with respect to the metric g_d, and $\fint$ denotes $\frac{1}{vol(M, g_d)} \int_M dv_d$.

Theorem 4.2. *If $k_d \leq 0$, the extremal metric g_d for the functional F corresponding to the conformal Laplacian operator L is unique.*

This uniqueness assertion is obtained as consequence of the convexity of the corresponding functionals. Applying the uniqueness result, we were able to identify some of the extremal metrics with known metric in special circumstances.

As a consequence of a general regularity result for minimizing $W^{2,2}$ solution of the equation:

$$\Delta\Delta\omega = c_1|\nabla\omega|^4 + c_2(\Delta\omega)^2 + c_3\Delta\omega|\nabla\omega|^2 + \text{lower order terms}, \tag{4.6}$$

we have the following regularity result:

Theorem 4.3. *([CGY]) Let $F[\omega]$ be as in Theorem 4.1, then $\sup_{\omega\in\omega^{2,2}} F[\omega]$, when attained, is a smooth function.*

The appearance of the Ricci tensor in the Euler equation (4.4) can be put to good use since such terms also arise in the Bochner formula for the Laplacian of the length of harmonic forms. Indeed Gursky has given ([Gu1]) beautiful application of the extremal metrics (also some modified version) to characterize certain special 4-dimensional conformal classes. To state his results, we need to make some definitions. On a compact manifold (M^n, g), define the Yamabe invariant of g as

$$Y(g) = \inf_{g_\omega = e^{2\omega}g} \mathrm{vol}(g_\omega)^{-\frac{n-2}{n}} \int R_{g_\omega} dv_{g_\omega}. \tag{4.7}$$

By the work of Yamabe, Trudinger, Aubin and Schoen mentioned in section 1, every compact manifold M^n admits a metric g_ω conformal to g which achieves $Y(g)$, hence g_ω has constant scalar curvature. We say (M^n, g) is of positive scalar class if $Y(g) > 0$.

On compact 4-manifolds, both $Y(g)$ and $\int Q_g dv_g$ are conformal invariants. The following result of Gursky [Gu1] indicates that these two conformal invariants constrain the topological type of M^4.

Theorem 4.4. *([Gu1]) Suppose (M^4, g) is a compact manifold with $Y(g) > 0$,*
(i) *If $\int Q_g dv_g > 0$, then M admits no non-zero harmonic 1-forms. In particular, the first Betti number of M vanishes.*
(ii) *If $\int Q_g dv_g = 0$, and if M admits a non-zero harmonic 1-form, then (M, g) is conformal equivalent to a quotient of the product space $S^3 \times \mathbb{R}$. In particular (M, g) is locally conformally flat.*

As a corollary of part (ii) of Theorem 4.4, one can characterize quotient of the product space $S^3 \times \mathbb{R}$ as compact, locally conformally flat 4-manifold with $Y(g) > 0$ and $\chi(M) = 0$.

A crucial step in the proof of theorem above is to show that for suitable choice of γ_1, γ_2, γ_3, the extremal metric g_d for the log-determinant functional $F[\omega]$ exists and is unique. Furthermore under the assumption $Y(g) > 0$, one has $R_{g_d} > 0$; if $Y(g) = 0$ then $R_{g_d} \equiv 0$. In the case $\int Q_g dv_g = 0$, existence of non-zero harmonic 1-form actually indicates that $R_{g_d} \equiv$ positive constant. It is curious that in characterizing these special conformal classes one cannot work directly with the $Q =$ constant metrics. Using similar ideas, Gursky [Gu1] has also characterized the conformal class of Kahler-Einstein surfaces:

Theorem 4.5. *([Gu1]) Suppose (M^4, g) is compact 4-manifold with non-negative scalar curvature, and suppose the self intersection form has a positive element, then*

$$\int |W_+|^2 dV \geq \frac{4\pi^2}{3}(2\chi + 3\sigma) \tag{4.8}$$

where W_+ is the self dual part of the Weyl tensor, χ is the euler number and σ is the signature. Furthermore,

(i) equality is achieved in (4.8) by some metric with $Y(g) > 0$ if and only if g is conformal to a Kahler-Einstein metric with positive scalar curvature,

(ii) equality is achieved in (4.8) by some metric with $Y(g) = 0$ if and only if g is conformal to a Kahler Ricci flat metric.

