ON A CLASS OF LOCALLY CONFORMALLY FLAT MANIFOLDS

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1. Introduction

In [SY], Schoen and Yau proved that if (M^n, g) $(n \geq 3)$ is a smooth compact locally conformally flat manifold with scalar curvature $R \geq 0$, and $\widetilde{M}$ is the universal covering space of M, then the developing map $\phi : \widetilde{M} \to S^n$ is a conformal embedding, in addition, the complement of the development image $\Lambda = S^n \backslash \phi\left(\widetilde{M}\right)$ has its Hausdorff dimension bounded by $\frac{n-2}{2}$. On the other hand, if the Ricci tensor is positive definite, the Bonnet-Myers theorem implies that $|\pi_1(M)| < \infty$ and hence ϕ is a diffeomorphism onto S^n. In another direction, [CQY] gives a criteria for the set Λ to consist of isolated points in terms of the finiteness of the Q curvature integral. The Q curvature is closely connected with the second symmetric function $\sigma_2(A)$ of the Schouten tensor. We recall the Schouten tensor is given by

$$A = \frac{1}{n-2}\left(Rc - \frac{R}{2(n-1)}g\right). \tag{1.1}$$

For $1 \leq k \leq n$, we denote by $\sigma_k(A)$ the kth elementary polynomial function of the eigenvalues of A (with respect to g). The fourth order Q curvature is defined as

$$Q = -\frac{1}{2(n-1)}\Delta R + \frac{n-4}{8(n-1)^2}R^2 + 4\sigma_2(A). \tag{1.2}$$

The purpose of this article is to show that the positivity of the quantity $\sigma_2(A)$ or that of the Q curvature give further control of the size of the complement Λ.

Theorem 1.1. *Let $\Omega \subset S^n$ $(n \geq 5)$ be an open connected subset and g_{S^n} be the standard metric on S^n. Assume we have a metric g on Ω such that (Ω, g) is complete, g is conformal to g_{S^n}, $|R| + |\nabla_g R|_g \leq c_0$ and $\sigma_1(A) \geq c_1 > 0$, $\sigma_2(A) \geq 0$, then* $\dim(S^n \backslash \Omega) < \frac{n-4}{2}$.

On the other hand, we have

Theorem 1.2. *Let $\Omega \subset S^n$ $(n \geq 3)$ be an open connected subset and g_{S^n} be the standard metric on S^n. Assume we have a metric g on Ω such that (Ω, g) is complete, g is conformal to g_{S^n}, $|Rc|_g + |\nabla_g R|_g \leq c_0$, $R \geq c_1 > 0$, $Q \geq c_2 > 0$, then* $\dim(S^n \backslash \Omega) < \frac{n-4}{2}$. *In particular, this means $\Omega = S^n$ when $n \leq 4$.*

If we replace $Q \geq c_2 > 0$ by $Q \geq 0$, then when $n \geq 5$, we have $\dim(S^n \backslash \Omega) \leq \frac{n-4}{2}$*; when $n = 3$, we have $\Omega = S^3$.*

We would like to point out that if the condition $|R| + |\nabla_g R|_g \leq c_0$ in Theorem 1.1 is dropped, then one has $\dim(S^n \backslash \Omega) \leq \frac{n-4}{2}$. Similarly, if the condition $|Rc|_g + |\nabla_g R|_g \leq c_0$ in Theorem 1.2 is dropped, then one has $\dim(S^n \backslash \Omega) \leq \frac{n-4}{2}$ when $n \geq 5$ and $\Omega = S^3$ when $n = 3$. These can be done by the proof presented below together with the coercivity of conformal factor proved in Section 8 and a method in [SY] of using capacity theory to estimate the dimension of Λ, which replaces the use of Lemma 2.1. In general the positivity of $\sigma_k(A)$ will yield corresponding size control of Λ. Indeed, when $k \geq \frac{n}{2}$, a calculation of [GVW] shows that when $\sigma_1(A) > 0, \cdots, \sigma_k(A) > 0$, the Ricci tensor is positive definite hence the complement Λ is empty. For the intermediate range $3 \leq k < \frac{n}{2}$, the question is addressed in the forthcoming thesis of M. Gonzalez.

It follows from the same consideration as in [SY] that the following hold.

Corollary 1.1. *Let (M^n, g) $(n \geq 5)$ be a smooth compact locally conformally flat Riemannian manifold such that $\sigma_1(A) > 0$ and $\sigma_2(A) \geq 0$, then for any $2 \leq i \leq \left[\frac{n}{2}\right] + 1$, $\pi_i(M) = 0$; for any $\frac{n}{2} - 1 \leq j \leq \frac{n}{2} + 1$, $H^j(M, \mathbb{R}) = 0$.*

Corollary 1.2. *Let (M^n, g) $(n \geq 3)$ be a smooth compact locally conformally flat Riemannian manifold such that $R > 0$ and $Q > 0$, then when $n \geq 5$, we have for any $2 \leq i \leq \left[\frac{n}{2}\right] + 1$, $\pi_i(M) = 0$; for any integer j with $\frac{n}{2} - 1 \leq j \leq \frac{n}{2} + 1$, $H^j(M, \mathbb{R}) = 0$; when $n = 3$ or 4, the universal cover of M is conformal isomorphic to S^n.*

We were informed recently that the result on vanishing of cohomology groups in Corollary 1.1 was also derived in [GLW].

In dimension four, one has the following slightly improved version of theorem 2 in [CQY].

Theorem 1.3. *Let $\Omega \subset S^4$ be an open connected subset and g_{S^4} be the standard metric on S^4. Assume we have a metric g on Ω such that (Ω, g) is complete, g is conformal to g_{S^4}, $|Rc|_g + |\nabla_g R|_g \leq c_0$, $R \geq c_1 > 0$, and $\int_\Omega Q^- d\mu_g < \infty$, here $Q^- = \max\{-Q, 0\}$, μ_g is the natural measure on Ω associated with Riemannian metric g, then $\int_\Omega |Q| \, d\mu_g < \infty$, in addition, $\Omega = S^4 \backslash \{p_1, \cdots, p_m\}$ for some $p_1, \cdots, p_m \in S^4$.*

Observe that for $\mathbb{R} \times S^3$, under the product metric, we have $R = 6$, $Q = 0$. By standard gluing method, we may find many examples of metrics satisfying the conditions in Theorem 1.3. We remark that if $\Omega \subset S^4$ is an open subset endowed with a complete metric g, which is conformal to g_{S^4}, such that $\sigma_1(A) \geq 0, \sigma_2(A) \geq 0$ on Ω, then we have $Rc \geq 0$. It then follows from the splitting theorem of Cheeger-Gromoll that Ω is S^4 or $S^4 \backslash \{p\}$ or $S^4 \backslash \{p_1, p_2\}$.

The above discussions and formula (1.2) indicate that there are strong relations between the positivity of $\sigma_2(A)$ and the positivity of the Paneitz operator. In four dimension, it was proved in [G] that if $\left(M^4, g\right)$ is a smooth compact Riemannian manifold with positive Yamabe invariant and $\int_M \sigma_2(A) \, d\mu \geq 0$, then the Paneitz operator $P \geq 0$ and $\ker P$ consists of constant functions. One of the interesting aspect of this result is that the assumptions are conformally invariant. In higher dimensions, the search of similar criterion has no success so far. Nevertheless, we have

Theorem 1.4. *Let (M^n, g) $(n \geq 5)$ be a smooth compact Riemannian manifold with $\sigma_1(A) \geq 0$ and $\sigma_2(A) \geq 0$. If (M, g) is not Ricci flat, then $P > 0$; otherwise, $P = \Delta^2 \geq 0$ and $\ker P = \{$all constant functions$\}$.*

The converse of Theorem 1.4 is not true. Indeed, in Section 7, we will present explicitly some conformal class of metrics with positive Yamabe invariant and positive Paneitz operator but no metric in that conformal class can have nonnegative $\sigma_1(A)$ and $\sigma_2(A)$, in particular, no conformal metric could have positive $\sigma_2(A)$.

The paper is written as follows. In Section 2, we shall list some standard formulas and prove two elementary lemmas for future use. From Section 3 to Section 5 we shall prove Theorem 1.1 to Theorem 1.3. In Section 6, we shall discuss the positivity of Paneitz operator in dimension greater than or equal to 5. In Section 7, we shall present examples which illustrate results above and give limitation to further improvements. Finally, in Section 8, motivated by the proof of Theorem 1.1 and Theorem 1.2, we study the coercivity of the conformal factor of a complete conformal metric on an open subset of S^n.

