Entire Solutions of a Fully Nonlinear Equation

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0 Introduction

In this paper we are concerned with the classification of entire solutions of an equation in conformal geometry. The result is motivated by the classical theorem of Obata [O] that any metric conformal to the standard metric on the n-sphere S^n with constant scalar curvature, is isometric to the standard metric. As we will recall in section 1 below, the main step in the original proof of Obata is the Bianchi identity for the scalar curvature: $divE = \frac{(n-2)}{2n}\nabla R$, where E denotes the traceless Ricci tensor and R the scalar curvature. Obata's result was later generalized by Caffarelli-Gidas-Spruck [CGS], where instead of considering metrics on S^n, they considered metrics $u^{\frac{4}{n-2}}|dx|^2$ conformal to the Euclidean metric $|dx|^2$ on $\mathbb{R}^n$ with constant scalar curvature, i.e. $u > 0$ satisfies the equation

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \qquad u > 0 \quad \text{on} \quad \mathbb{R}^n\ . \tag{0.1}$$

They classified all solutions of (0.1) via the method of moving planes. Namely, $u(x) = (\frac{2\lambda}{\lambda^2+|x-x_0|^2})^{\frac{2}{n-2}}$ for some $x_0 \in \mathbb{R}^n$, $\lambda > 0$; or equivalently the metric $u^{\frac{4}{n-2}}|dx|^2$ is isometric to the standard metric on S^n, where points in $\mathbb{R}^n$ are identified with points on S^n via stereographic projection.

In this note we will illustrate a method which reduces the proof of the above classification result of [CGS] to a "tail" term estimate using method of proof in [O]. We will then indicate that the estimate can easily be verified when the dimension n is equal to 3. and for $n \geq 4$ under the additional (strong) assumption that the volume of the metric is finite. We then apply the same scheme to classify entire solution of a fully non-linear equation studied in ([V-1], [V-2] and [CGY-1], [CGY-2]) which can be viewed as a fully nonlinear generalization of the Yamabe equation.

We now describe our main result:

Given a Riemannian manifold (M^n, g), we denote the Weyl-Schouten tensor $A_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij}$, where R_{ij} denotes the components of the Ricci tensor. Viewed as an endomorphism on the tangent bundle, $A = A_g$ has n real eigenvalues, and we let $\sigma_2(A_g)$ denote the second elementary symmetric function of the eigenvalues.

Theorem 0.1 *Let $g = v^{-2}|dx|^2$ be a conformal metric on $\mathbb{R}^n$, $n \geq 4$, satisfying*

$$\sigma_2(A_g) = \frac{1}{8}n(n-1)(n-2)^2. \tag{0.2}$$

If $n \geq 6$, assume in addition that

$$vol(g) = \int_{\mathbb{R}^n} v^{-n}dx < \infty. \tag{0.3}$$

Then $v = a|x|^2 + b_i x^i + c$ for constants a, b^i, c. In particular, g is obtained by pulling back to $\mathbb{R}^n$ the round metric on S^n.

Theorem 0.1 was proved by Viaclovsky [V-2] for all elementary symmetric functions $\sigma_k(A)$, but under the (strong) assumption that the metric is defined on S^n. This amounts to assuming that the singularity at infinity is removable. When $n = 4$ and $k = 2$ Theorem 0.1 is contained in [CGY-2]. We recently learned that A. Li and Y. Li have announced a version of this result which holds for all k in all dimensions ([LL]; see also the article of Guan and Wang [GW]). However, since our proof (which we have obtained a year ago) is quite different than the one announced in [LL], we decided to publish our result. The geometric nature of our argument also allows the possibility that it can be generalized to establish the uniqueness of solutions on general Einstein manifolds, like Obata's result which inspired it.

The note is organized as follows: In section 1, we describe Obata's proof and the "tail" term estimate required to modify Obata's proof to obtain the result in [CGS]. We then prove the tail estimate for dimension $n = 3$. In section 2, we establish a conservation law for the $\sigma_2(A)$ equation that is analogous to classical Bianchi identity. In section 3, we derive the "tail" estimate required for the σ_2 equation for $n = 4, 5$ and for $n \geq 6$ under the additional assumption (0.3). Finally in section 4, we prove the main classification result Theorem 0.1.

1 Obata's proof

In this section we will first recall the proof a result of Obata ([O]) that metrics defined on the n-sphere with constant scalar curvature and conformal to the standard metric is Einstein, hence isometric to the standard metric on S^n. We will then modify Obata's argument to show that one can reduce the proof of the main result in ([CGS], see Theorem 1.1 below) for metrics defined on $\mathbb{R}^n$ to a "tail" term estimate. We then establish the tail term estimate for the case $n = 3$ and under the additional volume bound condition for the cases $n \geq 4$.

To this end, suppose $g = v^{-2} g_0$ is a conformal metric on S^n, where g_0 is the round metric. Assume that g has constant scalar curvature.

To begin, we express the trace-free Ricci Tensor E in terms of v;

$$E = -(n-2) v \nabla_g^2 (v^{-1}) + \frac{(n-2)}{n} v \Delta_g (v^{-1}) g. \tag{1.1}$$

Note that the Hessian and Laplacian in (1.1) are with respect to g, not g_0. If we pair both sides of (1.1) with $v^{-1}E$ and integrate over S^n we obtain

$$\int_{S^n} |E|^2 v^{-1} dvol_g = -(n-2) \int_{S^n} g(E, \nabla_g^2 (v^{-1})) dvol_g.$$

Note that the second term vanishes *because E is trace-free.* We apply the divergence theorem to conclude

$$\int_{S^n} |E|^2 v^{-1} dvol_g = (n-2) \int_{S^n} g(\delta E, d(v^{-1})) dvol_g.$$

The contracted second Bianchi identity says that $divE = \frac{(n-2)}{2n} dR$, where R is the scalar curvature. Since R is constant, E *is divergence-free.* Thus

$$\int_{S^n} |E|^2 v^{-1} dvol_g = 0. \tag{1.2}$$

The uniqueness result follows, since (1.2) implies that $E \equiv 0$, i.e. g is Einstein.

We will now modify the argument above to metrics defined on $\mathbb{R}^n$ and prove the following result:

Theorem 1.1. *Let $g = v^{-2}|dx|^2$ be a conformal metric on $\mathbb{R}^n$, $n \geq 3$, whose scalar curvature R equals the constant $n(n-1)$. Assume in addition that g satisfies*

$$\int_{A_\rho} |\nabla_0 v|^2 v^{1-n} dx \lesssim \rho^2, \tag{1.3}$$

where ∇_0 denotes the Euclidean gradient, A_ρ denotes the annulus $B(2\rho) - B(\rho)$, and $B(\rho)$ denotes the Euclidean ball on $\mathbb{R}^n$ centered at 0 of radius $\rho > 0$.

Then $v = a|x|^2 + b_i x^i + c$ for constants a, b^i, c. In particular, g is obtained by pulling back to $\mathbb{R}^n$ the round metric on S^n.

