Extremal Functions for a Mean Field Equation in Two Dimension

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Abstract

Let Ω be a bounded piecewise C^2 simply-connected domain. In this article, we give necessary and sufficient conditions for the existence of maximizer of

$$J_{8\pi}(\phi) = \log\left(\fint_{\Omega} e^{\phi} dx\right) - \frac{1}{16\pi}\int_{\Omega} |\bigtriangledown \phi|^2 dx$$

for $\phi \in H_0^1(\Omega)$. We prove among other things that the maximizer exists provided that the regular part $\gamma(x)$ of the Green function for the Dirichlet problem has more than one maximum points in Ω.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^2$ and ρ be a constant of real numbers. In this paper, we consider the following nonlinear elliptic equation

$$\text{(1.1)} \qquad \begin{cases} \Delta u + \rho \frac{e^u}{\int_{\Omega} e^u dx} = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$ stands for the Laplacian operator of $\mathbf{R}^2$. Throughout the paper, the domain Ω is assumed to be piecewisely C^2. A domain Ω is called picewisely C^2 if its boundary is C^2 except at a finite vertex points $\{Q_1, \ldots, Q_N\}$ such that two conditions holds at each vertex Q_j.

(i) At each Q_j, the inner angle θ_j of $\partial\Omega$ at Q_j satisfies $0 < \theta_j \neq \pi < 2\pi$,

(ii) At each Q_j, there is a 1-1 conformal map form $B_{\delta_0}(Q_j) \cap \bar{\Omega}$ to the complex plane $\mathbf{C}$ such that the portion $\partial\Omega \cap B_{\delta_0}(Q_j)$ of the boundary is mapped onto a C^2 curve.

Clearly, non-smooth domains such as n-polygons satisfies conditions (i) and (ii) above. Hence, our theory developed in this article can be applied to n-polygons. See examples in section 6.

Equation (1.1) often appears in many different disciplines of mathematics. Recently, it has been derived in the context of statistical mechanics from the mean field limit of the Gibbs measure associated to vorticity fields of Euler flows, as studied by Caglioti, Lions, Marchioro and Pulvirenti [5, 6], and also by Kiessling [18], and Chanillo and Kiessling [8]. A related problem also appears in the self-dual condensate solutions of Chern-Simons Higgs model of superconductivity. See Ding, Jost, Li and Wang [15, 17], Nolasco and Tarantello [25, 26] and references therein. In many examples of application, one is interested in the case when $\rho = 8\pi$.

Associated with (1.1) is the nonlinear functional J_ρ:

$$J_\rho(\phi) = \log\left(⨍_\Omega e^\phi dx\right) - \frac{1}{2\rho}\int_\Omega |\nabla \phi|^2 dx$$

for $\phi \in H_0^1(\Omega)$. It is well-known that $J_\rho(\phi)$ is bounded above for $\phi \in H_0^1(\Omega)$ if and only if $\rho \leq 8\pi$. For $\rho < 8\pi$, the supreman of J_ρ is always attained by virtue of the Moser-Trudinger inequality [23]. However, for $\rho = 8\pi$, the existence of extremal functions, which attains the supreman of $J_{8\pi}$, is a more difficult problem and depends on the geometry of Ω in a subtle way. It was noted in [5, 6, 7] that when Ω is a ball, the supreman of $J_{8\pi}$ is never attained by a function ϕ in $H_0^1(\Omega)$. On the other hand, the extremal functions exists for a long and thin rectangle Ω. As far as the authors know, there are very few known results concerning either the existence of solutions of (1.1) with $\rho = 8\pi$ or the existence of the extremal functions of $J_{8\pi}$. Even now, the following basic question is not yet answered. For the simplicity, a domain Ω is called *type C* if the supreman of $J_{8\pi}$ can be attained.

[Q]. The domains of type C is open in the C^1-topology.

In this paper, we will give an affirmative answer to [Q] at least for simply-connected domains.