§5. P_3 – a boundary operator

There exist natural boundary operators for functions defined on the boundary of compact manifolds. We describe such operators on boundary of M^n for $n = 2$ and $n = 4$. Most of the material described in this section is contained in the joint work [CQ1], [CQ2]and [CQ3]. The reader is also referred to the lecture notes [C2] for a more detailed description of such operators derived in conjunction with the generalized formula [BO] [BCY] [CY] of Polyakov [Po] of zeta functional determinant for 4-manifolds with boundary. We start with terminology. On compact manifold (M^n, g) with boundary, we say a pair of operators (A, B) satisfy the *conformal assumptions* if:

Conformal Assumptions. *Both A and B are conformally covariant of bidegree (a_1, a_2) and (b_1, b_2) in the following sense*

$$A_w(f) = e^{-a_1 \omega} A(e^{a_2 \omega} f)$$
$$B_w(g) = e^{-b_1 \omega} B(e^{b_2 \omega} g),$$

for any $f \in C^\infty(M), g \in C^\infty(\partial M)$. Assume also that

$$B(e^{a_2 \omega} g) = 0 \ \textit{ if and only if } \ B_w(g) = 0,$$

for any $\omega \in C^\infty(\bar{M})$, where A_w, B_w denote the operator A, B respectively with respect to the conformal metric $g_w = e^{2w} g$.

Examples: The typical examples of pairs (A, B) which satisfy all three assumptions above are:

(i) when $n = 2$, $A = -\Delta, B = \frac{\partial}{\partial n}$ (negative of) the Laplacian operator and the Neumann operator respectively.

(ii) when $n = 4$, in [CQ-1] we have discovered a boundary operator P_3 conformal of bidegree (0,3) on the boundary of a compact 4-manifold. On 4-manifolds, (P_4, P_3) is a pair of operators satisfying the conformal covariant assumptions, which in the sense we shall describe below, is a natural analogue of the pair of operators $(-\Delta, \frac{\partial}{\partial n})$ defined on compact surfaces.

On compact surface M with boundary, the Gauss-Bonnet formula takes the form

$$2\pi\chi(M) = \int_M K dv + \oint_{\partial M} k d\sigma, \tag{5.1}$$

where k denotes the geodesic curvature of ∂M and $d\sigma$ the arc length measure on ∂M. Through conformal change of metric $g_w = e^{2w} g$ for w defined on $\bar{M}$, the Neumann operator $\frac{\partial}{\partial n}$ is related to the geodesic curvature k via the differential equation

$$-\frac{\partial w}{\partial n} + k_w e^w = k \ \ on \quad \partial M. \tag{5.2}$$

On 4-manifold with boundary there exists a boundary local invariant of order 3 and a conformal covariant operator P_3 of bidegree $(0,3)$, the relation of (Q,T) to (P_4, P_3) on 4-manifolds is parallel to that of (K,k) to $(\Delta, \frac{\partial}{\partial n})$ on compact surfaces.

$$P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + (\frac{1}{3}R - R_{\alpha N\alpha N})\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla}. \tag{5.3}$$

$$T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 - \frac{1}{3}\tilde{\Delta}H. \tag{5.4}$$

In particular, via the conformal change of metrics $g_w = e^{2w}g$, P_3 and T satisfies the equation:

$$-P_3 w + T_w e^{3w} = T \ \ \text{on} \quad \partial M, \tag{5.5}$$

and

$$(P_3)_w = e^{-3w} P_3 \ \ on \ \ \partial M. \tag{5.5}$$

The operator T and P_3 were discovered in [CQ2] through the Chern-Gauss-Bonnet formula for 4-manifolds with boundary:

$$\chi(M) = (32\pi^2)^{-1}\int_M (|C|^2 + 4Q)dx + (4\pi^2)^{-1}\oint_{\partial M}(T - \mathcal{L}_4 - \mathcal{L}_5)dy. \tag{5.7}$$