Proof To begin, fix $\rho > 1$ and let η denote a cut-off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\partial_i \eta| \lesssim \rho^{-1}$. Following the outline of Obata's argument above, we pair both sides of (1.1) with $v^{-1}E\eta^2$ and integrate over $\mathbb{R}^n$ to obtain

$$\int g(E,E) v^{-1} \eta^2 dvol(g) = -\int (n-2) g(E, \nabla_g^2(v^{-1})) \eta^2 dvol(g). \tag{1.4}$$

Note that in (1.4) we have used the fact that E is trace-free. Applying Bianchi identity $divE = \frac{(n-2)}{2n} dR$ as before, we obtain

$$\int g(E,E) v^{-1} \eta^2 dvol(g) = \int (n-2) g(\delta E, d(v^{-1})) \eta^2 dvol(g)$$
$$+ \int (n-2) E(\nabla_g(v^{-1}), \nabla_g(\eta^2)) dvol(g).$$

Since $R = R_g$ is constant, E is divergence-free. Thus

$$\int g(E,E) v^{-1} \eta^2 dvol(g) = \int (n-2) E(\nabla_g(v^{-1}), \nabla_g(\eta^2)) dvol(g)$$
$$\lesssim \int |E| |\nabla_g(v^{-1})| |\nabla_g(\eta^2)| dvol(g)$$

$$\lesssim \int |E||\nabla_g v||\nabla_g \eta| v^{-2}\eta dvol(g).$$

Applying the Schwartz inequality we conclude

$$\int g(E,E)v^{-1}\eta^2 dvol(g) \lesssim \left(\int\limits_{supp|\nabla\eta|} |g(E,E)|v^{-1}\eta^2 dvol(g)\right)^{\frac{1}{2}}$$

$$\times \left(\int |\nabla_g v|^2 |\nabla_g \eta|^2 v^{-3} dvol(g)\right)^{\frac{1}{2}}. \tag{1.5}$$

We now rewrite the integral on the Rhs of (1.5) in terms of the Euclidean metric, using the identities

$$|\nabla_g v|^2 = v^2|\nabla v|^2,$$

$$|\nabla_g \eta|^2 = v^2|\nabla \eta|^2,$$

$$dvol(g) = v^{-n}dx.$$

Thus,

$$\int |\nabla_g v|^2|\nabla_g\eta|^2 v^{-3}dvol(g) = \int |\nabla v|^2|\nabla \eta|^2 v^{1-n}dx.$$

Since $|\nabla\eta|^2 \lesssim \rho^{-2}$ and $supp\ \eta \subset A(\rho) = \{x\epsilon\mathbb{R}^n | \rho < |x| < 2\rho\}$, we conclude from assumption (1.3)

$$\int |\nabla_g v|^2|\nabla_g\eta|^2 v^{-3}dvol(g) \lesssim \rho^{-2}\int_{A_\rho} |\nabla v|^2 v^{1-n}dx < \infty.$$

Thus

$$\int_{\mathbb{R}^n} g(E,E)v^{-1}dvol(g) < \infty. \tag{1.6}$$

In particular,

$$\int_{supp|\nabla\eta|} g(E,E)v^{-1}dvol(g) = \int_{supp|\nabla\eta|} |g(E,E)|v^{-1}dvol(g) \to 0 \tag{1.7}$$

as $\rho \to \infty$. Now combining (1.7) with (1.5) and the boundedness of the integrals in (1.6), we conclude that

$$\int g(E,E)v^{-1}\eta^2 dvol(g) \to 0 \quad \text{as} \quad \rho \to \infty,$$

so $g(E,E) \equiv 0$ on $\mathbb{R}^n$. This implies $E \equiv 0$. The conclusion of Theorem 1.1 that v is a quadratic polynomial follows easily from the expression of E as in (1.1).

We will end this section by remarking that the assumption (1.3) in the statement of Theorem 1.1 can be easily established for metrics of constant scalar curvature when $n = 3$, but for $n \geq 4$ the same argument only establishes the inequality (1.3) under the additional assumption that volume of g is finite. We remark that the volume finiteness assumption is frequently harmless when the result is applied to the limiting case of a "blow up" argument for metrics defined on a compact manifold. We state the result in the following proposition.

To simplify the notations, we will hence forth denote ∇_0 by ∇, Δ_0 by Δ, etc.

Proposition 1.2 *Let $g = v^{-2}dx^2$ be a conformal metric on $\mathbb{R}^n$. Assume that there exists some positive constants $C_0 = C_0(n), C_1 = C_1(n)$, so that*

$$\text{when } n = 3 \qquad R_g \geq C_0,$$

and

$$\text{when } n \geq 4 \qquad \begin{cases} \frac{1}{C_1} \leq R_g \leq C_1 \\ \int_{A_\rho} v^{-n} dx \leq C_1 \end{cases}$$

where $A(\rho) = \{x \epsilon \mathbb{R}^n, \rho \leq |x| \leq 2\rho\}$, for all $\rho >> 1$. Then there is a constant $C_2 = C_2(C_0, C_1, n)$, so that

$$\int_{A_\rho} |\nabla v|^2 v^{1-n} dx \leq C_2 \rho^2 \quad \text{for all} \quad \rho >> 1.$$

To prove the proposition, we begin with a technical Lemma which is a well known result (c.f. [KMPS, Lemma 1]). We

Lemma 1.3 *Suppose $g = v^{-2}ds^2$ is a conformal metric with $R = R_g \geq C_3 \geq 0$, then there is some constant C_4 so that $v(x) \leq C_4|x|^2$ for all $|x|$ sufficiently large.*

Proof. Denote $g = u^{\frac{4}{n-2}}dx^2$; i.e. $u = v^{-\frac{n-2}{2}}$, then the scalar curvature equation is of the familiar form

$$-\Delta u = \frac{n-2}{4(n-1)} R\ u^{\frac{n+2}{n-2}}.$$

This equation is invariant under the Kelvin transform: let us denote

$$\hat{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right),\ \hat{R}(x) = R\left(\frac{x}{|x|^2}\right)$$

then $\hat{u}$ satisfies

$$-\Delta \hat{u} = \frac{n-2}{4(n-1)} \hat{R} \hat{u}^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n - \{0\}.$$

Since $\hat{R} \geq 0$, $-\Delta \hat{u} \geq 0$ on $\mathbb{R}^n - \{0\}$ implies that $-\Delta \hat{u} \geq 0$ on $\mathbb{R}^n$ in the distribution sense. Hence $\hat{u}$ is superharmonic near $x = 0$; thus $\hat{u}(x) \geq C_4$ for some C_4 for all $|x|$ sufficiently small. This is equivalent to the statement that $v(x) \leq C_4 |x|^2$ for $|x|$ sufficiently large.

We now prove Proposition 1.2.