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2. Some preparations

First let us recall the Paneitz operator and the Q curvature (see [B] and [P]). Let (M^n, g) be a smooth Riemannian manifold with dimension $n \geq 3$. We denote

$$J = \frac{R}{2(n-1)}, \quad Q = -\Delta J - 2|A|^2 + \frac{n}{2}J^2. \tag{2.1}$$

The Paneitz operator is defined by

$$P\varphi = \Delta^2\varphi + \operatorname{div}\left(4A(\nabla\varphi, e_i)e_i - (n-2)J\nabla\varphi\right) + \frac{n-4}{2}Q\cdot\varphi, \tag{2.2}$$

where $e_1, \cdots, e_n$ is an orthonormal frame. It has the following conformal covariant property, namely

$$P_{e^{2w}g}\varphi = e^{-\frac{n+4}{2}w}P_g\left(e^{\frac{n-4}{2}w}\varphi\right) \tag{2.3}$$

for any $w \in C^\infty(M)$ and $\varphi \in C^\infty(M)$.

Let $\widetilde{g} = e^{2w}g$ with $w \in C^\infty(M)$. It follows from (2.3) that

$$\widetilde{Q} = \frac{2}{n-4}e^{-\frac{n+4}{2}w}P_g\left(e^{\frac{n-4}{2}w}\right) \quad \text{for } n \neq 4 \tag{2.4}$$

and

$$\widetilde{Q} = e^{-4w}(P_g w + Q) \quad \text{for } n = 4. \tag{2.5}$$

Under an orthonormal frame with respect to g, for the scalar curvature, standard calculation shows

$$\widetilde{J}e^{2w} = J - \Delta w - \frac{n-2}{2}|\nabla w|^2. \tag{2.6}$$

For Schouten tensor, we have

$$\widetilde{A}_{ij} = A_{ij} - w_{ij} + w_i w_j - \frac{|\nabla w|^2}{2}g_{ij}, \tag{2.7}$$

Hence

$$\begin{aligned} & 2e^{4w}\sigma_2\left(\widetilde{A}\right) \\ = \; & 2\sigma_2(A) + (\Delta w)^2 - \left|D^2 w\right|^2 + (n-3)|\nabla w|^2\Delta w + 2w_{ij}w_i w_j \\ & + 2A_{ij}w_{ij} - 2J\Delta w + \frac{(n-1)(n-4)}{4}|\nabla w|^4 - 2A_{ij}w_i w_j \\ & - (n-3)J|\nabla w|^2. \end{aligned} \tag{2.8}$$

We will need the following simple lemma later.

Lemma 2.1. *Let F be a compact subset of $\mathbb{R}^n$. Define $\delta_F(x) = \operatorname{dist}(x, F)$ for $x \in \mathbb{R}^n$. Assume for some $r > 0$ and $\alpha \geq 1$, we have*

$$F \subset B_r \quad \text{and} \quad \int_{B_r \backslash F} \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) < \infty,$$

here $\mathcal{H}^n$ is the standard Hausdorff measure on $\mathbb{R}^n$, then $\mathcal{H}^{n-\alpha}(F) = 0$. In addition, if $\alpha \geq n$, then $F = \emptyset$.

Proof. We first observe that $\mathcal{H}^n(F) = 0$. In fact, when $n = 1$, the conclusion is trivial. Now one just let $x = (x', x^n)$, it follows easily from Fubini theorem and the assertion in dimension 1 that for $\mathcal{H}^{n-1}$ a.e. x', $(\{x'\} \times \mathbb{R}) \cap F = \emptyset$. Hence $\mathcal{H}^n(F) = 0$. Going back we see $\int_{B_r} \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) < \infty$.

If $\alpha \geq n$, then F must be empty. Otherwise, let $x_0 \in F$, then

$$\int_{B_r} \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) \geq \int_{B_r} |x - x_0|^{-\alpha} \, d\mathcal{H}^n(x) = \infty,$$

a contradiction.

Next, assume $1 \leq \alpha < n$. For any $\varepsilon > 0$, we may find a bounded open set $U \supset F$ such that

$$\int_U \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) < \varepsilon.$$

By Vitali covering theorem (see [EG]) we may find finitely many points $x_1, \cdots, x_m \in F$ and $r_1, \cdots, r_m > 0$ such that $\overline{B_{r_i}(x_i)} \subset U$, $\overline{B_{r_i}(x_i)} \cap \overline{B_{r_j}(x_j)} = \emptyset$ for $i \neq j$ and $F \subset \cup_{i=1}^m \overline{B_{5r_i}(x_i)}$. Hence we have

$$\begin{aligned} \mathcal{H}_\infty^{n-\alpha}(F) &\leq c(n) \sum_{i=1}^m r_i^{m-\alpha} \leq c(n) \sum_{i=1}^m \int_{B_{r_i}(x_i)} \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) \\ &\leq c(n) \int_U \delta_F(x)^{-\alpha} \, d\mathcal{H}^n(x) \leq c(n) \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we see $\mathcal{H}_\infty^{n-\alpha}(F) = 0$. Hence $\mathcal{H}^{n-\alpha}(F) = 0$. ∎

Next we recall an elementary algebraic lemma. For reader's convenience, we present the proof here.

Lemma 2.2. *Let B be a real $n \times n$ symmetric matrix such that $\sigma_1(B) \geq 0$ and $\sigma_2(B) \geq 0$, then*

$$\frac{2-n}{n} \sigma_1(B) \cdot I \leq B \leq \sigma_1(B) \cdot I,$$

here I is the $n \times n$ identity matrix.

Proof. Since

$$\begin{aligned} \sigma_2(B) &= \frac{1}{2} \left(\sigma_1(B)^2 - |B|^2 \right) \\ &= \frac{1}{2} \left(\frac{n-1}{n} \sigma_1(B)^2 - \left| B - \frac{\sigma_1(B)}{n} I \right|^2 \right) \geq 0, \end{aligned}$$

we see

$$\left| B - \frac{\sigma_1(B)}{n} I \right|^2 \leq \frac{n-1}{n} \sigma_1(B)^2.$$

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $B - \frac{\sigma_1(B)}{n} I$. Without losing of generality, we may assume $|\lambda_i| \leq |\lambda_1|$ for $1 \leq i \leq n$. Then

$$\left| B - \frac{\sigma_1(B)}{n} I \right|^2 = \lambda_1^2 + \sum_{i=2}^{n} \lambda_i^2 \geq \lambda_1^2 + \frac{1}{n-1} \left(\sum_{i=2}^{n} \lambda_i \right)^2 = \frac{n}{n-1} \lambda_1^2.$$

Hence

$$|\lambda_i| \leq |\lambda_1| \leq \frac{n-1}{n} \sigma_1(B).$$

This implies

$$\frac{1-n}{n} \sigma_1(B) \cdot I \leq B - \frac{\sigma_1(B)}{n} I \leq \frac{n-1}{n} \sigma_1(B) \cdot I.$$

The lemma follows easily. ∎

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. By rotation, we may assume the north pole $\mathrm{N} \in \Omega$. Let $\pi_{\mathrm{N}} : S^n \backslash \{\mathrm{N}\} \to \mathbb{R}^n$ be the stereographic projection. Then we may write

$$\left(\pi_{\mathrm{N}}^{-1}\right)^* (g) = \widetilde{g} = e^{2w} g_{\mathbb{R}^n}.$$

Here $g_{\mathbb{R}^n}$ is the Euclidean metric. Define the bad set $\mathcal{B} = \pi_{\mathrm{N}} (S^n \backslash \Omega)$. Then $\mathcal{B}$ is a compact subset of $\mathbb{R}^n$. Fix a $r > 0$ such that $\mathcal{B} \subset B_r$.

Next we recall some basic estimates for w observed in the proof of proposition 2.6 in [SY]. By Lemma 2.2, we see $Rc \geq \frac{4-n}{2n} Rg$, this plus the bounds on R shows the total Riemann curvature is bounded. Because $g_{\mathbb{R}^n} = e^{-2w} \widetilde{g}$, we see

$$0 = -\frac{4(n-1)}{n-2} \widetilde{\Delta} \left(e^{-\frac{n-2}{2} w} \right) + \widetilde{R} e^{-\frac{n-2}{2} w}.$$

It follows from the gradient estimate of Cheng-Yau that we have

$$\left| \nabla_{\widetilde{g}} \log \left(e^{-\frac{n-2}{2} w} \right) \right|_{\widetilde{g}} \leq c(g) \quad \text{on } B_r \backslash \mathcal{B}.$$

Changing back to $g_{\mathbb{R}^n}$, we get

$$|\nabla w| \leq c(g) e^w \quad \text{on } B_r \backslash \mathcal{B}. \tag{3.1}$$

It follows from this and the completeness of (Ω, g) that

$$e^{w(x)} \geq c(g) \delta_{\mathcal{B}}(x)^{-1} \quad \text{for } x \in B_r \backslash \mathcal{B}. \tag{3.2}$$

Here $\delta_{\mathcal{B}}(x) = \mathrm{dist}(x, \mathcal{B})$. In particular, we know $w(x) \to \infty$ as $\delta_{\mathcal{B}}(x) \to 0$.