For any bounded smooth domain Ω, we set

$$I_{8\pi}(\Omega) = \sup_{\phi \in H_0^1(\Omega)} J_{8\pi}(\phi). \tag{1.2}$$

In the case when the domain is not of type C, we can compute $J_{8\pi}(\Omega)$ explicitly, as done in [5, 6]. We denote $G(x, y)$ to be the Green function and $\tilde{G}(x, y)$ to be the regular part of G:

$$\tilde{G}(x, y) = G(x, y) + \frac{1}{2\pi} \log |x - y|. \tag{1.3}$$

Set

$$\gamma(x) = \tilde{G}(x, x). \tag{1.4}$$

When there are no confusions, $\gamma(x)$ is sometimes called the regular part of the Green function. Now suppose Ω is not of type C. Then in [5], $I_{8\pi}(\Omega)$ was computed and its value is equal to

$$I_{8\pi}(\Omega) = 1 + 4\pi \sup_{\bar{\Omega}} \gamma(x) + \log \frac{|B_1|}{|\Omega|}. \tag{1.5}$$

where B_1 is the unit ball and $|\Omega|$ stands for the area of Ω. See Theorem 7.1 of [5] for a proof of (1.5). Surprisingly, the converse holds.

Theorem 1.1. *Let Ω be a bounded piecewise C^2 simply-connected domain. Then Ω is of type C if and only if*

$$I_{8\pi}(\Omega) > 1 + 4\pi \sup_{\bar{\Omega}} \gamma(x) + \log \frac{|B_1|}{|\Omega|}. \tag{1.6}$$

Readily, Theorem 1.1 give a positive answer to [Q] for simply-connected domains.

Corollary 1.2. *The simply-connected domains of type C is open in the C^1 topology.*

We note that the "if" part of Theorem 1.1 has already been proved in [5]. The crucial step of our proof of the "only if" part is to study the behavior of a sequence of blowup solutions u_k of (1.1) with $\rho = \rho_k$, where ρ_k satisfies

$$\lim_{k \to +\infty} \rho_k = 8\pi. \tag{1.7}$$

Under the condition (1.7), the set of blowup points of u_k consists of only one point p, and the location of p can be determined apriori by

$$\nabla \gamma(p) = 0. \tag{1.8}$$

See Nagasaki and Suzuki [24], Li [19], Chen and Lin [10, 12] and the references therein. It is a fundamentally important question how to determine the sign of $\rho_k - 8\pi$. In [11], the second and third authors have studied the question for the general case of multi-bubble solutions. For the case of single blowup point, the asymptotic formulas in [11] can be reduced to

$$\rho_k - 8\pi = c\Big[(\Delta \log h(p)) + o(1)\Big]\varepsilon_k \log \frac{1}{\varepsilon_k} \tag{1.9}$$

for some positive constant $c > 0$, where

$$\varepsilon_k = \rho_k \left(\int_\Omega h(x) e^{u_k(x)} dx \right)^{-1}, \tag{1.10}$$

and u_k is a solution of

$$\begin{cases} \Delta u_k + \rho_k \frac{h(x)e^{u_k(x)}}{\int h(x)e^{u_k(x)}dx} = 0 \ \text{ in } \ \Omega, \\ u_k = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{1.11}$$

If the function $\log h$ is a harmonic function at p, then the formulas (1.9) does not yield any information for the sign of $\rho_k - 8\pi$. In this case, we should refine our previous works in order to answer the questions here.

Theorem 1.3. *Let Ω be a bounded piecewise C^2 domain in $\mathbf{R}^2$ and $h(x)$ be a C^2 positive function in $\bar{\Omega}$. Suppose $\log h(x)$ is a harmonic function in Ω and u_k is a sequence of blowup solutions of (1.11) with $\lim_{h\to+\infty} \rho_k = 8\pi$. Then*

$$\rho_k - 8\pi = \frac{8}{\pi h(p)} \left(\int_\Omega \frac{H(y,0)}{|y-p|^4} dy - \int_{\Omega^c} \frac{dy}{|y-p|^4} + o(1) \right) \varepsilon_k, \tag{1.12}$$

where p is the blowup point, $\Omega^c = \mathbf{R}^2 \backslash \Omega$ and $H(y,0)$ is defined by

$$H(y,0) = \frac{h(y)}{h(p)} e^{8\pi(\tilde{G}(y,p) - \gamma(p))} - 1 \ \text{ for } \ y \in \Omega. \tag{1.13}$$

In many cases, it is convenient to work with the formulas (1.12). However, for some situations such as when the deformation of domains is required, it is suitable to work through the conformal mapping of the unit ball onto Ω while Ω is a simply connected domain. We explain it in the followings.