Proof. We first recall the scalar curvature equation for the metric $g = v^{-2} dx^2$:

$$-\Delta v + \frac{n}{2} v^{-1} |\nabla v|^2 + \frac{1}{2(n-1)} R v^{-1} = 0. \tag{1.8}$$

Choose $\rho > 1$ and η a cut off function supported with $\eta \equiv 1$ on $A(\rho)$ and with η supported on $B(\frac{5}{2}\rho) -\ B(\frac{1}{2}\rho)$. Multiply the equation (1.8) by $v^{-n+2}\eta^4$ and integrate by parts, we get

$$(\frac{n}{2} - n + 2) \int |\nabla v|^2 v^{-n+1} \eta^4 dx + \frac{1}{2(n-1)} \int R v^{-n+1} \eta^4 dx = -\int \nabla v \cdot \nabla \eta^4 v^{-n+2} dx. \tag{1.9}$$

n=3 case: We have

$$Lhs\ of\ (1.9) = \frac{1}{2} \int |\nabla v|^2 v^{-2} \eta^4 dx + \frac{1}{4} \int R v^{-2} \eta^4 dx$$

$$\begin{aligned} Rhs\ of\ (1.9) &\lesssim \frac{1}{\rho} \left(\int |\nabla v|^2 v^{-2} \eta^4 dx \right)^{\frac{1}{2}} (\int \eta^2 dx)^{\frac{1}{2}}, \\ &\lesssim \frac{1}{\rho} (\int |\nabla v|^2 v^{-2} \rho^4 dx)^{\frac{1}{2}} \rho^{\frac{2}{3}}, \end{aligned}$$

which in turn, under the assumption that $R \geq 0$, implies that

$$\int |\nabla v|^2 v^{-2} \eta^4 dx \lesssim \rho \lesssim \rho^2$$

as claimed.

$n \geq 5$ **cases:** We rewrite (1.9) as

$$(\frac{n}{2}-2)\int |\nabla v|^2 v^{-n+1}\eta^4 dx = \frac{1}{2(n-1)}\int R v^{-n+1}\eta^4 dx + \int \nabla v \cdot \nabla(\eta^4) v^{-n+2} dx. \qquad (1.10)$$

Thus under the additional assumptions that there is a positive number that $\frac{1}{C_1} \leq R \leq C_1$ and $\int_{A_\rho} v^{-n} \leq C_1$, we may apply Lemma 1.3 to obtain from (1.10) that

$$\int |\nabla v|^2 v^{-n+1}\eta^4 dx \lesssim \rho^2 + (\int |\nabla v|^2 v^{-n+1}\eta^4 dx)^{\frac{1}{2}}\rho.$$

Thus

$$\int_{A_\rho} |\nabla v|^2 v^{-n+1} dx \lesssim \rho^2$$

as desired.

n=4 case: The proof is slightly more complicated. We now mutiply the equation (1.8) by $v^{-n+\alpha}\eta^4$ and integrate by parts, for $n=4$ we get

$$(\alpha-2)\int |\nabla v|^2 v^{-5+\alpha}\eta^4 dx + \int \nabla v \cdot \nabla \eta^4 v^{-4+\alpha} dx + \frac{1}{6}\int R v^{-5+\alpha}\eta^4 dx = 0. \qquad (1.11)$$

We now choose $\alpha = 1$ and conclude from (1.11) that

$$\int |\nabla v|^2 v^{-4}\eta^4 dx = \frac{1}{6}\int R v^{-4}\eta^4 dx + \int \nabla v \cdot \nabla \eta^4 v^{-3} dx$$

$$\lesssim \int R v^{-4}\eta^4 dx + \frac{1}{\rho}(\int |\nabla v|^2 v^{-4}\eta^4 dx)^{\frac{1}{2}}(\int v^{-4}\eta^4 dx)^{\frac{1}{4}}|supp\eta|^{\frac{1}{4}}. \qquad (1.12)$$

Thus from our assumption that $R \leq C$ and $\int_{supp\eta} v^{-4} dx \leq C$, we conclude from (1.12) that

$$\int |\nabla v|^2 v^{-4}\eta^4 dx \lesssim C.$$

Applying Lemma 1.3, we then conclude that

$$\int |\nabla v|^2 v^{-3}\eta^4 dx \lesssim \rho^2 \int |\nabla v|^2 v^{-4}\eta^4 dx \lesssim \rho^2.$$

This finishes the proof for the case $n=4$, hence the proof of the proposition.

2 A conservation law

In this section, we will derive tensor estimates for a tensor which plays the same role for our σ_2 equation as the trace-free Ricci tensor does in Obata's proof. It turns out that such a tensor has been described in four dimensions by Gursky [Gu] and in general by Viaclovsky [V-3].

Recall that on a n-dimensional manifold (M, g), we denote the Weyl-Schouten tensor by $A_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij}$, where R_{ij} denotes the Ricci curvature, and R the scalar curvature of the metric g. We also denote the second elementary symmetric function of the eigenvalues of the tensor A by $\sigma_2(A_g) = \frac{1}{2}((TraceA)^2 - |A|^2)$.

Proposition 2.1 *Suppose (M, g) is locally conformally flat. Define the symmetric two-tensor L by*

$$L = \frac{2}{n}\sigma_2(A_g) - \sigma_1(A)A + A^2. \tag{2.1}$$

Then L satisfies

$$tr_g L = 0, \tag{2.2}$$

$$\delta L = \left(\frac{n-2}{n}\right) d\sigma_2(A). \tag{2.3}$$

Remark Therefore, when $\sigma_2(A_g)$ is constant, L is both trace-free and divergence-free.

For the proof, we will need two additional sharp inequalities involving L:

Proposition 2.2 *Assume $\sigma_2(A) > 0$. Then*

$$(i) \quad - g(L, E) \geq 0, \tag{2.4}$$

with equality if and only if $E = 0$.

$$(ii) \quad |L|^2 \leq \frac{-2(n-2)}{n}\sigma_1(A) g(L, E). \tag{2.5}$$

Proof (i) This follows from [Vi-1, Lemma 23]. If we define the second Newton transformation by

$$T_2(A) = \sigma_2(A)g - \sigma_1(A)A + A^2,$$

Then

$$L = T_2(A) - (trT_2)g = T_2(A) - \frac{(n-2)}{n}\sigma_2(A)g.$$

Thus, as E is trace-free,

$$
\begin{aligned}
-g(L,E) &= -g(L,A)\\
&= -g(T_2(A) - \frac{(n-2)}{n}\sigma_2(A)g, A)\\
&= -g(T_2(A),A) + \frac{(n-2)}{n}\sigma_2(A)\sigma_1(A).
\end{aligned}
$$

Now, according to [Vi,Lemma 23], if $\sigma_2(A) > 0$, and $\sigma_1(A) > 0$

$$
g(T_2(A),A) \leq \frac{(n-2)}{n}\sigma_2(A)\sigma_1(A)
$$

with equality if and only if $E = 0$. This implies (2.4).