In the boundary integral above the invariants $\mathcal{L}_4$ and $\mathcal{L}_5$ involve the ambient curvature tensor and the second fundamental form L_{ab}. We shall use an orthonormal frame and use the latin indices to run through the ambient indices and the Greek indices to only run through the boundary directions.

$$\mathcal{L}_4 = -\frac{RH}{3} + R_{\alpha N\alpha N}H - R_{\alpha N\beta N}L_{\alpha\beta} + R_{\gamma\alpha\gamma\beta}L_{\alpha\beta}, \tag{5.8}$$

and

$$\mathcal{L}_5 = -\frac{2}{9}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\gamma} + L_{\alpha\alpha}L_{\beta\gamma}L_{\beta\gamma} - L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}. \tag{5.9}$$

Analogous to the Weyl term, $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order 3 which are pointwise invariant under conformal change of metrics. Hence for a fixed conformal class of metrics,

$$\frac{1}{2}\int_M Q dv + \oint_{\partial M} T ds$$

is a fixed constant. We remark that in the original Chern-Gauss-Bonnet formula T differs from the present form by $\frac{1}{3}\tilde{\Delta}H$, which does not affect the integration formula (5.7).

Thus on 4-manifolds with boundary it is natural to study the energy functional

$$E[w] = \frac{1}{4}\int wP_4w + \frac{1}{2}\oint_{\partial M} wP_3w. \tag{5.10}$$

In view of the complicated expressions of the operators P_4, P_3, Q and T, it is difficulty to study the functional $E[w]$ defined as above on general compact manifolds. We mention some special situations that allows an understanding of the basic situation. In the case of (B^4, S^3) with the standard metrics, we have

$$P_4 = (\Delta)^2, P_3 = -\frac{1}{2}N\Delta - \tilde{\Delta}N - 2\tilde{\Delta} \quad \text{and} \quad Q = 0, \quad \text{and} \quad T = 3, \tag{5.11}$$

where $\tilde{\Delta}$ denotes the the Laplacian operator Δ on (S^3, g). Thus the expression in $E[w]$ becomes relatively simple. In this special case, we are able to study the functional $E[w]$. The main analytic tool is the following sharp inequality of Lebedev-Milin type on (B^4, S^3).

Theorem 5.1. *Suppose $w \in C^\infty(\bar{B}^4)$. Then*

(5.12)
$$\log\left\{\frac{1}{2\pi^2}\oint_{S^3} e^{3(w-\bar{w})}dy\right\} \le \frac{3}{4\pi^2}\left\{\frac{1}{4}\int_{B^4} w\Delta^2 w + \oint_{S^3}\frac{1}{2}wP_3w - \frac{1}{4}\frac{\partial w}{\partial n} + \frac{1}{4}\frac{\partial^2 w}{\partial n^2}\right\},$$

under the boundary assumptions $\int_{S^3}\tau[w]ds[w] = 0$ where τ is the scalar curvature of S^3. Moreover the equality holds if and only if $e^{2w}g$ on B^4 is isometric to the canonical metric g.

§6 An extension of the Cohn-Vossen/Huber/Finn inequality

We first recall the Cohn-Vossen ([CV]) inequality for complete surfaces. Suppose (M, g) is a complete surface with Gauss curvature K in L^1, then

$$\int_M K dA \leq 2\pi\chi. \tag{6.1}$$

In fact, Huber ([H]) has shown that such a surface has a conformal compactification $M = \tilde{M}\backslash\{P_1, ...P_n\}$ where $\tilde{M}$ is a compact Riemann surface. At each puncture P_i by inverting a conformal disc $D_i\backslash\{P_i\}$, Finn ([Fn]) has considered the isoperimetric ratio $\nu_i = \lim_{r\to\infty} \frac{(Length(\partial D_r))^2}{2Area(D_r)}$, and accounted for the deficit in the inequality above:

$$2\pi\chi - \int_M K dA = \sum_{i=1}^{n} \nu_i. \tag{6.2}$$

A completely analogous situation holds in dimension four provided we restrict ourselves to conformally flat 4-manifolds of positive scalar curvature. Let us first recall that Schoen-Yau ([SY]) has demonstrated that for such manifolds, the holonomy cover of such manifolds embed conformally as domain $\tilde{M}$ in S^4 with a boundary which has Hausdorff dimension less than one. Thus by going to a covering of such manifolds we may assume that we are dealing with domains in R^4.