Let $z = x_1+ix_2$ denote the complex variable associated with the point $x = (x_1, x_2)$, and $z = f(w)$ be the conformal map from the unit ball $\{w \mid |w| < 1\}$ onto Ω. Let $w = g(z)$ stands for the inverse function of f. Obviously, the Green function $G(x, p) = -\frac{1}{2\pi} \log |g(z)|$ and $\tilde{G}(x, p) = -\frac{1}{2\pi} \log(\frac{|g(z)|}{|z-p|})$. Hence, $\gamma(p)$ can be expressed in term of f by

$$\gamma(p) = \tilde{G}(p, p) = \frac{1}{2\pi} \log |f'(0)| = \frac{1}{2\pi} \log |a_1|, \tag{1.14}$$

where $f(w)$ is written as

$$f(w) = f(0) + \sum_{n=1}^{\infty} a_n w^n. \tag{1.15}$$

Furthermore, due to the symmetry of $\tilde{G}(x, y) = \tilde{G}(y, x)$, condition (1.8) is equivalent to $\bigtriangledown_x \tilde{G}(x, p) = 0$ at $x = p$. Let $\frac{\partial}{\partial z}$ denote the derivative with respect to the complex variable z. Then at $x = p$,

$$\begin{aligned} \frac{\partial}{\partial z}\tilde{G}(z,p)\Big|_{z=p} &= -\frac{1}{4\pi}\frac{\partial}{\partial z}\log\left(\frac{|g(z)|^2}{|z-p|^2}\right)\Bigg|_{z=p} \\ &= -\frac{1}{4\pi}\left\{\frac{g'(z)}{g(z)} - \frac{1}{z-p}\right\}\Bigg|_{z=p} \\ &= -\frac{1}{4\pi}\left\{\frac{1}{wf'(w)} - \frac{1}{f(w)-p}\right\}\Bigg|_{w=0} \\ &= \frac{a_2}{4\pi a_1^2}. \end{aligned}$$

Hence, (1.8) is equivalent to $a_2 = 0$. Hence, for any critical point p of γ, we set

$$D(p) = \sum_{n=3}^{\infty} \frac{n^2}{n-2}|a_n|^2 - |a_1|^2. \tag{1.16}$$

Obviously, $D(p)$ is well-defined because $a_2 = 0$. Since the area $|\Omega|$ of $\Omega = \pi \sum_{n=1}^{\infty} |a_n|^2 n$, the series of (1.16) is absolutely convergent. By using the conformal map f, the asymptotic formulas (1.12) can be simplified to

Theorem 1.4. *Suppose f and Ω be described as above and u_k is a sequence of blowup solution of (1.1) and (1.7). Let p be the blowup point of u_k. Then*

$$\rho_k - 8\pi = \pi(D(p) + o(1))\varepsilon_k, \tag{1.17}$$

where ε_k and $D(p)$ are given in (1.10) and (1.16).

We note that when Ω is smooth, the existence of solutions with single one blowup point has been obtained by Weston and Moseley in 70s. In [33] and [21, 22], Weston and Moseley apply the Liouville theorem for (1.1) to construct approximation solutions, where the error of solutions and its approximations can be estimated within the range ε_k^l for some large l. Then they use the implicit function theorem to obtain the solution with sufficiently small ε_k. For solutions with multi-bubbles, we refer the recent works Barakat and Pacard [3], Chen and Lin [12] and the references therein. However, in order to construct solutions successfully, Weston and Moseley have to assume Ω to satisfy a nondegenerate condition at p, that is, p is a nondegenerate critical point of $\gamma(x)$. In terms of f, the nondegenerate condition is equivalent to

$$\frac{|a_3|}{|a_1|} \neq \frac{1}{3}. \tag{1.18}$$

Under the condition (1.18), Theorem 1.4 has been proved by Suzuki and Nagasaki [31] for those *special* solutions u_k obtained by Weston and Moseley, because the estimates of the errors of solutions and its approximation is used in their proof. Our theorem holds for any blowup solutions in any simply-connected domain. Our proof of Theorem 1.3 are based on the estimates of the error term of solutions and its approximation bubble, which has been done for the general case in [11]. To see how $D(p)$ is related to our problem, we employ the conformal map to construct a test function. Set $v_\varepsilon(w) = 2\log(\frac{1+\varepsilon}{\varepsilon+|w|^2})$ for $|w| \leq 1$ and $u_\varepsilon(z) = v_\varepsilon(g(z))$ for $z \in \Omega$, where g is the inverse function of f. Then by a straightforward computation (see section 5 for details of computation), one has

$$J_{8\pi}(u_\varepsilon) = 1 + 4\pi\gamma(p) + \log\frac{|B_1|}{|\Omega|} + |a_1|^{-2}D(p)\varepsilon + O(|\varepsilon|^2). \tag{1.19}$$

Clearly, if $D(p) > 0$ for some maximum point p of γ, then $I_{8\pi}(\Omega) > 1 + 4\pi \max_{\bar{\Omega}} \gamma + \log \frac{|B_1|}{|\Omega|}$. Hence by the "if" part of Theorem 1.1, we conclude that Ω is of type C. In this paper, we will prove that the converse holds also.