(ii) In terms of the trace-free Ricci tensor, we have

$$
\begin{aligned}
A &= E + \frac{1}{n}\sigma_1(A)g,\\
A^2 &= E^2 + \frac{2}{n}\sigma_1(A)E + \frac{1}{n^2}\sigma_1(A)^2 g,\\
L &= -\frac{1}{n}|E|^2 g - \frac{(n-2)}{n}\sigma_1(A)E + E^2.
\end{aligned}
$$

Therefore,

$$
|L|^2 = |E^2|^2 - \frac{1}{n}|E|^4 - \frac{2(n-2)}{n}\sigma_1(A) trE^3 + \frac{(n-2)^2}{n^2}\sigma_1(A)^2|E|^2. \tag{2.6}
$$

Similarly,

$$
-g(L,E) = \frac{(n-2)}{n}\sigma_1(A)|E|^2 - trE^3,
$$

or

$$
trE^3 = g(L,E) + \frac{(n-2)}{n}\sigma_1(A)|E|^2,
$$

where $trE^3 = E_i^k E_k^j E_j^i$. Substituting this into (2.6) gives

$$
|L|^2 = |E^2|^2 - \frac{1}{n}|E|^4 - \frac{(n-2)^2}{n^2}\sigma_1(A)^2|E|^2 - \frac{2(n-2)}{n}\sigma_1(A)g(L,E). \tag{2.7}
$$

Lemma 2.3 *For an $n \times n\,(n \geq 3)$ traceless symmetric matrix E, we have*

$$|E^2|^2 \leq \frac{n^2-3n+3}{n(n-1)}|E|^4 \tag{2.8}$$

and equality holds if and only if E is of the form

$$E = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & -(n-1)\lambda \end{pmatrix}$$

Proof We begin by observing that for $n = 3$, the ratio $\frac{|E^2|^2}{|E|^4}$ is a constant given by $\frac{3^2-3.3+3}{3(3-1)} = \frac{1}{2} \leq \frac{n^2-3n+3}{n(n-1)}$ for $n > 3$.

In general we write, for $\lambda \in \mathbb{R}^{n-1}$,

$$E_\lambda = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & -\sum\limits_1^{n-1} \lambda_k \end{pmatrix}$$

View the function $f(E) = |E^2|^2$ as a smooth function on the hypersurface $\{\lambda \epsilon R^{n-1} || E| = 1\}$. At the maximum value of f, we find a Lagrange multiplier μ:

$$\lambda_i^3 + (\Sigma\lambda_k)^3 = \mu(\lambda_i + \Sigma\lambda_k) \tag{2.9}$$

for each $i = 1,\, 2,\, ...n-1$, where $\Sigma\lambda_k = \sum\limits_1^{n-1} \lambda_k$.

The general case is modeled after the case $n = 4$, which we will first consider in detail. To solve for μ in (3.11) we first assume that

$$\lambda_i + \Sigma\lambda_k \neq 0 \quad \text{for } i = 1, 2, 3. \tag{2.10}$$

Then we find, using the common value of μ,

$$\lambda_i^2 - \lambda_i(\Sigma\lambda_k) + (\Sigma\lambda_k)^2 = \lambda_j^2 - \lambda_j(\Sigma\lambda_k) + (\Sigma\lambda_k)^2 \text{ for } i \neq j.$$

Thus

$$\lambda_1(\lambda_2+\lambda_3)=\lambda_2(\lambda_1+\lambda_3)=\lambda_3(\lambda_1+\lambda_2).$$

If any of the $\lambda_i = 0$, we find $f(\lambda)=\frac{1}{2}$ as in the case $n=3$. When none of λ_i is zero, we find $\lambda_1=\lambda_2=\lambda_3$, $f(\lambda)=\frac{7}{12}$

Returning to the assumption (2.10), if for some i say $i=3$, we have

$$\lambda_3+\Sigma\lambda_k=0,$$

then rewriting (2.9) for $i=1,2$, we find

$$\lambda_i^3-\lambda_3^3=\mu(\lambda_i-\lambda_3).$$

If $\lambda_1=\lambda_3$, we find $\lambda_2=-3\lambda_3$. In either case E is conjugate to the matrix

$$E=\begin{pmatrix}\lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & -3\lambda\end{pmatrix}.$$

If $\lambda_1\neq\lambda_3$ and $\lambda_2\neq\lambda_3$, we find using the common value of μ,

$$\lambda_2(\lambda_2+\lambda_3)=\lambda_1(\lambda_1+\lambda_3)$$

Hence either $\lambda_1=\lambda_2$, so that $\lambda_3=-\lambda_1$ and $f(\lambda)=\frac{1}{4}<\frac{7}{12}$; or $\lambda_1+\lambda_2=-\lambda_3=0$, $f(\lambda)=\frac{1}{2}\leq\frac{7}{12}$. Thus we have determined the maximum and the minimum value of $f(E)$ in dimension four.

For $n>4$, we apply (2.9) to find

$$\lambda_i^3-\mu\lambda_i=\mu(\Sigma\lambda_k)-(\Sigma\lambda_k)^3 \tag{2.11}$$

As the right hand side is independent of i, we conclude that there are at most three values for λ_i, the roots of the cubic equation (2.11). Thus by relabeling if necessary, we find

$$\lambda_1=\lambda_2\ldots=\lambda_{l_1}=\Lambda_1$$

$$\lambda_{l+1}=\ldots=\lambda_{l_1+l_2}=\Lambda_2$$

$$\lambda_{l_1+l_2+1}=\ldots=\lambda_{l_1+l_2+l_3}=\Lambda_3$$

where $l_1+l_2+l_3=n-1$.

Consider once again the function

$$f(E_\Lambda) = l_1\Lambda_1^4 + l_2\Lambda_2^4 + l_3\Lambda_3^4 + (l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3)^4$$

on the hypersurface $\{\Lambda \in \mathbb{R}^3 | \|E_\Lambda\| = 1\}$. At a maximum we have a Lagrange multiplier μ:

$$\Lambda_i^3 + (\Sigma l_k\Lambda_k)^3 = \mu(\Lambda_i + \Sigma l_k\Lambda_k) \quad i = 1, 2, 3.$$

Assuming that

$$\Lambda_i + \Sigma_k\Lambda_k \neq 0 \quad \text{for all} \quad i = 1, 2, 3$$

we find

$$\Lambda_i^2 - \Lambda_i(\Sigma l_k\Lambda_k) = \Lambda_j^2 - \Lambda_j(\Sigma l_k\Lambda_k) \quad \text{for} \quad i \neq j. \tag{2.12}$$

Since the quadratic equation in Λ_i for a given $\Sigma l_k\Lambda_k$ has at most two roots, we conclude that there are only two possibly distinct values for Λ_i. We may without loss of generality assume that

$$\lambda_1 = \ldots = \lambda_{l_1} = \Lambda_1$$

$$\lambda_{l_1+1} = \ldots = \lambda_{l_2} = \Lambda_2 \tag{2.13}$$

and $l_1 + l_2 = n - 1$. To determine the maximum possible value of the function

$$f(E_\lambda) = l_1\Lambda_1^4 + l_2\Lambda_2^4 + (l_1\Lambda_1 + l_2\Lambda_2)^4$$

under the constraints: $l_1 + l_2 = n - 1$

$$l_1\Lambda_1^2 + l_2\Lambda_2^2 + (l_1\Lambda_1 + l_2\Lambda_2)^2 = 1$$

and

$$\Lambda_1^2 - \Lambda_1(l_1\Lambda_1 + l_2\Lambda_2) = \Lambda_2^2 - \Lambda_2(l_1\Lambda_1 + l_2\Lambda_2).$$