For $\lambda > \max_{\partial B_r} w$, let

$$\Omega_\lambda = \{x \in B_r \backslash \mathcal{B} : w(x) < \lambda\}.$$

In view of (3.2) and Sard's theorem, we know for generic λ, Ω_λ is a bounded smooth open subset of $\mathbb{R}^n$, $\overline{\Omega_\lambda} \subset \overline{B_r} \backslash \mathcal{B}$ and

$$\partial \Omega_\lambda = \partial B_r \cup \{x \in B_r \backslash \mathcal{B} : w(x) = \lambda\}. \tag{3.3}$$

By (2.8), we see for any $\alpha \in \mathbb{R}$,

$$
\begin{aligned}
& 2\int_{\Omega_\lambda} e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n \\
= & \int_{\Omega_\lambda} e^{\alpha w}\left((\Delta w)^2 - \left|D^2 w\right|^2\right) d\mathcal{H}^n + (n-3)\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^2 \Delta w d\mathcal{H}^n \\
& +2\int_{\Omega_\lambda} e^{\alpha w} w_{ij} w_i w_j d\mathcal{H}^n + \frac{(n-1)(n-4)}{4}\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^4 d\mathcal{H}^n.
\end{aligned}
$$

Since

$$
\begin{aligned}
& e^{\alpha w}|\Delta w|^2 - e^{\alpha w}\left|D^2 w\right|^2 \\
= & \left(e^{\alpha w}\Delta w \cdot w_i\right)_i - \left(e^{\alpha w} w_{ij} w_i\right)_j - \alpha e^{\alpha w}|\nabla w|^2 \Delta w + \alpha e^{\alpha w} w_{ij} w_i w_j,
\end{aligned}
$$

we see

$$
\begin{aligned}
& 2\int_{\Omega_\lambda} e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n \\
= & \int_{\partial\Omega_\lambda} e^{\alpha w}\left(\Delta w \cdot \frac{\partial w}{\partial \nu} - w_{ij} w_i \nu_j\right) d\mathcal{H}^{n-1} + (n-3-\alpha)\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^2 \Delta w d\mathcal{H}^n \\
& +(\alpha+2)\int_{\Omega_\lambda} e^{\alpha w} w_{ij} w_i w_j d\mathcal{H}^n + \frac{(n-1)(n-4)}{4}\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^4 d\mathcal{H}^n.
\end{aligned}
$$

Here ν is the outer normal direction. Substituting the identity

$$
e^{\alpha w} w_{ij} w_i w_j = \frac{1}{2}\left(e^{\alpha w}|\nabla w|^2 w_i\right)_i - \frac{1}{2}e^{\alpha w}|\nabla w|^2 \Delta w - \frac{\alpha}{2}e^{\alpha w}|\nabla w|^4,
$$

we find

$$
\begin{aligned}
& 2\int_{\Omega_\lambda} e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n \\
= & \int_{\partial\Omega_\lambda} e^{\alpha w}\left(\Delta w \cdot \frac{\partial w}{\partial \nu} - w_{ij} w_i \nu_j + \frac{\alpha+2}{2}|\nabla w|^2 \frac{\partial w}{\partial \nu}\right) d\mathcal{H}^{n-1} \\
& +\left(n-4-\frac{3}{2}\alpha\right)\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^2 \Delta w d\mathcal{H}^n \\
& +\left(\frac{(n-1)(n-4)}{4} - \frac{\alpha(\alpha+2)}{2}\right)\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^4 d\mathcal{H}^n.
\end{aligned}
$$

By (2.6), we see

$$
\Delta w = -\widetilde{J} e^{2w} - \frac{n-2}{2}|\nabla w|^2, \tag{3.4}
$$

hence

$$
\begin{aligned}
& 2\int_{\Omega_\lambda} e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n \\
= & \int_{\partial\Omega_\lambda} e^{\alpha w}\left(-\widetilde{J} e^{2w}\frac{\partial w}{\partial \nu} - w_{ij} w_i \nu_j - \frac{n-4-\alpha}{2}|\nabla w|^2 \frac{\partial w}{\partial \nu}\right) d\mathcal{H}^{n-1} \\
& -\left(n-4-\frac{3}{2}\alpha\right)\int_{\Omega_\lambda} \widetilde{J} e^{(\alpha+2)w}|\nabla w|^2 d\mathcal{H}^n \\
& -\frac{1}{2}\left(\alpha - \frac{n-4}{2}\right)(\alpha-(n-3))\int_{\Omega_\lambda} e^{\alpha w}|\nabla w|^4 d\mathcal{H}^n.
\end{aligned}
$$

In view of (3.3) and the fact that on $\{x\in B_r\backslash\mathcal{B}:w(x)=\lambda\}$, the outer normal $\nu=\frac{\nabla w}{|\nabla w|}$, and $\frac{\partial w}{\partial\nu}=|\nabla w|$, we see

$$\begin{aligned}
&2\int_{\Omega_\lambda}e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right)d\mathcal{H}^n\\
=&\int_{\{x\in B_r\backslash\mathcal{B}:w(x)=\lambda\}}e^{\alpha w}\left(-\widetilde{J}e^{2w}\frac{\partial w}{\partial\nu}-w_{ij}w_i\nu_j-\frac{n-4-\alpha}{2}|\nabla w|^2\frac{\partial w}{\partial\nu}\right)d\mathcal{H}^{n-1}\\
&-\left(n-4-\frac{3}{2}\alpha\right)\int_{\Omega_\lambda}\widetilde{J}e^{(\alpha+2)w}|\nabla w|^2d\mathcal{H}^n\\
&-\frac{1}{2}\left(\alpha-\frac{n-4}{2}\right)(\alpha-(n-3))\int_{\Omega_\lambda}e^{\alpha w}|\nabla w|^4d\mathcal{H}^n+c(\alpha,g).
\end{aligned}$$

But using (2.7), we see

$$\begin{aligned}
&-\widetilde{J}e^{2w}\frac{\partial w}{\partial\nu}-w_{ij}w_i\nu_j-\frac{n-4-\alpha}{2}|\nabla w|^2\frac{\partial w}{\partial\nu}\\
=&\frac{1}{|\nabla w|}\left(\widetilde{A}_{ij}w_iw_j-\widetilde{J}e^{2w}|\nabla w|^2-\frac{n-3-\alpha}{2}|\nabla w|^4\right).
\end{aligned}$$

Since $\sigma_1\left(\widetilde{A}\right)>0,\sigma_2\left(\widetilde{A}\right)\geq0$, by Lemma 2.2 we have

$$\widetilde{A}\leq\left(\operatorname{tr}\widetilde{A}\right)\cdot\widetilde{g}=\widetilde{J}e^{2w}g,$$

this implies

$$\widetilde{A}_{ij}w_iw_j\leq\widetilde{J}e^{2w}|\nabla w|^2.$$

Summing up, we get the following

$$\begin{aligned}
&2\int_{\Omega_\lambda}e^{(\alpha+4)w}\sigma_2\left(\widetilde{A}\right)d\mathcal{H}^n\\
\leq&-\frac{n-3-\alpha}{2}\int_{\{x\in B_r\backslash\mathcal{B}:w(x)=\lambda\}}e^{\alpha w}|\nabla w|^3d\mathcal{H}^{n-1}\\
&-\left(n-4-\frac{3}{2}\alpha\right)\int_{\Omega_\lambda}\widetilde{J}e^{(\alpha+2)w}|\nabla w|^2d\mathcal{H}^n\\
&-\frac{1}{2}\left(\alpha-\frac{n-4}{2}\right)(\alpha-(n-3))\int_{\Omega_\lambda}e^{\alpha w}|\nabla w|^4d\mathcal{H}^n+c(\alpha,g).
\end{aligned}\tag{3.5}$$

On the other hand, it follows from (3.4) that for any $\alpha\in\mathbb{R}$,

$$\int_{\Omega_\lambda}\widetilde{J}e^{(\alpha+2)w}d\mathcal{H}^n=-\int_{\partial\Omega_\lambda}e^{\alpha w}\frac{\partial w}{\partial\nu}d\mathcal{H}^{n-1}+\left(\alpha-\frac{n-2}{2}\right)\int_{\Omega_\lambda}e^{\alpha w}|\nabla w|^2d\mathcal{H}^n.$$

This implies

$$\begin{aligned}
&\int_{\Omega_\lambda}\widetilde{J}e^{(\alpha+2)w}d\mathcal{H}^n\\
=&\left(\alpha-\frac{n-2}{2}\right)\int_{\Omega_\lambda}e^{\alpha w}|\nabla w|^2d\mathcal{H}^n-\int_{\{x\in B_r\backslash\mathcal{B}:w(x)=\lambda\}}e^{\alpha w}|\nabla w|d\mathcal{H}^{n-1}\\
&+c(\alpha,g).
\end{aligned}\tag{3.6}$$

Taking $\alpha = \frac{n-4}{2}$ in (3.5), we get

$$\int_{\Omega_\lambda} e^{\frac{n+4}{2}w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n + \int_{\Omega_\lambda} \widetilde{J} e^{\frac{n}{2}w} \left|\nabla w\right|^2 d\mathcal{H}^n \le c\left(g\right).$$