Theorem 6.1. *([CQY1])*
Let $e^{2w}|dx|^2$ be a complete metric on $\Omega = R^4\backslash\{P_1, ..., P_n\}$ with nonnegative scalar curvature near the punctures. Suppose in addition that Q is integrable. Then we have

$$\chi(\Omega) - \frac{1}{8\pi^2}\int_\Omega Q dV = \sum_{i=1}^{n} \nu_i \tag{6.3}$$

where at each puncture P_i a conformal disk $D_i\backslash\{P_i\}$ is inverted and

$$\nu_i = \lim_{r\to\infty} \frac{(vol(\partial B_r))^{4/3}}{4(2\pi^2)^{1/3} vol(Br)}. \tag{6.4}$$

The finiteness of the Q integral together with the embedding result of Schoen-Yau has strong implication for the underlying topology:

Theorem 6.2. *([CQY2]) Let (M^4, g) be a simply connected complete conformally flat manifold satisfying scalar curvature $R \geq c > 0$, $Ric \geq -c$, and $\int |Q| dv < \infty$; then M is conformally equivalent to $R^4\backslash\{P_1, ..., P_n\}$. In case M^4 is not assumed simply connected, under the additional assumption that M^4 is geometrically finite as a Kleinian manifold, then M is conformally equivalent to $\tilde{M}\backslash\{P_1, ..., P_n\}$, where $\tilde{M}$ is a compact conformally flat manifold.*

Remarks

1. An important ingredient in the proof of theorem is the consideration of the boundary P_3 operator. The local expression for the quantity T allows us to interpret the T integral as controlling the growth of volume. The idea is that the finiteness of the Q integral does not allow large growth of volume hence constrain the number of ends.

2. As a consequence of this finiteness criteria, we can classify the complete conformal metrics defined on domains in S^4, which satisfy the curvature conditions in the statement of Theorem 6.2, and in addition has constant Q curvature which are integrable. There are only three such metrics: the standard metric on S^4, the flat metric on $\mathbb{R}^4$ and the cylindrical metric on $\mathbb{R}^4 \backslash \{0\}$.

§7 A conformal variational problem for 4-manifold with boundary

In this section we consider conformal variations of Neumann type on a 4-manifold with boundary $(M, \partial M)$. That is the background metric g on the pair $(M, \partial M)$ which satisfy the property that the scalar curvature of M is constant and the mean curvature H vanishes on the boundary. This condition can always be achieved according to Escobar's solution of the Yamabe problem on manifolds with boundary ([E]). We consider conformal change of metric $g_w = e^{2w} g = u^2 g$ that satisfy the constraints

$$\begin{aligned} vol(M)[w] &= vol(M)[0] \\ H[w] &= 0. \end{aligned} \tag{7.1}$$

For convenience we set

$$Y(g) = inf \int_M Lu \cdot udv$$

where the infimum is taken over positive conformal factors satisfying the constraint (7.1). The variational functional we shall consider is an analogue of the functional II discussed in section three with additional terms to accommodate the presence of boundary:

$$J[w] = \frac{1}{4} \int \{wP_4 w + \frac{1}{2} wQ\} + \frac{1}{2} \oint_{\partial M} wP_3 w - (1/8 \int Q) log \int e^{4w}. \tag{7.2}$$

In order to formulate criterion for existence of extremal metrics for the functional J let us define the natural invariant

$$k_p(M, \partial M) = \frac{1}{2} \int_M Qdv + \oint_{\partial M} Tds \tag{7.3}$$

Then the analysis of ([CQ3]) yields the following:

Theorem 7.1. *Suppose $k_p(M, \partial M) < 16\pi^2$ and the operator pair (P_4, P_3) is positive except on constants, then the functional J under the constraint (7.1) achieves its minimum.*

In fact it is easy to see that the functional J is convex under the stronger assumption $k_p(M, \partial M) \leq 0$ and hence:

Theorem 7.2. *Suppose $k_p(M, \partial M) \leq 0$ and the operator pair (P_4, P_3) is positive except on constants, then the functional J under the constraint (7.1) has a unique critical point which is the minimum.*

In the following we impose additional boundary conditions on the pair $(M, \partial M)$ in order to formulate the simplest boundary value problem for the operator pair (P_4, P_3). It would be interesting to relax this boundary condition to accommodate more general situations.