We notice that we may enlarge the consideration to allow $l_1 = t$ and $l_2 = n - 1 - t$ to range over $0 \leq t \leq n - 1$ and fix (Λ_1, Λ_2), viewing $f(E_\Lambda)$ as a function of t, we find

$$\begin{aligned}
\frac{d}{dt} \quad f(E_\Lambda, t) &= \Lambda_1^4 - \Lambda_2^4 + 4(t\Lambda_1 + (n-1-t)\Lambda_2)^3(\Lambda_1 - \Lambda_2) \\
&= (\Lambda_1 - \Lambda_2)(\Lambda_1^3 + \Lambda_1^2\Lambda_2 + \Lambda_1\Lambda_2^2 + \Lambda_2^3 + 4(t\Lambda_1 + (n-1-t)\Lambda_2)^3). \\
\frac{d^2}{dt^2} \quad f(E_\Lambda t) &= 12(\Lambda_1 - \Lambda_2)^2(t\Lambda_1 + (n-1-t)\Lambda_2)^2 \geq 0
\end{aligned}$$

Hence the maximum is achieved at $t = 0$ or $t = n - 1$. That is either $l_1 = 0$ or $l_2 = 0$ and $f(E_\Lambda) = \frac{n^2-3n+3}{n(n-2)}$.

Finally we consider the case when

$$\Lambda_i + \Sigma_k \Lambda_k \neq 0$$

fails to hold. There are three possibilities: In case $\Lambda_i + \Sigma_k \Lambda_k = 0 \quad \text{for} \quad i = 1, 2, 3$, we find $\Lambda_1 = \Lambda_2 = \Lambda_3$. In case $\Lambda_1 + \Sigma_k \Lambda_k = 0 \quad \text{for} \quad i = 1, 2$ then we find $\Lambda_1 = \Lambda_2$, so that there are only two common values and the previous consideration shows $f(E_\Lambda) \leq \frac{n^2-3n+3}{n(n-1)}$. In the remaining case, say

$$(1 + l_3)\Lambda_3 = -(l_1\Lambda_1 + l_2\Lambda_2),$$

we find

$$f(E_\Lambda) = l_1\Lambda_1^4 + l_2\Lambda_2^4 + (1 + l_3)\left(\frac{l_1\Lambda_1 + l_2\Lambda_2}{1 + l_3}\right)^4.$$

Again we enlarge the consideration for a fixed l_3, allow l_1 to run between $0 \leq l_1 \leq n - 1 - l_3$, and $l_2 = n - 1 - l_1 - l_3$. Differentiate twice $f(E_\Lambda, l_1)$ with respect to l_1, we find f is again strictly convex in l_1, hence its maximum value is attained at the end points, that is either $l_1 = 0$ or $l_1 = n - 1 - l_3$ hence $l_2 = 0$. Thus we are reduced to our previous consideration when there are at most two dinstinct eigenvalues and the maximum value of $f(E_\Lambda)$ is less than or equal to $\frac{n^2-3n+3}{n(n-20}$ as desired. ■

Using (2.8), from (2.7) we conclude

$$\begin{aligned} |L|^2 \leq & \left[\frac{(n^2 - 3n + 3)}{n(n-1)} - \frac{(n-1)}{n(n-1)}\right]|E|^4 - \frac{(n-2)^2}{n^2}\sigma_1(A)^2|E|^2 \\ & -\frac{2(n-2)}{n}\sigma_1(A)g(L, E) \\ = & \frac{(n-2)^2}{n(n-1)}|E|^4 - \frac{(n-2)^2}{n^2}\sigma_1(A)^2|E|^2 \\ & -\frac{2(n-2)}{n}\sigma_1(A)g(L, E) \\ = & -\frac{(n-2)^2}{n(n-1)}|E|^2\left[-|E|^2 + \frac{(n-1)}{n}\sigma_1(A)^2\right] \\ & -\frac{2(n-2)}{n}\sigma_1(A)g(L, E). \end{aligned}$$

However, $-|E|^2 + \frac{(n-1)}{n}\sigma_1(A)^2 = 2\sigma_2(A)$, so (2.5) follows. ■

3 Estimates for the tail term

In this section, we will establish some technical results which will be used in the proof of the main theorem in section four.

Proposition 3.1 *Let $g = v^{-2}|dx|^2$ be a conformal metric on $\mathbb{R}^n$. Assume that there exists some positive constants $C_0 = C_0(n), C_1 = C_1(n)$, so that*

$$\text{when } n = 4, 5 \qquad \sigma_2(A_g) \geq C_0,$$

and

$$\text{when } n \geq 6 \qquad \begin{cases} \frac{1}{C_1} \leq \sigma_2(Ag) \leq C_1 \\ \int_{A_\rho} v^{-n} \leq C_1 \end{cases}$$

where $A(\rho) = \{x \epsilon \mathbb{R}^n, \rho \leq |x| \leq 2\rho\}$, for all $\rho >> 1$.

Then there is a constant $C_2 = C_2(C_0, C_1, n)$, so that

$$\int_{A_\rho} R|\nabla v|^2 v^{1-n} \leq C_2 \rho^2 \quad \text{for all} \quad \rho >> 1.$$

Proof. Since

$$\sigma_2(A_g) = \sigma_2((n-2)v^{-1}\nabla^2_{ij}v - \frac{(n-2)}{2}v^{-2}|\nabla v|^2\delta_{ij}))$$

we have

$$\sigma_2(A_g) = \frac{1}{2}v^4(n-2)^2\{-v^{-2}|\nabla^2 v|^2 + v^{-2}(\Delta v)^2 - (n-1)v^{-3}\Delta v|\nabla v|^2 + \frac{1}{4}n(n-1)|\nabla v|^4 v^{-4}\}.$$

Using the formula

$$\frac{1}{2}\Delta|\nabla v|^2 = |\nabla^2 v|^2 + < \nabla v, \nabla\Delta v >,$$

we rewrite the above equation in the form:

$$2\sigma_2(A_g)v^{-n} = (n-2)^2 \left\{ -\frac{1}{2}v^{2-n}\Delta|\nabla v|^2 + v^{2-n} < \nabla v, \nabla\Delta v > \right.$$
$$\left. + v^{2-n}(\Delta v)^2 - (n-1)v^{1-n}\Delta v|\nabla v|^2 + \frac{1}{4}n(n-1)v^{-n}|\nabla v|^4 \right\}. \qquad (3.1)$$