Theorem 1.5. *A bounded piecewise C^2 simply-connected domain Ω is of type C if and only if*

$$D(p) > 0 \tag{1.20}$$

for some maximum point p of γ.

We use two deep results to prove Theorem 1.5. The first one is Theorem 1.3 or Theorem 1.4, that is, the asymptotic formulas (1.12) and (1.17). The other is the uniqueness theorem for solutions to equation (1.1) with $\rho \leq 8\pi$ and for a simply connected domain Ω. By employing Bol's isoperimetric inequality, T. Suzuki [29] beautifully proved the uniqueness theorem for $\rho < 8\pi$. Together with Theorem 1.3, this uniqueness theorem for $\rho < 8\pi$ heuristically establishes that if $D(p) < 0$ for all maximum points of γ, then Ω is not of type C. Because if $D(p) < 0$ for a maximum point p of γ, then by (1.18), p is a nondegenerate critical point of γ. Thus, there exists a sequence of solution u_k of (1.1) with $\rho = \rho_k$, which blows up at p. The existence of such a sequence of solutions was due to Weston [33]. By Theorem 1.4, we have $\rho_k < 8\pi$. Now, the uniqueness theorem of Suzuki yields that u_k is the maximizer of J_{ρ_k}. Since the maximizer u_k blows up at p as $\rho_k \uparrow 8\pi$, it is plausible to guess that the maximizer of $J_{8\pi}$ does not exist, that is, Ω is not of type C. In fact, this is a corollary of the following stronger result.

Theorem 1.6. *Suppose that Ω is a bounded piecewise C^2 simple-connected domain, then equation (1.1) possesses at most one solution for each $\rho \leq 8\pi$. Furthermore, the linearized equation of (1.1) with $\rho \leq 8\pi$ at the unique solution has a positive first eigenvalue.*

As we noted before, Theorem 1.6 was proved by Suzuki [29] for $\rho < 8\pi$. For this paper, the uniqueness theorem for $\rho = 8\pi$ plays an even more important role. In fact, by the argument above, Theorem 1.6 for $\rho = 8\pi$ implies that Ω is not of type C if $D(p) < 0$ for all maximum point p of γ. It is easy to see that domains such as balls and cubes belong to this class. However, for domains with $D(p) = 0$ for some maximum point p of γ, the above argument to prove that Ω is not type C simply fails, because Theorem 1.3 does not yield any information of sign of $\rho_k - 8\pi$ in this case. Surprisingly, the following theorem show that the existence of critical point p of γ with $D(p) \leq 0$ set a strong restriction on the "geometry" of Ω.

Theorem 1.7 *Suppose that Ω is bounded piecewise C^2 simply-connected domain and p is a critical point of γ with $D(p) \leq 0$. Then p is the unique maximum point of γ. Furthermore, the unique solution $u_\rho(x)$ of (1.1) with $\rho < 8\pi$ must blow up at p as $\rho \uparrow 8\pi$.*

Together with Theorem 1.5, we have

Theorem 1.8. *Suppose that Ω is a bounded piecewise C^2 simply-connected domain and $\gamma(x)$ has more then one maximum point. Then Ω is of type C.*