Let $\lambda \to \infty$, it becomes

$$\int_{B_r \backslash \mathcal{B}} e^{\frac{n+4}{2}w}\sigma_2\left(\widetilde{A}\right) d\mathcal{H}^n + \int_{B_r \backslash \mathcal{B}} e^{\frac{n}{2}w} \left|\nabla w\right|^2 \widetilde{J} d\mathcal{H}^n \le c\left(g\right) < \infty.$$

In view of the fact $\widetilde{J} \ge c_1 > 0$, we see

$$\int_{B_r \backslash \mathcal{B}} e^{\frac{n}{2}w} \left|\nabla w\right|^2 d\mathcal{H}^n < \infty.$$

Using this inequality, the fact $\widetilde{J} \ge c_1 > 0$ and taking $\alpha = \frac{n}{2}$ in (3.6), we see

$$\int_{B_r \backslash \mathcal{B}} e^{\frac{n+4}{2}w} d\mathcal{H}^n < \infty.$$

By (3.2) the above inequality implies

$$\int_{B_r \backslash \mathcal{B}} \delta_{\mathcal{B}}^{-\frac{n+4}{2}} d\mathcal{H}^n < \infty.$$

It follows from this and Lemma 2.1 that $\mathcal{H}^{\frac{n-4}{2}}\left(\mathcal{B}\right) = 0$, and hence $\dim \mathcal{B} \le \frac{n-4}{2}$. To get the strict less sign, we put $\alpha = \frac{n-4}{2} + \varepsilon$ in (3.5) for $\varepsilon > 0$ very small, then using (3.1) and the lower bound for $\widetilde{J}$ we see

$$\int_{\Omega_\lambda} e^{\left(\frac{n}{2}+\varepsilon\right)w} \left|\nabla w\right|^2 d\mathcal{H}^n \le c\left(g\right) + c\left(g\right)\varepsilon \int_{\Omega_\lambda} e^{\left(\frac{n}{2}+\varepsilon\right)w} \left|\nabla w\right|^2 d\mathcal{H}^n.$$

This shows $\int_{\Omega_\lambda} e^{\left(\frac{n}{2}+\varepsilon\right)w} \left|\nabla w\right|^2 d\mathcal{H}^n \le c\left(g\right)$ when ε is tiny enough. By letting $\lambda \to \infty$, we get $\int_{B_r \backslash \mathcal{B}} e^{\left(\frac{n}{2}+\varepsilon\right)w} \left|\nabla w\right|^2 d\mathcal{H}^n < \infty$. Let $\alpha = \frac{n}{2} + \varepsilon$ in (3.6), we get $\int_{B_r \backslash \mathcal{B}} e^{\left(\frac{n+4}{2}+\varepsilon\right)w} d\mathcal{H}^n < \infty$, and this implies $\dim \mathcal{B} \le \frac{n-4}{2} - \varepsilon < \frac{n-4}{2}$.

4. Proof of Theorem 1.2

In this section, we shall show that similar methods for proving Theorem 1.1 would give us Theorem 1.2. As in the beginning of Section 3, we may assume the north pole $\mathrm{N} \in \Omega$ and using the stereographic projection, we write

$$\left(\pi_{\mathrm{N}}^{-1}\right)^*\left(g\right) = \widetilde{g} = e^{2w} g_{\mathbb{R}^n}.$$

We also have the corresponding $\mathcal{B} = \pi_{\mathrm{N}}\left(S^n \backslash \Omega\right)$ and $r > 0$ such that $\mathcal{B} \subset B_r$. Again, estimates (3.1) and (3.2) remain true. It follows from (3.4) that for any $\alpha \in \mathbb{R}$,

$$\Delta\left(e^{\alpha w}\right) = -\alpha \widetilde{J} e^{(\alpha+2)w} + \alpha\left(\alpha - \frac{n-2}{2}\right)\left|\nabla w\right|^2 e^{\alpha w}. \tag{4.1}$$

Assume $n \ge 5$. Then it follows from (4.1), the fact $\widetilde{R} \ge c_1 > 0$ and (3.2) that

$$\Delta\left(e^{\frac{n-4}{2}w}\right)\left(x\right) \to -\infty \quad \text{as } \delta_{\mathcal{B}}\left(x\right) \to 0. \tag{4.2}$$

For $\lambda < \min_{\partial B_r} \Delta \left(e^{\frac{n-4}{2}w}\right)$, we denote

$$\Omega_\lambda = \left\{x \in B_r \backslash \mathcal{B} : \Delta \left(e^{\frac{n-4}{2}w}\right)(x) > \lambda\right\}.$$

For generic λ, using (4.2) and Sard's theorem we see Ω_λ is a bounded smooth open subset with $\overline{\Omega}_\lambda \subset \overline{B_r} \backslash \mathcal{B}$ and

$$\partial\Omega_\lambda = \partial B_r \cup \left\{x \in B_r \backslash \mathcal{B} : \Delta \left(e^{\frac{n-4}{2}w}\right)(x) = \lambda\right\}. \tag{4.3}$$

It follows from (2.4) that

$$\frac{n-4}{2} e^{\frac{n+4}{2}w}\widetilde{Q} = \Delta^2 \left(e^{\frac{n-4}{2}w}\right),$$

hence for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\frac{n-4}{2}\int_{\Omega_\lambda} e^{\left(\frac{n+4}{2}+\alpha\right)w}\widetilde{Q}d\mathcal{H}^n \\ = \; & -\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}} e^{\alpha w}\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right| d\mathcal{H}^{n-1} \\ & -\alpha\int_{\Omega_\lambda} e^{\alpha w} w_i \left(\Delta\left(e^{\frac{n-4}{2}w}\right)\right)_i d\mathcal{H}^n + c(\alpha, g). \end{aligned} \tag{4.4}$$

Taking $\alpha = 0$ in (4.4), we see

$$\frac{n-4}{2}\int_{\Omega_\lambda} e^{\frac{n+4}{2}w}\widetilde{Q}d\mathcal{H}^n \leq c(g).$$

Letting $\lambda \to -\infty$, we get

$$\int_{B_r\backslash\mathcal{B}} e^{\frac{n+4}{2}w}\widetilde{Q}d\mathcal{H}^n \leq c(g) < \infty.$$

In view of the fact $\widetilde{Q} \geq c_2 > 0$, it implies

$$\int_{B_r\backslash\mathcal{B}} e^{\frac{n+4}{2}w}d\mathcal{H}^n < \infty.$$

Similar arguments in Section 3 shows $\dim \mathcal{B} \leq \frac{n-4}{2}$. To get the strict inequality, we observe that it follows from (3.4), (2.7), (3.1), (4.4) and the lower bound of $\widetilde{Q}$ that for $\varepsilon > 0$ small, we have

$$\int_{\Omega_\lambda} e^{\left(\frac{n+4}{2}+\varepsilon\right)w}d\mathcal{H}^n \leq c(g) + c(g)\,\varepsilon\int_{\Omega_\lambda} e^{\left(\frac{n+4}{2}+\varepsilon\right)w}d\mathcal{H}^n,$$

and hence

$$\int_{\Omega_\lambda} e^{\left(\frac{n+4}{2}+\varepsilon\right)w}d\mathcal{H}^n \leq c(g)$$

when ε is very tiny. This shows $\int_{B_r\backslash\mathcal{B}} e^{\left(\frac{n+4}{2}+\varepsilon\right)w}d\mathcal{H}^n < \infty$ and $\dim \mathcal{B} \leq \frac{n-4}{2} - \varepsilon < \frac{n-4}{2}$ as before.

Assume $n = 4$. Then it follows from (3.4), the fact $\widetilde{R} \geq c_1 > 0$ and (3.2) that $\Delta w \to -\infty$ as $\delta_{\mathcal{B}}(x) \to 0$. By (2.5) we see

$$e^{4w}\widetilde{Q} = \Delta^2 w.$$

For $\lambda < \min_{\partial B_r} \Delta w$, define

$$\Omega_\lambda = \{x \in B_r\backslash\mathcal{B} : \Delta w(x) > \lambda\}.$$

As before, we may prove that for generic λ,

$$\int_{\Omega_\lambda} e^{4w}\widetilde{Q}d\mathcal{H}^4 \leq c(g). \tag{4.5}$$

Letting $\lambda \to -\infty$, we get

$$\int_{B_r\backslash\mathcal{B}} e^{4w}\widetilde{Q}d\mathcal{H}^4 \leq c(g) < \infty.$$

This plus the lower bound of $\widetilde{Q}$ implies $\int_{B_r\backslash\mathcal{B}} \delta_{\mathcal{B}}^{-4}d\mathcal{H}^4 < \infty$ and hence $\mathcal{B}=\emptyset$ by Lemma 2.1.