Fix $\rho >> 1$ and let η be a cut-off function supported on $B(\frac{5}{2}\rho) - B(\frac{\rho}{2}), \eta \equiv 1$ on $A(\rho) = B(2\rho) - B(\rho)$ and $|\nabla^k \eta| \leq C_k \rho^{-k}$ on $B(\frac{5}{2}\rho)$. We multiply both sides of (3.1) by $\eta^4 v^\alpha$ and integrate over $\mathbb{R}^n$. For the first two terms in the right hand side of (3.1), we integrate by parts and arrive at:

$$
\begin{aligned}
\int \eta^4 v^{2+\alpha-n}\Delta|\nabla v|^2 &= \int |\nabla v|^2 \Delta(\eta^4 v^{2+\alpha-n}) \\
&= \int \{ |\nabla v|^2 \eta^4 \Delta(v^{2+\alpha-n}) + 2|\nabla v|^2 \nabla(v^{2+\alpha-n}) \nabla \eta^4 + |\nabla v|^2 v^{2+\alpha-n} \Delta \eta^4 \} \\
&= \int (2+\alpha-n) v^{1+\alpha-n} \Delta v |\nabla v|^2 \eta^4 + \int (2+\alpha-n)(1+\alpha-n) v^{\alpha-n} |\nabla v|^4 \eta^4 \\
&+ 2(2+\alpha-n) \int v^{1+\alpha-n} \nabla v \cdot \nabla \eta^4 |\nabla v|^2 + \int v^{2+\alpha-n} |\nabla v|^2 \Delta \eta^4
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int \eta^4 v^{2+\alpha-n} < \nabla v, \nabla \Delta v > \\
&= \int \{ -v^{2+\alpha-n} (\Delta v)^2 - (2+\alpha-n) v^{1+\alpha-n} |\nabla v|^2 \Delta v \eta^4 - v^{2+\alpha-n} \nabla v \cdot \nabla \eta^4 \Delta v \}
\end{aligned}
$$

For the last term in the line above, we integrate by parts again to obtain

$$
\begin{aligned}
\int v^{2+\alpha-n} \nabla v \nabla \eta^4 \Delta v &= \int v^{2+\alpha-n} \partial_i v \partial_i \eta^4 \ \partial_k \partial_k v \\
&= -\int \partial_k (v^{2+\alpha-n}) \partial_i v \partial_i \eta^4 \partial_k v - v^{2+\alpha-n} \partial_k \partial_i v \partial_i \eta^4 \partial_k v - v^{2+\alpha-n} \partial_i v \partial_k \partial_i \eta^4 \partial_k v \} \\
&= -(2+\alpha-n) \int v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 - \int v^{2+\alpha-n} \nabla^2 \eta^4 (\nabla v, \nabla v) \\
&- \frac{1}{2} \int v^{2+\alpha-n} < \nabla |\nabla v|^2, \nabla \eta^4 > \\
&= -\frac{1}{2}(2+\alpha-n) \int v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 - \int v^{2+\alpha-n} \nabla^2 \eta^4 (\nabla v, \nabla v) \\
&+ \frac{1}{2} \int v^{2+\alpha-n} |\nabla v|^2 \Delta \eta^4
\end{aligned}
$$

Finally we substitute these identities into the left hand side of (3.1), and obtain

$$
\begin{aligned}
\frac{2}{(n-2)^2} \int \sigma_2(A_g) v^{\alpha-n} \eta^4 &= (-(n-1) - \frac{3}{2}(2+\alpha-n)) \int v^{1+\alpha-n} |\nabla v|^2 (\Delta v) \eta^4 \\
&+ (\frac{1}{4} n(n-1) - \frac{1}{2}(2+\alpha-n)(1+\alpha-n)) \int v^{\alpha-n} |\nabla v|^4 \eta^4 \\
&- \frac{1}{2}(2+\alpha-n) \int v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 + \int v^{2+\alpha-n} \nabla^2 \eta^4 (\nabla v, \nabla v) \\
&- \int v^{2+\alpha-n} |\nabla v|^2 \Delta \eta^4 \qquad (3.2)
\end{aligned}
$$

Recall that when $g = v^{-2}|dx|^2$, the scalar curvature of g is given by (1.8)

$$-\Delta v + \frac{n}{2} v^{-1} |\nabla v|^2 + \frac{1}{2(n-1)} R v^{-1} = 0.$$

Equivalently,

$$\Delta v = \frac{n}{2} v^{-1} |\nabla v|^2 + \frac{1}{2(n-1)} R v^{-1}.$$

Finally, substituting this into (3.2) gives the identity

$$\frac{2}{(n-2)^2} \int \sigma_2(A_g) v^{\alpha-n} \eta^4 = a_{\alpha,n} \int v^{\alpha-n} R |\nabla v|^2 \eta^4 + b_{\alpha,n} \int v^{\alpha-n} |\nabla v|^4 \eta^4 + T_1 + T_2 \qquad (3.3)$$

where

$$\begin{aligned} a_{\alpha,n} &= \frac{1}{4(n-1)}(n-4-3\alpha) \\ b_{\alpha,n} &= n(\frac{1}{4} + \frac{\alpha}{4}) - \frac{1}{2}(2+\alpha)(1+\alpha) \\ T_1 &= -\frac{1}{2}(2+\alpha-n) \int v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 \\ T_2 &= \int v^{2+\alpha-n} \nabla^2 \eta^4 < \nabla v, \nabla v > - \int v^{2+\alpha-n} \Delta \eta^4 |\nabla v|^2. \end{aligned}$$

Thus

$$|T_1| \lesssim \frac{1}{\rho} \int |\nabla v|^3 \eta^3 v^{1+\alpha-n} \lesssim \frac{1}{\rho} \left(\int |\nabla v|^4 v^{\alpha-n} \eta^4 \right)^{\frac{3}{4}} \left(\int_{supp\ \eta} v^{4+\alpha-n} \right)^{\frac{1}{4}} \qquad (3.4)$$

$$|T_2| \lesssim \frac{1}{\rho^2} \int |\nabla v|^2 v^{2+\alpha-n} \eta^2 \lesssim \frac{1}{\rho^2} \left(\int |\nabla v|^4 v^{\alpha-n} \eta^4 \right)^{\frac{1}{2}} \left(\int_{supp\ \eta} v^{4+\alpha-n} \right)^{\frac{1}{2}} \qquad (3.5)$$

We make the following choice of α according to the dimension n of the manifold.