The paper is organized as follows. In order to prove (1.5), we have to show apriori that for a sequence of blowup solution, any blowup point must be inside of the domain Ω. This has been done for C^2 domain. This holds also for a peiecewisely C^2 domain. In section 2, we give a grief account. In section 3, we first prove Theorem 1.3. Here we have to use estimates in [11] for the error term η_k of the solution u_k and its approximation. By using this estimate, we will derive the asympototic formulas (1.12). This asymptotic formulas is more useful when we come to computation for concrete examples. By using the conformal map f, (1.17) can be derived easily from (1.12). As we note, Theorem 1.6, the uniqueness theorem, plays an important role in our paper. We present its proof in section 4. Here, we employ the symmetrization and the Bol inequality to reduce our problem to the case of radial functions. We treat the radial case in a more elementary and direct way than in [29], where the uniqueness theorem for $\rho < 8\pi$ was proved in smooth domains. Our argument becomes lengthy when non-smooth domains are considered. However, we want to include its proof here because the uniqueness theorem for non-smooth domains has its own interest. As we mentioned, the asymptotic formulas (1.17) is employed in the argument when the deformation of domains is required. This is the case for the proof of Theorem 1.7, whose proof are given in section 5. In section 6, we first give a proof of Theorem 1.5 and by using Theorem 1.5, we prove Theorem 1.1. In the final section, we present several examples to discuss the type of domains, as applications of our theorems in this paper.

Ackowledgement. Part of this work has been done while the second author visited National Center for Theoretical Sciences of NSC in Taiwan. He would like to thank NCTS for the warm hospitality and the stimulating environment.

2 Non-smooth domains

Since in section 6, we will consider non-smooth domains such as n-polygons. We will give a brief account concerning the regularity of solutions of (1.1) in our setting. First, we note that in order to derive (1.5) as done in the proof of Theorem 7.1 in [5], we have to prove the blowup point could not occur on the boundary. This is known when Ω is a C^2 domain. For a piecewise C^2 domain, this holds true as shown in the followings.

Lemma 2.1. *Let $u \in H_0^1(\Omega)$ be a weak solution of (1.1). Then $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$. Furthermore if u_k is a sequence of blowup solutions of (1.1), then all the blowup points are located inside of Ω.*

Proof. By the Moser-Trudinger inequality, $e^{u(x)} \in L^p(\bar{\Omega})$ for any $p > 1$. Thus, $u(x) \in C^\infty(\Omega)$ follows readily from the standard regularity theorems of elliptic equation. As for the boundary regularity, u is $C^{1,\alpha}$ locally near $x_0 \in \partial\Omega$ for any $\alpha < 1$ provided $\partial\Omega$ has a C^2 regularity at x_0. Now suppose x_0 is one of vertex $\{Q_1, \ldots, Q_N\}$, say, $x_0 = 0 \in \Gamma_1 \cap \Gamma_2$ and θ is the inner angle of Γ_1 and Γ_2 at 0, where Γ_j is the C^2 connected component of $\partial\Omega$. Let $w = g(z)$ be the conformal map which maps $\Gamma_1 \cup \Gamma_2$ to be a C^2 curve near $g(0)$. Set $g(0) = 0$ and $z = f(w)$ be to be the inverse map of g. Then $v(w) = u(f(w))$ satisfies

$$(2.1) \qquad \begin{cases} \Delta v + |f'(w)|^2 e^v = 0 \text{ in } \tilde{\Omega}, \\ v(w) = 0 \text{ for } w \in \partial'\tilde{\Omega} = g(\Gamma_1 \cup \Gamma_2), \end{cases}$$

where $\tilde{\Omega} = g(B_{\delta_0}(0) \cap \Omega)$. Clearly, $|f(w)| = |w|^{\frac{\theta}{\pi}}(1 + o(1))$ with $o(1) \to 0$ as $|w| \to 0$. Hence, if $\theta > \pi$, then $|f'(w)|^2$ is a Hölder function of exponent α_0 for some $\alpha_0 > 0$. By the regularity theorem, $v \in C^{1,\alpha}$ near 0 for any $o < \alpha < 1$. In particular, $u(x)$ is continuous at 0.

If $0 < \theta < \pi$, then $|f'(w)|^2 e^v \in L^{p_0}$ for some $p_0 > 1$ due to the Moser-Trudinger inequality. Since $\partial\tilde{\Omega}$ is C^2 at 0, $v \in W^{2,p_0}$ locally near 0. By the Sobolev embedding, v is Hölder locally near 0. Hence the continuity of u at each vortex is proved.