Definition. *We say ∂M is umbilic if the second fundamental form has all equal principal eigenvalues.*

Remark. The umbilicity condition is conformally invariant, thus under this assumption, the solution of Yamabe problem with minimal boundary actually makes the boundary totally geodesic.

Proposition 7.3. *Suppose (M^4, g) has constant positive scalar curvature and ∂M has zero mean curvature, then $k_p(M, \partial M) \leq 16\pi^2$; and equality holds if and only if $(M, \partial M)$ is conformally equivalent to the upper hemisphere (S^4_+, S^3).*

Proof: The solution of the Yamabe problem with boundary provides a conformal metric with constant scalar curvature with xero mean curvature $H = 0$ on the boundary. Hence the boundary contribution in the definition of k_p vanishes and hence

$$\begin{aligned} k_p(M, \partial M) &= \int (-|E|^2 + \frac{1}{12} R^2) dV \\ &\leq \int \frac{1}{12} R^2 dV \\ &\leq 16\pi^2 \end{aligned}$$

Of course, equality can hold if and only if the Yamabe constant is equal to that of the hemisphere, in which case the positive mass theorem (see [E]) asserts that $(M, \partial M)$ is conformally equivalent to the hemisphere (S^4_+, S^3).

In order to circumvent the positivity requirement on the pair (P_4, P_3), we add an extra term to the functional J and consider

$$F[w] = \beta \int R^2 dV + J[w]. \tag{7.4}$$

The argument in the proof of Theorem 2.1 in ([CQ3]) shows that as long as $\beta > 0$, and $k_p \leq 16\pi^2$, we have $W^{2,2}$ compactness of the minimizing sequence for the functional F under the constraint (7.1). Hence we can minimize the functional F to produce a minimizing extremal metric. The extremal metric satisfy the Euler equation

$$\Delta R = \lambda + \gamma(-|E|^2 + \frac{1}{12} R^2), \text{ where } \lambda \leq 0, \ \gamma = \frac{1}{4}(3\beta + \frac{1}{12})^{-1}, \tag{7.5}$$

and the boundary condition $H = 0$. Since the boundary condition is of Neumann type, the regularity theory developed in ([CGY2]) goes through without essential change. Hence we may assume the solution is smooth up to the boundary.

Lemma 7.4. *If* $0 < \beta \le \frac{95}{432}$ *then the extremal metric satisfies*

$$\Delta R = \lambda + \gamma(-|E|^2 + \frac{1}{12}R^2), \text{ where } \lambda \le 0, \tag{7.6}$$

and

$$\beta \partial_n R = 0 \text{ on} \partial M. \tag{7.7}$$

In addition, we have $R > 0$ *on* M.

Proof. Let ϕ be the principal eigenfunction of the pair (P_4, P_3): $L\phi = \mu_1 \phi$ with the boundary condition $\partial_n \phi = 0$. Consider the function

$$F = \frac{R}{\phi}.$$

Then we find gives

$$\begin{aligned}\Delta F = \lambda - \gamma \frac{|E|^2}{\phi} + (\gamma - \frac{1}{6} + \mu_1)F - 2 < \nabla F, \frac{\nabla \phi}{\phi} > \\ \le \mu_1 F - 2 < \nabla F, \frac{\nabla \phi}{\phi} >\end{aligned}$$

and

$$\partial_n F = \frac{\partial_n R}{\phi} - F \frac{\partial_n \phi}{\phi} \text{ on } \partial M.$$

The strong maximum principle and the boundary Hopf lemma then shows that the minimum must occur on the interior and F is positive there.

Now we are in a position to single out the pairs $(M, \partial M)$ with $Q = 0$ and $T = 0$.