(a) When $n = 4$ or 5, we choose $\alpha = 1$. Notice that for $\alpha = 1$, $a_{1,n} = \frac{n-7}{4(n-1)}$, $b_{1,n} = \frac{1}{2}(n-6)$. Thus for both $n = 4, 5$, $\quad a_{1,n} < 0, \quad b_{1,n} < 0$.

n=4 case: Under the assumption $\sigma_2(Ag) \geq C_0 > 0$, we have $R_g \geq \sqrt{24}\sqrt{C_0} > 0$; thus $v(x) \lesssim \rho^2$ for $x \in$ supp η, and ρ large by Lemma 1.3. Thus

$$|T_1| \lesssim \frac{1}{\rho} \left(\int |\nabla v|^4 v^{-3} \eta^4 \right)^{\frac{3}{4}} \left(\int_{supp\ \eta} v \right)^{\frac{1}{4}}$$

$$\lesssim \rho^{\frac{1}{2}} \left(\int |\nabla v|^4 v^{-3} \eta^4 \right)^{\frac{3}{4}}$$

$$\lesssim \epsilon \int |\nabla v|^4 v^{-3} \eta^4 + C_\epsilon \rho^2.$$

Also

$$|T_2| \lesssim \rho \left(\int |\nabla v|^4 v^{-3} \eta^4 \right)^{\frac{1}{2}}$$

$$\lesssim \epsilon \int |\nabla v|^4 v^{-3} \eta^4 + C_\epsilon \rho^2$$

for any $\epsilon > 0$ and some constant $C_\epsilon = C(\epsilon)$. Thus if we choose $\epsilon \leq \frac{1}{4}(-b_{1,4})$, we obtain from (3.3) that

$$\int R |\nabla v|^2 v^{-3} \eta^4 \lesssim \rho^2$$

Hence

$$\int_{A(\rho)} R |\nabla v|^2 v^{-3} \lesssim \rho^2.$$

n=5 case: We argue in the same way as in the case of $n = 4$; except that

$$|T_1| \lesssim \frac{1}{\rho} \left(\int |\nabla v|^4 v^{-4} \eta^4 \right)^{\frac{3}{4}} \rho^{\frac{5}{4}}$$

$$\lesssim \epsilon \int |\nabla v|^4 v^{-4} \eta^4 + C_\epsilon \rho$$

Similarly

$$|T_2| \leq \frac{1}{\rho^2} \left(\int |\nabla v|^4 v^{-4} \eta^4 \right)^{\frac{1}{2}} \rho^{\frac{5}{2}} \leq \epsilon \int |\nabla v|^4 v^{-4} \eta^4 + C_\epsilon \rho.$$

Thus

$$\int_{A(\rho)} R |\nabla v|^2 v^{-4} \lesssim \rho \leq \rho^2 \text{ when } \rho \geq 1.$$

(b) $n \geq 6$ **cases.** *Here we choose $\alpha = 0$; notice that for $n > 5$*

$$a_{0,n} = \frac{n-4}{4(n-1)} > 0, \ b_{0,n} = \frac{1}{4}(n-4) > 0.$$

Thus for $\sigma_2(A_g) \leq C_0$, we have from (3.3)

$$\frac{2C_0}{(n-2)^2}\int v^{-n}\eta^4 \geq a_{0,n}\int R|\nabla v|^2 v^{-n}\eta^4 + b_{0,n}\int |\nabla v|^4 v^{-n}\eta^4 - |T_1| - |T_2|, \tag{3.6}$$

while

$$\begin{aligned} |T_1| &\leq \frac{1}{\rho}\left(\int |\nabla v|^4 v^{-n}\eta^4\right)^{\frac{3}{4}}\left(\int_{supp\ \eta} v^{4-n}\right)^{\frac{1}{4}} \\ &\lesssim \frac{1}{\rho}\left(\int |\nabla v|^4 v^{-n}\eta^4\right)^{\frac{3}{4}}\left(\int_{supp\ \eta} v^{-n}\right)^{\frac{n-4}{4n}}(\rho^n)^{\frac{1}{n}} \end{aligned}$$

Thus under the additional assumption that $\int_{A(\rho)} v^{-n} \leq C_1$, for C_1 independent of ρ, we get

$$|T_1| \lesssim (\int |\nabla v|^4 v^{-n}\eta^4)^{\frac{3}{4}}$$

$$|T_2| \lesssim (\int |\nabla v|^4 v^{-n}\eta^4)^{\frac{1}{2}}.$$

Thus it follows from (3.6) that

$$\int R|\nabla v|^2 v^{-n}\eta^4 \leq C_2$$

for some $C_2 = C(C_0, C_1, n)$. We apply Lemma 1.3, to conclude

$$\int_{A(\rho)} R|\nabla v|^2 v^{1-n} \leq \int R|\nabla v|^2 v^{1-n}\eta^4 < C_2\rho^2.$$

Thus we have finished the proof of the proposition.

As a consequence of the computation in the proof of the above proposition, we have:

Corollary 3.2 *When $n = 4$, and $g = v^{-2}ds^2$ is a conformal metric on $\mathbb{R}^4$ with $\sigma_2(A_g) \geq C_0 > 0$, then $vol(g) = \int_{\mathbb{R}^4} v^{-4}dx \lesssim \frac{1}{C_0}$.*

Proof. Fix $\rho > 0$ and take η to be a cut off function with $\eta \equiv 1$ on B and $\eta \equiv 0$ off $B_{2\rho}$. Let $\alpha \equiv 0$ in the formula (3.3). We observe that $a_{0,4} = b_{0,4} = 0$.

Hence

$$\frac{1}{2}\int \sigma_2(A_g)v^{-4}\eta^4 = T_1 + T_2, \tag{3.7}$$

where

$$T_1 = \int v^{-3}|\nabla v|^2 \nabla v \cdot \nabla \eta^4$$

$$T_2 = \int v^{-2}\nabla^2\eta^4(\nabla v, \nabla v) - \int v^{-2}\Delta\eta^4|\nabla v|^2.$$

We first observe that for (3.3) we have when ($n = 4$)

$$\frac{1}{6}Rv^{-2} = v^{-1}\Delta v - 2v^{-2}|\nabla v|^2,$$

hence for $R = R_g \gtrsim \sqrt{C_0} > 0$, we have

$$2\int v^{-2}|\nabla v|^2\eta^2 \le \int v^{-1}\Delta v\eta^2 \le \int |\nabla v|^2 v^{-2}\eta^2 + \int |\nabla v||v^{-1}||\nabla\eta^2|.$$

Thus,

$$\int v^{-2}|\nabla v|^2\eta^2 \le \left(\int |\nabla v|^2\eta^2 v^{-2}\right)^{\frac{1}{2}} \times \frac{1}{\rho} \times \rho^2,$$

and hence $\int v^{-2}|\nabla v|^2\eta^2 \lesssim \rho^2$. We conclude

$$|T_2| \lesssim \frac{1}{\rho^2}\int v^{-2}|\nabla v|^2\eta^2 \lesssim \text{ constant.} \tag{3.8}$$

We now claim that $T_1 \le 0$. To see this we integrate by parts to rewrite T_1 as

$$T_1 = -\int(\Delta v|\nabla v|^2 + 2\nabla^2 v(\nabla v, \nabla v))v^{-3}\eta^4 + 3\int |\nabla v|^4 v^{-3}\eta^4. \tag{3.9}$$

We observe that when $n = 4$, $g = v^{-2}ds^2$, the components of the Ricci tensor $Ric(g)$ are given by

$$R_{ij} = 2v^{-1}\nabla_i\nabla_j v + v^{-1}\Delta v\delta_{ij} - 3v^{-2}|\nabla v|^2\delta_{ij}.$$

Moreover, since $\sigma_2(A_g) > 0$ on $\mathbb{R}^n$, we have $R > 0$ (e.g. CGY-2, Lemma 3.5) it follows that $R_{ij} > 0$ (see [CGY1, Lemma 1.2]). Therefore, rewriting (3.9) in terms of Ric we see that

$$T_1 = \int -Ric(\nabla v, \nabla v)v^{-2}\eta^4 \le 0. \tag{3.10}$$

Combine (3.7), (3.8) and (3.10) we have

$$\int_{B_\rho} v^{-4}dx \le \int v^{-4}\eta^4 \le \frac{1}{C_0}.$$

Letting $\rho \to \infty$, we obtain the volume bound of $\int_{\mathbb{R}^4} v^{-4}$ as claimed.