Assume $n=3$. Then it follows from (4.1) that

$$\Delta\left(e^{-\frac{1}{2}w}\right) = \frac{\widetilde{J}}{2}e^{\frac{3}{2}w} + \frac{1}{2}|\nabla w|^2 e^{-\frac{1}{2}w} \to \infty \quad \text{as } \delta_{\mathcal{B}}(x) \to 0. \tag{4.6}$$

It follows from (2.4) that

$$e^{\frac{7}{2}w}\widetilde{Q} = -2\Delta^2\left(e^{-\frac{1}{2}w}\right). \tag{4.7}$$

Using this equation, and for $\lambda > \max_{\partial B_r}\Delta\left(e^{-\frac{1}{2}w}\right)$, redefine Ω_λ by

$$\Omega_\lambda = \left\{x \in B_r\backslash\mathcal{B} : \Delta\left(e^{-\frac{1}{2}w}\right) < \lambda\right\},$$

then similar arguments will give us

$$\int_{\Omega_\lambda} e^{\frac{7}{2}w}\widetilde{Q}d\mathcal{H}^3 \leq c(g).$$

Letting $\lambda \to \infty$, we see $\int_{B_r\backslash\mathcal{B}} e^{\frac{7}{2}w}\widetilde{Q}d\mathcal{H}^3 \leq c(g) < \infty$. Using the lower bound for $\widetilde{Q}$ we get $\int_{B_r\backslash\mathcal{B}} \delta_{\mathcal{B}}^{-\frac{7}{2}}d\mathcal{H}^3 < \infty$. This clearly implies $\mathcal{B}=\emptyset$ in view of Lemma 2.1.

Now let us turn to the case when we only have $Q \geq 0$. First assume $n \geq 5$. By doing integration by part one more time in (4.4), we get for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\frac{n-4}{2}\int_{\Omega_\lambda} e^{\left(\frac{n+4}{2}+\alpha\right)w}\widetilde{Q}d\mathcal{H}^n \\
=\;& -\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}} e^{\alpha w}\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right| d\mathcal{H}^{n-1} \\
&+\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}} \Delta\left(e^{\frac{n-4}{2}w}\right)\left(\nabla e^{\alpha w}\cdot\frac{\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)}{\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right|}\right) d\mathcal{H}^{n-1} \\
&+\int_{\Omega_\lambda}\Delta\left(e^{\alpha w}\right)\Delta\left(e^{\frac{n-4}{2}w}\right)d\mathcal{H}^n + c(\alpha,g).
\end{aligned} \tag{4.8}$$

For $\varepsilon > 0$ small, it follows from (4.1) that

$$\Delta\left(e^{-\varepsilon w}\right) = \varepsilon\widetilde{J}e^{(2-\varepsilon)w} + \varepsilon\left(\frac{n-2}{2}+\varepsilon\right)|\nabla w|^2 e^{-\varepsilon w} > 0, \tag{4.9}$$

and

$$\Delta\left(e^{\frac{n-4}{2}w}\right) = -\frac{n-4}{2}\widetilde{J}e^{\frac{n}{2}w} - \frac{n-4}{2}|\nabla w|^2 e^{\frac{n-4}{2}w}. \tag{4.10}$$

Claim 4.1. $\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}}\nabla e^{-\varepsilon w}\cdot\frac{\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)}{\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right|}d\mathcal{H}^{n-1}\geq 0.$

Proof. For a generic $\tau>\max_{x\in\overline{\Omega}_\lambda}w(x)$, the open set

$$U=\left\{x\in B_r\backslash\mathcal{B}:w(x)<\tau\text{ and }\Delta\left(e^{\frac{n-4}{2}w}\right)(x)<\lambda\right\}$$

is smooth, in addition it satisfies $\overline{U}\subset\overline{B_r}\backslash\mathcal{B}$ and

$$\partial U=\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}\cup\left\{x\in B_r\backslash\mathcal{B}:w(x)=\tau\right\}.$$

By (4.9), we see

$$\begin{aligned}0 &\leq \int_U\Delta\left(e^{-\varepsilon w}\right)d\mathcal{H}^n=\int_{\partial U}\frac{\partial e^{-\varepsilon w}}{\partial\nu}d\mathcal{H}^{n-1}\\ &= -\varepsilon\int_{\{x\in B_r\backslash\mathcal{B}:w(x)=\tau\}}e^{-\varepsilon w}\left|\nabla w\right|d\mathcal{H}^{n-1}\\ &\quad+\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}}\nabla e^{-\varepsilon w}\cdot\frac{\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)}{\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right|}d\mathcal{H}^{n-1}.\end{aligned}$$

The claim follows easily. ∎

Taking $\alpha=-\varepsilon$ in (4.8), we get

$$\begin{aligned}0 &\leq \frac{n-4}{2}\int_{\Omega_\lambda}e^{\left(\frac{n+4}{2}-\varepsilon\right)w}\widetilde{Q}d\mathcal{H}^n\\ &= -\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}}e^{-\varepsilon w}\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right|d\mathcal{H}^{n-1}\\ &\quad+\lambda\int_{\left\{x\in B_r\backslash\mathcal{B}:\Delta\left(e^{\frac{n-4}{2}w}\right)(x)=\lambda\right\}}\left(\nabla e^{-\varepsilon w}\cdot\frac{\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)}{\left|\nabla\Delta\left(e^{\frac{n-4}{2}w}\right)\right|}\right)d\mathcal{H}^{n-1}\\ &\quad+\int_{\Omega_\lambda}\Delta\left(e^{-\varepsilon w}\right)\Delta\left(e^{\frac{n-4}{2}w}\right)d\mathcal{H}^n+c(\varepsilon,g).\end{aligned}$$

Using Claim 4.1 we see

$$-\int_{\Omega_\lambda}\Delta\left(e^{-\varepsilon w}\right)\Delta\left(e^{\frac{n-4}{2}w}\right)d\mathcal{H}^n\leq c(\varepsilon,g).$$

In view of (4.9) and (4.10) and the fact $R\geq c_1>0$, we see

$$\int_{\Omega_\lambda}e^{\left(\frac{n+4}{2}-\varepsilon\right)w}d\mathcal{H}^n\leq c(\varepsilon,g).$$

Letting $\lambda\to-\infty$, we see

$$\int_{B_r\backslash\mathcal{B}}e^{\left(\frac{n+4}{2}-\varepsilon\right)w}d\mathcal{H}^n\leq c(\varepsilon,g)<\infty.$$

As in Section 3 this implies $\dim\mathcal{B}\leq\frac{n-4}{2}+\varepsilon$. Taking $\varepsilon\to 0^+$, we see $\dim\mathcal{B}\leq\frac{n-4}{2}$.

Next we look at the case $n=3$. For $\varepsilon>0$ small, it follows from (4.1) that

$$\Delta\left(e^{-\varepsilon w}\right)=\varepsilon\widetilde{J}e^{(2-\varepsilon)w}+\varepsilon\left(\varepsilon+\frac{1}{2}\right)\left|\nabla w\right|^2e^{-\varepsilon w}. \tag{4.11}$$

This implies $\Delta\left(e^{-\varepsilon w}\right)(x) \to \infty$ as $\delta_{\mathcal{B}}(x) \to 0$. Using this, similar to the proof of Claim 4.1, one easily shows

$$\int_{\left\{x \in B_r \backslash \mathcal{B}: \Delta\left(e^{-\frac{1}{2} w}\right)(x)=\lambda\right\}} \nabla e^{-\varepsilon w} \cdot \frac{\nabla \Delta\left(e^{-\frac{1}{2} w}\right)}{\left|\nabla \Delta\left(e^{-\frac{1}{2} w}\right)\right|} d\mathcal{H}^2 \leq 0. \tag{4.12}$$

Using (4.7), we have for $\varepsilon > 0$ small,

$$\begin{aligned}
0 &\leq \int_{\Omega_\lambda} e^{\left(\frac{7}{2}-\varepsilon\right) w} \widetilde{Q} d\mathcal{H}^3 = -2 \int_{\Omega_\lambda} e^{-\varepsilon w} \Delta^2\left(e^{-\frac{1}{2} w}\right) d\mathcal{H}^3 \\
&= -2 \int_{\left\{x \in B_r \backslash \mathcal{B}: \Delta\left(e^{-\frac{1}{2} w}\right)(x)=\lambda\right\}} e^{-\varepsilon w}\left|\nabla \Delta\left(e^{-\frac{1}{2} w}\right)\right|^2 d\mathcal{H}^2 \\
&\quad +2\lambda \int_{\left\{x \in B_r \backslash \mathcal{B}: \Delta\left(e^{-\frac{1}{2} w}\right)(x)=\lambda\right\}} \nabla e^{-\varepsilon w} \cdot \frac{\nabla \Delta\left(e^{-\frac{1}{2} w}\right)}{\left|\nabla \Delta\left(e^{-\frac{1}{2} w}\right)\right|} d\mathcal{H}^2 \\
&\quad -2 \int_{\Omega_\lambda} \Delta\left(e^{-\varepsilon w}\right) \Delta\left(e^{-\frac{1}{2} w}\right) d\mathcal{H}^3 + c(\varepsilon, g).
\end{aligned}$$

In view of (4.12), (4.6), (4.11) and the fact $R \geq c_1 > 0$, this implies

$$\int_{\Omega_\lambda} e^{\left(\frac{7}{2}-\varepsilon\right) w} d\mathcal{H}^3 \leq c(\varepsilon, g).$$

Letting $\lambda \to \infty$, we see $\int_{B_r \backslash \mathcal{B}} e^{\left(\frac{7}{2}-\varepsilon\right) w} d\mathcal{H}^3 \leq c(\varepsilon, g) < \infty$. By (3.2) we get $\int_{B_r \backslash \mathcal{B}} \delta_{\mathcal{B}}(x)^{-\frac{7}{2}+\varepsilon} d\mathcal{H}^3(x) < \infty$. Using Lemma 2.1, we see $\mathcal{B} = \emptyset$, and hence $\Omega = S^3$.