Proposition 7.5. *If* M^4 *is locally conformally flat with umbilic boundary* ∂M*, assume* $Y(g) > 0$ *and* $\chi(M)$=0*, then either* $(M, \partial M) = (S^1 \times S^3_+, S^1 \times S^2)$*, or* $(M, \partial M) = (I \times S^3, \partial I \times S^3)$ *where* I *is an interval.*

Proof: We first calculate the Euler number to see that either $b_1 = 1, b_3 = 0$ or $b_1 = 0, b_3 = 1$. This is due to the vanishing of the second homology since the argument of Bourguignon ([Bo]) still applies in our situation: if $b_2 \neq 0$, there is a harmonic two form ω satisfying the absolute boundary condition $i(\partial n)(\omega) = 0$ and $i(\partial n)(d\omega) = 0$. The Bochner formula gives

$$\begin{aligned}\frac{1}{2}\Delta|\omega|^2 = |\nabla \omega|^2 + \frac{2}{3}R|\omega|^2 \\ \partial_n |\omega|^2 = 0 \text{ on } \partial M.\end{aligned}$$

The strong maximum principle and the Hopf boundary lemma then shows ω must be identically zero. Thus not both b_1 and b_3 can vanish.

In the first case, there is a harmonic 1-form ω on M^4 satisfying the absolute boundary conditions. Choose $\beta = \frac{1}{36}$ in (7.5) for the functional F, we have the existence of conformal metric satisfying the equation

$$\Delta R = -\frac{3}{2}|E|^2 + \frac{1}{8}R^2 \tag{7.8}$$

with the boundary condition

$$\partial_n R = 0. \tag{7.9}$$

Consider the function $G = \frac{|\omega|}{R}$. The calculation of Gursky then gives

$$\Delta G + 2 < \nabla G, \frac{\nabla R}{R} >\geq \frac{3}{2}\frac{G}{R}(|E| - \frac{\sqrt{3}}{6}R)^2 + G|\omega|^{-2}(|\nabla\omega|^2 - |\nabla|\omega||^2) \tag{7.10}$$

on the set $\Omega = \{x|\omega(x) \neq 0\}$; and equality can hold at a point if and only if E has the form

$$E = \begin{pmatrix} -3\nu & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix} \tag{7.11}$$

Since the boundary condition assures that $\partial_n G = 0$, the Hopf boundary lemma shows that the maximum of G cannot take place there unless ω vanishes identically contradicting its non-triviality. The strong maximum principle then asserts that G is constant so that $|E|^2 = \frac{1}{12}R^2$. Then we find $\Delta R = 0$, and the boundary condition $\partial_n R = 0$ implies that R is constant. This implies $|\omega|$ is constant, and hence $\nabla\omega = 0$. We find equality holds everywhere in (7.10). This shows that the Ricci tensor is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4\nu & 0 & 0 \\ 0 & 0 & 4\nu & 0 \\ 0 & 0 & 0 & 4\nu \end{pmatrix}$$

This shows that M is locally conformally equivalent to $\mathbb{R} \times S^3$ with $\omega = 0$ displaying the S^3 factor. The boundary condition $\omega(\partial_n) = 0$ shows that the boundary respect the product structure and the one dimensional factor is tangent to the boundary, but the total geodesic condition implies that the other factor is a constant positive Gauss curvature surface, hence is S^2. Thus globally we have $(M, \partial M) = (S^1 \times S^3_+, S^1 \times S^2)$.

In the second case, we have a harmonic 3-form ψ satisfying the absolute boundary conditions. Hence its dual 1-form $\omega = *\psi$ is harmonic and satisfies the relative boundary conditions. The foregoing argument still shows that M is locally of the form $\mathbb{R} \times S^3$, but now the one dimensional factor is perpendicular to the boundary, which being totally geodesic must be of constant sectional curvature. Thus $(M, \partial M)$ is conformally equivalent to $(I \times S^3/\Gamma, \partial I \times S^3/\Gamma)$.

Remark: An alternative method would be to prove this proposition directly by doubling the manifold, by reflecting across the boundary, this is possible because all boundary data are of Neumann type. However we present this argument since it offers possibility to generalize to more complicated situations.

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