Remark. In the case $n \ge 5$, the result of above corollary does not hold. That is, there exist metrics $g = v^{-2}|dx|^2$ with $R_g > 0$ and $\sigma_2(g) \ge C_0 > 0$, while $vol(g)$ is not uniformly bounded. For the cylindrical metric on $S^{n-1} \times \mathbb{R}$, i.e. $v(x) = |x|$, we have $\sigma_2(A_g) = \frac{1}{8}(n-1)(n-2)^2(n-4)$. Thus, if perturb $v(x)$ by $v_\epsilon(x) = (|x|^2 + \epsilon^2)^{\frac{1}{2}}$, we have for $g_\epsilon = v_\epsilon^{-2}ds^2$ that $\sigma_2(A_{g_\epsilon}) \ge C_n > 0$ while $vol(g_\epsilon) \to +\infty$ as $\epsilon \to 0$ for $n \ge 5$.

4 Classifying the entire solutions

In this section we prove our main result Theorem 0.1. That is; we provide a classification of all conformal metrics $g = v^{-2}|dx|^2$ on $\mathbb{R}^n$ which satisfies the equation

$$\sigma_2(A_g) = \frac{1}{8}n(n-1)(n-2)^2,$$

when $n = 4$ or 5, or when $n \geq 6$ under the additional assumption that the volume of g is finite. More precisely, all such solutions are obtained by pulling back the round metric on the sphere (and its images under the conformal group) by stereographic projection. In dimension $n \geq 5$ we get the same conclusion provided we assume in addition that the volume is bounded:

$$\sigma_2(A_g) = \frac{1}{8}n(n-1)(n-2)^2,$$

$$vol(g) = \int_{\mathbb{R}^n} v^{-n}dx < \infty.$$

Proof To begin, fix $\rho > 1$ and let η denote a cut-off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\partial_i \eta| \lesssim \rho^{-1}$. As outlined in section 1 above, we write the formula for the trace-free Ricci tensor E of g in terms of v as in (1.1):

$$E = -(n-2)v\nabla_g^2(v^{-1}) + \frac{(n-2)}{n}v\Delta_g(v^{-1})g.$$

Notice that in (1.1), that Hessian and Laplacian are with respect to g, not the Euclidean metric. Next we pair both sides with $v^{-1}\eta^2 L$ to get

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) = \int (n-2)g(L, \nabla_g^2(v^{-1}))dvol(g). \tag{4.1}$$

Note that in (4.1) we have used the fact that L is trace-free. Applying the divergence theorem we find

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) = \int -(n-2)g(\delta L, d(v^{-1}))\eta^2 dvol(g)$$
$$- \int (n-2)L(\nabla_g(v^{-1}), \nabla_g(\eta^2))dvol(g).$$

Since $\sigma_2(A_g)$ is constant, (2.3) implies that L is divergence-free. Thus

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) = \int -(n-2)L(\nabla_g(v^{-1}), \nabla_g(\eta^2))dvol(g)$$

$$\lesssim \int |L||\nabla_g(v^{-1})||\nabla_g(\eta^2)|dvol(g)$$

$$\lesssim \int |L||\nabla_g v||\nabla_g \eta|v^{-2}\eta dvol(g).$$

Using inequality (2.5) we conclude

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) \lesssim \int R^{\frac{1}{2}}|g(L,E)|^{\frac{1}{2}}|\nabla_g v||\nabla_g \eta|v^{-2}\eta dvol(g).$$

By the Schwartz inequality,

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) \lesssim \left(\int\limits_{supp|\nabla\eta|} |g(L,E)|v^{-1}\eta^2 dvol(g)\right)^{\frac{1}{2}}$$

$$\times \left(\int R|\nabla_g v|^2|\nabla_g \eta|^2 v^{-3} dvol(g)\right)^{\frac{1}{2}}. \tag{4.2}$$

By inequality (2,4), $-g(L,E) \geq 0$. Also, *supp* $|\nabla\eta| \subseteq$ *supp* η, so (4.2) implies

$$0 \leq \int -g(L,E)v^{-1}\eta^2 dvol(g) \lesssim \int R|\nabla_g v|^2|\nabla_g \eta|^2 v^{-3} dvol(g). \tag{4.3}$$

We now rewrite the integral on the Rhs of (4.3) in terms of the Euclidean metric, using the identities $|\nabla_g v|^2 = v^2|\nabla v|^2$,$|\nabla_g \eta|^2 = v^2|\nabla\eta|^2$ and $dvol(g) = v^{-n}dx$ as in the proof of Theorem 1.1, we get

$$\int R|\nabla_g v|^2|\nabla_g \eta|^2 v^{-3} dvol(g) = \int R|\nabla v|^2|\nabla\eta|^2 v^{1-n} dx.$$

Since $|\nabla\eta|^2 \lesssim p^{-2}$ and *supp* $\eta \subset A(\rho) = \{x \in \mathbb{R}^n \,|\rho < |x| < 2\rho\}$, we conclude

$$\int R|\nabla_g v|^2|\nabla_g \eta|^2 v^{-3} dvol(g) \lesssim \rho^{-2}\int R|\nabla v|^2 v^{1-n} dx, \tag{4.4}$$

and this will suffice: By Proposition 3.1 the Rhs of (4.4) is bounded independent of ρ. This implies via (4.3) that

$$\int_{\mathbb{R}^n} -g(L,E)v^{-1}dvol(g) < \infty.$$

In particular,

$$\int_{supp|\nabla\eta|} -g(L,E)v^{-1}dvol(g) = \int_{supp|\nabla\eta|} |g(L,E)|v^{-1}dvol(g) \to 0 \tag{4.6}$$

as $\rho \to \infty$. Now combining (4.6) with (4.3) and the boundedness of the integrals in (4.5), we conclude that

$$\int -g(L,E)v^{-1}\eta^2 dvol(g) \to 0 \quad \text{as} \quad \rho \to \infty,$$

so $-g(L,E) \equiv 0$ on $\mathbb{R}^n$. By (2.4) this implies $E \equiv 0$. The conclusion of Theorem 0.1 follows.

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