5. Proof of Theorem 1.3

Now we may prove Theorem 1.3. We use the same notations as in Section 4. Under the condition of Theorem 1.3, the proof in Section 4 shows (4.5) is still true. That is

$$\int_{\Omega_\lambda} e^{4w} \widetilde{Q} d\mathcal{H}^4 \leq c(g).$$

Since $\int_\Omega Q^- d\mu_g < \infty$, we see

$$\int_{B_r \backslash \mathcal{B}} e^{4w}\left(\widetilde{Q}\right)^- d\mathcal{H}^4 = \int_{B_r \backslash \mathcal{B}} \left(\widetilde{Q}\right)^- d\mu_{\widetilde{g}} < \infty.$$

Let $\left(\widetilde{Q}\right)^+ = \max\left\{\widetilde{Q}, 0\right\}$, then we have

$$\int_{\Omega_\lambda} e^{4w}\left(\widetilde{Q}\right)^+ d\mathcal{H}^4 \leq c(g) + \int_{\Omega_\lambda} e^{4w}\left(\widetilde{Q}\right)^- d\mathcal{H}^4 \leq c(g),$$

letting $\lambda \to -\infty$, we see

$$\int_{B_r \backslash \mathcal{B}} e^{4w}\left(\widetilde{Q}\right)^+ d\mathcal{H}^4 \leq c(g) < \infty.$$

Hence $\int_{B_r \backslash \mathcal{B}} e^{4w}\left|\widetilde{Q}\right| d\mathcal{H}^4 < \infty$ and this clearly implies $\int_\Omega |Q| \, d\mu_g < \infty$. Using theorem 2 of [CQY], we see $\Omega = S^4 \backslash \{p_1, \cdots, p_m\}$ for $p_1, \cdots, p_m \in S^4$.

6. The positivity of Paneitz operator

In this section, we shall study the positivity of Paneitz operator on a smooth compact Riemannian manifold (M^n, g). It follows easily from (1.2) and (2.1) that

$$Q = -\Delta J + \frac{n-4}{2} J^2 + 4\sigma_2(A).$$

Using this and (2.2), we see for any $\varphi \in C^\infty(M)$,

$$\begin{aligned}
&\int_M P\varphi \cdot \varphi d\mu \\
= &\int_M \left[(\Delta\varphi)^2 - 4A(\nabla\varphi, \nabla\varphi) + (n-2) J |\nabla\varphi|^2 + \frac{n-4}{2} Q\varphi^2 \right] d\mu \\
= &\int_M \left[(\Delta\varphi)^2 - 4A(\nabla\varphi, \nabla\varphi) + (n-2) J |\nabla\varphi|^2 - \frac{n-4}{2} \Delta J \cdot \varphi^2 \right. \\
&\left. + \frac{(n-4)^2}{4} J^2\varphi^2 + 2(n-4)\sigma_2(A)\varphi^2 \right] d\mu.
\end{aligned} \tag{6.1}$$

By the Bochner formula, we have

$$\begin{aligned}
\int_M (\Delta\varphi)^2 d\mu &= \int_M \left|D^2\varphi\right|^2 d\mu + \int_M Rc(\nabla\varphi, \nabla\varphi) d\mu \\
&= \int_M \left|D^2\varphi\right|^2 d\mu + (n-2)\int_M A(\nabla\varphi, \nabla\varphi) d\mu + \int_M J|\nabla\varphi|^2 d\mu.
\end{aligned}$$

Under any local orthonormal frame on M, we have

$$\begin{aligned}
\Delta J \cdot \varphi^2 &= J_{ii}\varphi^2 = \left(J_i\varphi^2\right)_i - 2J_i\varphi\varphi_i = \left(J_i\varphi^2\right)_i - 2A_{ijj}\varphi\varphi_i \\
&= \left(J_i\varphi^2\right)_i - 2\left(A_{ij}\varphi\varphi_i\right)_j + 2A_{ij}\varphi_i\varphi_j + 2A_{ij}\varphi_{ij}\varphi,
\end{aligned}$$

and this implies

$$\int_M \Delta J \cdot \varphi^2 d\mu = 2\int_M A(\nabla\varphi, \nabla\varphi) d\mu + 2\int_M \left\langle A, D^2\varphi \right\rangle \varphi d\mu.$$

Plug these identities into (6.1) we get

$$\begin{aligned}
&\int_M \left[(\Delta\varphi)^2 - 4A(\nabla\varphi, \nabla\varphi) + (n-2) J |\nabla\varphi|^2 + \frac{n-4}{2} Q\varphi^2 \right] d\mu \\
= &\int_M \left[\left|D^2\varphi\right|^2 - (n-4)\left\langle A, D^2\varphi \right\rangle \varphi + \frac{(n-4)^2}{4} |A|^2 \varphi^2 \right] d\mu \\
&+ \int_M \Big((n-3) J |\nabla\varphi|^2 + 2(Jg - A)(\nabla\varphi, \nabla\varphi) \Big) d\mu \\
&+ \frac{n(n-4)}{2} \int_M \sigma_2(A)\varphi^2 d\mu.
\end{aligned} \tag{6.2}$$

By approximation, we know the above identity remains true for $\varphi \in H^2(M)$.

Proof of Theorem 1.4. It follows from (6.2) that for any $\varphi \in H^2(M)$,

$$\begin{aligned}
&\int_M \left[(\Delta\varphi)^2 - 4A(\nabla\varphi,\nabla\varphi) + (n-2)J|\nabla\varphi|^2 + \frac{n-4}{2}Q\varphi^2\right]d\mu \qquad (6.3)\\
\geq & \int_M \left[(n-3)J|\nabla\varphi|^2 + 2(Jg-A)(\nabla\varphi,\nabla\varphi)\right]d\mu \\
&+\frac{n(n-4)}{2}\int_M \sigma_2(A)\varphi^2 d\mu + \int_M \left(\left|D^2\varphi\right| - \frac{n-4}{2}|A||\varphi|\right)^2 d\mu.
\end{aligned}$$

Since $\sigma_1(A) \geq 0$ and $\sigma_2(A) \geq 0$, we have $Jg \geq A$. It follows easily from (6.3) that $P \geq 0$. Assume (M,g) is not Ricci flat, then A does not vanish identically. Given any $\varphi \in \ker P$, we know $\varphi \in C^\infty(M)$ and $P\varphi = 0$. It follows from (6.3) that $J|\nabla\varphi| = 0$ and $\left|D^2\varphi\right| - \frac{n-4}{2}|A||\varphi| = 0$. Let U be the open subset on which A does not vanish, then it follows from $\sigma_2(A) \geq 0$ that $J > 0$ on U. Hence $\nabla\varphi|_U = 0$. This would imply $\varphi|_U = 0$. By unique continuation property of solutions of elliptic equations, we see $\varphi \equiv 0$ on M. ∎

7. Examples

In this section, we shall present examples related to the results above. Let k and l be natural numbers and $n = k + l$. Denote $\mathbb{H}^l$ as the half space $(0,\infty) \times \mathbb{R}^{l-1}$ with the hyperbolic metric

$$g_{\mathbb{H}^l} = \frac{1}{r^2}\left(dr \otimes dr + \sum_{j=1}^{l-1} dy^j \otimes dy^j\right).$$

Here $\left(r, y^1, \cdots, y^{l-1}\right)$ is the coordinate on $(0,\infty) \times \mathbb{R}^{l-1}$. Endow the space $S^k \times \mathbb{H}^l$ with the product metric $g_{S^k \times \mathbb{H}^l}$. By abusing a little bit notations, we have

$$\begin{aligned}
g_{S^k \times \mathbb{H}^l} &= g_{S^k} + g_{\mathbb{H}^l} = g_{S^k} + \frac{1}{r^2}\left(dr \otimes dr + \sum_{j=1}^{l-1} dy^j \otimes dy^j\right) \\
&= \frac{1}{r^2}\left(dr \otimes dr + r^2 g_{S^k} + \sum_{j=1}^{l-1} dy^j \otimes dy^j\right) \\
&= \frac{1}{r^2}\left(g_{\mathbb{R}^{k+1}} + \sum_{j=1}^{l-1} dy^j \otimes dy^j\right).
\end{aligned}$$

Note here we have used the polar coordinate on $\mathbb{R}^{k+1}\backslash\{0\}$. The above calculation shows $S^k \times \mathbb{H}^l$ is conformal isomorphic to $\left(\mathbb{R}^{k+1}\backslash\{0\}\right) \times \mathbb{R}^{l-1} = \mathbb{R}^n \backslash \left(\{0\} \times \mathbb{R}^{l-1}\right)$, which is conformal isomorphic to $\Omega = S^n \backslash S^{l-1}$ by the stereographic projection. Note that $\dim(S^n \backslash \Omega) = l - 1$.

On the other hand, we have

$$\begin{aligned}
Rc &= (k-1)g_{S^k} - (l-1)g_{\mathbb{H}^l}, \quad R = (n-1)(k-l), \quad J = \frac{k-l}{2}, \\
A &= \frac{1}{2}g_{S^k} - \frac{1}{2}g_{\mathbb{H}^l}, \quad \sigma_2(A) = \frac{1}{8}\left((k-l)^2 - n\right), \quad Q = \frac{n}{8}\left((k-l)^2 - 4\right).
\end{aligned}$$

Let N^l be a l dimensional smooth compact oriented Riemannian manifold whose universal covering space is isometric to $\mathbb{H}^l$. Endow $M^n = S^k \times N^l$ with the product metric $g = g_{S^k} + g_N$. For $S^5 \times N^2$, we have $R = 18$, $\sigma_2(A) = \frac{1}{4}$, $Q = \frac{35}{8}$, $\pi_5(S^5 \times N^2) \cong \mathbb{Z}$, $H^2(S^5 \times N^2, \mathbb{R}) \cong \mathbb{R}$ and $H^5(S^5 \times N^2, \mathbb{R}) \cong \mathbb{R}$. This should be compared with Corollary 1.1 and Corollary 1.2. More generally, if we let k be the minimal integer larger than or equal to $\frac{n+\sqrt{n}}{2}$, then $l \le \frac{n-\sqrt{n}}{2}$, it follows that $\sigma_1(A) > 0$, $\sigma_2(A) \ge 0$ and $\dim(S^n \backslash \Omega) = l - 1 > \frac{n-\sqrt{n}}{2} - 2$, this should be compared with Theorem 1.1. If $n \ge 5$ is even, then let $k = \frac{n+4}{2}$, $l = \frac{n-4}{2}$, we see $R > 0$, $Q > 0$ and $\dim(S^n \backslash \Omega) = l - 1 = \frac{n-6}{2}$; let $k = \frac{n+2}{2}$, $l = \frac{n-2}{2}$, we see $R > 0$, $Q = 0$ and $\dim(S^n \backslash \Omega) = l - 1 = \frac{n-4}{2}$. If $n \ge 5$ is odd, then let $k = \frac{n+3}{2}$, $l = \frac{n-3}{2}$, we see $R > 0$, $Q > 0$ and $\dim(S^n \backslash \Omega) = l - 1 = \frac{n-5}{2}$. These should be compared with Theorem 1.2.

Assume $n \ge 5$. For any $\varphi \in H^2(M)$, the quadratic form associated with the Paneitz operator on M is

$$\begin{aligned}
&\int_M \left[(\Delta\varphi)^2 - 4A(\nabla\varphi, \nabla\varphi) + (n-2) J |\nabla\varphi|^2 + \frac{n-4}{2} Q\varphi^2 \right] d\mu \\
= &\int_M \left[(\Delta\varphi)^2 - 2|\nabla_{S^k}\varphi|^2 + 2|\nabla_N\varphi|^2 + \frac{(n-2)(k-l)}{2} |\nabla\varphi|^2 \right. \\
&\left. + \frac{n(n-4)}{16} \left((k-l)^2 - 4 \right) \varphi^2 \right] d\mu.
\end{aligned}$$

In particular, if $k - l \ge 3$, then we know M has positive scalar curvature and positive Paneitz operator.

For any $w \in C^\infty(M, \mathbb{R})$, we let $g_w = e^{2w} g$, then it follows from (2.8) that (using orthonormal frame with respect to g)

$$\begin{aligned}
&2 \int_M e^{4w} \sigma_2(A_w)\, d\mu \\
= &\int_M \left((\Delta w)^2 - |D^2 w|^2 \right) d\mu + \int_M \left((n-3) |\nabla w|^2 \Delta w + 2 w_{ij} w_i w_j \right) d\mu \\
&+ \frac{(n-1)(n-4)}{4} \int_M |\nabla w|^4 d\mu + 2 \int_M \sigma_2(A)\, d\mu + 2 \int_M A_{ij} w_{ij} d\mu \\
&- 2 \int_M J \Delta w d\mu - 2 \int_M A(\nabla w, \nabla w)\, d\mu - (n-3) \int_M J |\nabla w|^2 d\mu.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_M \left((\Delta w)^2 - |D^2 w|^2 \right) d\mu &= (n-2) \int_M A(\nabla\varphi, \nabla\varphi)\, d\mu + \int_M J |\nabla\varphi|^2 d\mu, \\
\int_M w_{ij} w_i w_j d\mu &= -\frac{1}{2} \int_M |\nabla w|^2 \Delta w d\mu, \\
\int_M A_{ij} w_{ij} d\mu &= -\int_M \langle \nabla J, \nabla w \rangle\, d\mu = 0,
\end{aligned}$$

and

$$\int_M J \Delta w d\mu = J \int_M \Delta w d\mu = 0,$$

hence we have

$$
\begin{aligned}
& 2\int_M e^{4w}\sigma_2\left(A_w\right)d\mu \\
= & (n-4)\int_M |\nabla w|^2\Delta w d\mu+\frac{(n-1)(n-4)}{4}\int_M |\nabla w|^4 d\mu+2\int_M \sigma_2(A)\,d\mu \\
& +(n-4)\int_M A(\nabla w,\nabla w)\,d\mu-(n-4)\int_M J|\nabla w|^2 d\mu .
\end{aligned}
$$

Using (2.6) we see

$$
\Delta w=J-J_w e^{2w}-\frac{n-2}{2}|\nabla w|^2 .
$$

Hence

$$
\begin{aligned}
(7.1)\quad & 2\int_M e^{4w}\sigma_2\left(A_w\right)d\mu \\
= & -(n-4)\int_M e^{2w}|\nabla w|^2 J_w d\mu-\frac{(n-3)(n-4)}{4}\int_M |\nabla w|^4 d\mu+2\int_M \sigma_2(A)\,d\mu \\
& +(n-4)\int_M A(\nabla w,\nabla w)\,d\mu \\
= & -(n-4)\int_M e^{2w}|\nabla w|^2 J_w d\mu-\frac{(n-3)(n-4)}{4}\int_M |\nabla w|^4 d\mu \\
& -\frac{1}{4}\int_M\left(n-(k-l)^2\right)d\mu+\frac{n-4}{2}\int_M |\nabla_{S^k}w|^2 d\mu-\frac{n-4}{2}\int_M |\nabla_N w|^2 d\mu \\
= & -(n-4)\int_M e^{2w}|\nabla w|^2 J_w d\mu-\frac{(n-3)(n-4)}{4}\int_M\left(|\nabla_{S^k}w|^2-\frac{1}{n-3}\right)^2 d\mu \\
& -\frac{(n-3)(n-4)}{4}\int_M\left(|\nabla_N w|^4+2|\nabla_{S^k}w|^2|\nabla_N w|^2\right)d\mu \\
& -\frac{n-4}{2}\int_M |\nabla_N w|^2 d\mu-\frac{1}{4}\int_M\left(\frac{(n-2)^2}{n-3}-(k-l)^2\right)d\mu .
\end{aligned}
$$

For convenience, we write $k=\frac{n+a}{2}$, $l=\frac{n-a}{2}$. Fix a such that $3\le a\le\frac{n-2}{\sqrt{n-3}}$, then we know the scalar curvature and the Paneitz operator is always positive. In addition, if for some w, $\sigma_1(A_w)\ge 0$ and $\sigma_2(A_w)\ge 0$, then it follows from (7.1) that $J_w|\nabla w|=0$ and $|\nabla_{S^k}w|=\frac{1}{\sqrt{n-3}}$. Hence $J_w=0$. This contradicts with the fact $J>0$, which implies that the Yamabe constant is positive. The arguments show that we can not find any conformal metric g_w with $\sigma_1(A_w)\ge 0$ and $\sigma_2(A_w)\ge 0$. On the other hand, if for some w, we have $\sigma_2(A_w)>0$, then $J_w^2-|A_w|^2>0$ and this implies either $J_w>0$ or $J_w<0$. Since $J>0$, we see $J_w>0$. This shows $\sigma_1(A_w)>0$ and $\sigma_2(A_w)>0$. It contradicts with the former conclusion. Hence we can not find any conformal metric with positive σ_2 neither. This should be compared with Theorem 1.4.

8. Coercivity of the conformal factor

Motivated by the proofs of Theorem 1.1 and Theorem 1.2, we propose the following problem :

Let $\Omega \subset S^n (n \geq 3)$ be an open subset, $w_0 \in C^\infty(\Omega, \mathbb{R})$ such that $e^{2w_0} g_{S^n}$ is complete on Ω, when would we have $w_0(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$?

The following result is an answer to the question. One may apply this result to relax the assumptions in Theorem 1.1 and Theorem 1.2, as mentioned in the introduction.

Proposition 8.1. *Let $\Omega \subset S^n\ (n \geq 3)$ be an open subset, $w_0 \in C^\infty(\Omega, \mathbb{R})$ such that $\left(\Omega, e^{2w_0} g_{S^n}\right)$ is complete. If $R_{e^{2w_0} g_{S^n}} \geq -c_1$ for some $c_1 > 0$, then $w_0(x) \to \infty$ as* $\operatorname{dist}(x, \partial\Omega) \to 0$.

Proof. By rotation, we may assume $\mathrm{N} \in \Omega$. We may write

$$g = \left(\pi_{\mathrm{N}}^{-1}\right)^* \left(e^{2w_0} g_{S^n}\right) = e^{2w} g_{\mathbb{R}^n}$$

and $U = \pi_{\mathrm{N}}(\Omega \backslash \{\mathrm{N}\})$. Denote $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ as the conformal Laplacian, then

$$0 = L_{g_{\mathbb{R}^n}} 1 = L_{e^{-2w} g} 1 = e^{\frac{n+2}{2} w} L_g\left(e^{-\frac{n-2}{2} w}\right).$$

This implies

$$-\frac{4(n-1)}{n-2}\Delta_g\left(e^{-\frac{n-2}{2} w}\right) + R_g e^{-\frac{n-2}{2} w} = 0.$$

Since U is a domain in $\mathbb{R}^n$, by the conformal covariant property of the conformal Laplacian operator, one easily deduces that for any $\varphi \in C_c^\infty(U)$,

$$\int_U \left(\frac{4(n-1)}{n-2}\left|\nabla_g \varphi\right|_g^2 + R_g \varphi^2\right) d\mu_g \geq c(n)\left(\int_U |\varphi|^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}},$$

here $c(n) > 0$. That is, we have the Sobolev inequality. Hence we may start the procedure of Moser iteration. From now on, for convenience, we use g as back ground metric and omit the g in all notations. Let $u = e^{-\frac{n-2}{2} w}$, then

$$-\frac{4(n-1)}{n-2}\Delta u + R^+ u = R^- u \leq c_1 u.$$

For $\beta > 0$, $\eta \in C_c^\infty(U)$, $0 \leq \eta \leq 1$. Let $\varphi = \eta^2 u^\beta$, since

$$\int_U \left(\frac{4(n-1)}{n-2}\nabla u \cdot \nabla\varphi + R^+ u\varphi\right) d\mu \leq c_1 \int_U u\varphi d\mu,$$

and

$$\nabla\varphi = 2\eta u^\beta \nabla\eta + \beta\eta^2 u^{\beta-1}\nabla u,$$

we see

$$\begin{aligned} & c(n)\beta\int_U \eta^2 u^{\beta-1}|\nabla u|^2 d\mu + c(n)\int_U \eta u^\beta \nabla u\cdot\nabla\eta d\mu + \int_U R^+\eta^2 u^{\beta+1} d\mu \\ \leq\ & c_1\int_U \eta^2 u^{\beta+1} d\mu. \end{aligned}$$

Hence

$$\begin{aligned} & c(n)\beta\int_U \eta^2 u^{\beta-1}|\nabla u|^2 d\mu + \int_U R^+\eta^2 u^{\beta+1} d\mu \\ \leq\ & c_1\int_U \eta^2 u^{\beta+1} d\mu + \frac{c(n)}{\beta}\int_U |\nabla\eta|^2 u^{\beta+1} d\mu. \end{aligned}$$

In another way, it is

$$
\begin{aligned}
& \frac{c(n)\beta}{(\beta+1)^2}\int_U \eta^2 \left|\nabla u^{\frac{\beta+1}{2}}\right|^2 d\mu + \int_U R^+\eta^2 u^{\beta+1} d\mu \\
\leq\ & c_1 \int_U \eta^2 u^{\beta+1} d\mu + \frac{c(n)}{\beta}\int_U |\nabla\eta|^2 u^{\beta+1} d\mu.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int_U \eta^2 \left|\nabla u^{\frac{\beta+1}{2}}\right|^2 d\mu + \frac{c(n)(\beta+1)^2}{\beta}\int_U R^+\eta^2 u^{\beta+1} d\mu \\
\leq\ & c(n,c_1)\left(\beta+\frac{1}{\beta^2}\right)\int_U \left(\eta^2+|\nabla\eta|^2\right) u^{\beta+1} d\mu.
\end{aligned}
$$

But

$$
\int_U \left|\nabla\left(\eta u^{\frac{\beta+1}{2}}\right)\right|^2 d\mu \leq 2\int_U \eta^2\left|\nabla u^{\frac{\beta+1}{2}}\right|^2 d\mu + 2\int_U u^{\beta+1}|\nabla\eta|^2 d\mu,
$$

we get

$$
\begin{aligned}
& \frac{4(n-1)}{n-2}\int_U \left|\nabla\left(\eta u^{\frac{\beta+1}{2}}\right)\right|^2 d\mu + \int_U R^+\eta^2 u^{\beta+1} d\mu \\
\leq\ & c(n,c_1)\left(\beta+\frac{1}{\beta^2}\right)\int_U \left(\eta^2+|\nabla\eta|^2\right) u^{\beta+1} d\mu.
\end{aligned}
$$

Observing that

$$
\begin{aligned}
& \frac{4(n-1)}{n-2}\int_U \left|\nabla\left(\eta u^{\frac{\beta+1}{2}}\right)\right|^2 d\mu + \int_U R^+\eta^2 u^{\beta+1} d\mu \\
\geq\ & \frac{4(n-1)}{n-2}\int_U \left|\nabla\left(\eta u^{\frac{\beta+1}{2}}\right)\right|^2 d\mu + \int_U R\eta^2 u^{\beta+1} d\mu \\
\geq\ & c(n)\left|\eta u^{\frac{\beta+1}{2}}\right|^2_{L^{\frac{2n}{n-2}}(U)}.
\end{aligned}
$$

Hence

$$
\left|\eta u^{\frac{\beta+1}{2}}\right|^2_{L^{\frac{2n}{n-2}}(U)} \leq c(n,c_1)\left(\beta+\frac{1}{\beta^2}\right)\int_U \left(\eta^2+|\nabla\eta|^2\right) u^{\beta+1} d\mu.
$$

Choose a point $x_0 \in U$ such that $D_1^g(x_0) = \{x \in U : d_g(x,x_0) \leq 1\}$ is compact. For $k \in \mathbb{N}$, let $r_k = \frac{1}{2} + \frac{1}{2^k}$, by choosing suitable η we have

$$
\left|u^{\frac{\beta+1}{2}}\right|^2_{L^{\frac{2n}{n-2}}\left(B^g_{r_{k+1}}\right)} \leq c(n)\beta\cdot 4^k \int_{B^g_{r_k}} u^{\beta+1} d\mu.
$$

In another way, it is

$$
|u|^{\beta+1}_{L^{\frac{n}{n-2}(\beta+1)}\left(B^g_{r_{k+1}}\right)} \leq c(n)\beta\cdot 4^k \int_{B^g_{r_k}} u^{\beta+1} d\mu.
$$

Choose $\beta = \frac{2n}{n-2}\left(\frac{n}{n-2}\right)^{k-1} - 1$, then we get

$$
\begin{aligned}
|u|_{L^{\frac{2n}{n-2}\left(\frac{n}{n-2}\right)^k}\left(B_{r_{k+1}}\right)} \leq\ & c(n,c_1)^{\frac{n-2}{2n}\left(\frac{n-2}{n}\right)^{k-1}}\left(\frac{2n}{n-2}\left(\frac{n}{n-2}\right)^{k-1}\right)^{\frac{n-2}{2n}\left(\frac{n-2}{n}\right)^{k-1}}\cdot \\
& 4^{\frac{n-2}{2n}k\left(\frac{n-2}{n}\right)^{k-1}}|u|_{L^{\frac{2n}{n-2}\left(\frac{n}{n-2}\right)^{k-1}}\left(B_{r_k}\right)}.
\end{aligned}
$$

Iteration shows

$$|u|_{L^\infty\left(B^g_{1/2}\right)} \leq c\,(n, c_1)\, |u|_{L^{\frac{2n}{n-2}}\left(B^g_1\right)}.$$

In particular, we have

$$\begin{aligned} e^{-\frac{n-2}{2}w(x_0)} &\leq c\,(n,c_1)\left| e^{-\frac{n-2}{2}w}\right|_{L^{\frac{2n}{n-2}}\left(B^g_1(x_0)\right)} \\ &= c\,(n,c_1)\left(\int_{B^g_1(x_0)} e^{-nw}d\mu\right)^{\frac{n-2}{2n}} \\ &= c\,(n,c_1)\left(\mathcal{H}^n\left(B^g_1\left(x_0\right)\right)\right)^{\frac{n-2}{2n}} \to 0 \end{aligned}$$

as $x_0 \to \partial U$, in view of the Lebesgue dominated convergence theorem. ∎

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