Now suppose the exists a sequence of solutions u_k of (1.1) with $\rho_k \to \rho$. It suffices for us to show that $u_k(x)$ can not blow up at any vertex point Q. If the angle θ at Q is less than π, then the domain Ω is converx in a neighborhood of 0. Applying the method of moving planes to Ω, the blowup points must be away from Q. If $\theta > \pi$, then we can use the conformal map $f(w) = w^{\frac{\theta}{\theta_1}}$, $Q = 0$ and $f(0) = 0$ is assumed where $\theta_1 < \pi$. Clearly, $f(w)$ map a curve C onto $\Gamma_1 \cup \Gamma_2$ and the inner angle of C at 0 is equal to θ_1. Let $\tilde{\Omega}$ be the domain such that $f(\tilde{\Omega}) = \Omega \cap B_{\delta_0}(0)$. Then $\tilde{\Omega}$ is convex at 0, and $v(w) = u(f(w))$ satisfies

$$\begin{cases} \Delta v + |w|^{2(\frac{\theta}{\theta_1}-1)} e^v = 0 \ \text{ in } \ \tilde{\Omega}, \\ v = 0 \ \text{ on } \ \partial'\tilde{\Omega} = g(\Gamma_1 \cup \Gamma_2), \end{cases}$$

where $g = f^{-1}$ is the inverse function. Let $w = (w_1, w_2)$. By a rotation, we may assume the axis $w_1 = 0$ intersects with $\partial'\tilde{\Omega}$ non-tangentially and $\tilde{\Omega} \subset \{w \mid w_2 < 0\}$. Since $\frac{\theta}{\theta_1} - 1 > 0$, $|w|^{2(\frac{\theta}{\theta_1}-1)}$ is increasing in $|w_2|$ for $w \in \tilde{\Omega}$, Thus, we can apply the method of moving plane to show $v(w) > v(w^\lambda)$ for $w \in \tilde{\Omega}$, $w_2 < \lambda < 0$ and w^λ is the reflection point of w with respect to $w_2 = \lambda$, provided that $\tilde{\Omega} \cap \{w \mid w_2 = \lambda\}$ is non-empty. Then it implies that the blowup points must be away from Q. This completes the proof of Lemma 2.1. Q.E.D.

Corollary 2.2. *Suppose that Ω is a bounded piecewise C^2 domain. If Ω is not type C, then*

$$I_{8\pi}(\Omega) = 1 + 4\pi \sup_{\bar{\Omega}} \gamma(x) + \log \frac{|B_1|}{|\Omega|}. \tag{2.2}$$

Proof. This immediately follows from the original proof of (2.2) for the smooth domains and Lemma 2.1, which states the blowup point must be inside of Ω. We skip the proof here and refer the detail of the proof to Theorem 7.1 of [5]. Q.E.D.

3 Asymptotic expansion of $\rho_k - 8\pi$

In this section, we are going to prove the asymptotic formulas (1.12) for a piecewisely C^2 doamin Ω. Since all the estimates of solutions are local in nature, for the simplicity, we may assume our domain Ω is C^2 throughout this section. Let Ω be a C^2 bounded domain and $h(x)$ be a positive C^2 function defined in $\bar{\Omega}$. Consider a sequence of blowup solution u_k of (1.11) such that

$$\lim_{k\to+\infty} \rho_k = 8\pi. \tag{3.1}$$

Set

$$\begin{cases} \lambda_k = u_k(p_k) = \max_{\bar{\Omega}} u_k(x) \to +\infty \quad \text{as} \quad k \to +\infty, \\ p = \lim_{k\to+\infty} p_k. \end{cases} \tag{3.2}$$

To describe the behavior of u_k, we collect some well-known facts in the following lemmas.

Lemma 3.1. *By passing to a subsequence, solutions $u_k(x)$ converges to the Green function $G(x,p)$ in $C^2_{loc}(\bar{\Omega}\backslash\{p\})$. Furthermore, the blowup point p satisfies*

$$\bigtriangledown \left(\log h(x) + 8\pi\tilde{G}(x,p)\right) = 0 \quad \text{at} \quad x = p. \tag{3.3}$$

Lemma 3.1 was proved by Nagasaki and Suzuki [24] and the identity (3.3) can be derived from the Pohozaev identity. Since the proofs are standard and the results are well-known now, we skip the proofs here. To describe the bubbling behavior of u_k near p, we have to quote a result due to Li [19]. We denote $\tilde{u}_k$ by

$$\tilde{u}_k = u_k(x) - d_k, d_k = \log\left(\int_\Omega h(x)e^{u_k}dx\right). \tag{3.4}$$

Then $\tilde{u}_k(x)$ satisfies

$$\begin{cases} \Delta\tilde{u}_k + \rho_k h(x)e^{\tilde{u}_k(x)} = 0 \quad \text{in} \quad \Omega, \\ \tilde{u}_k(x) = -d_k \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

Since $u_k(x)$ uniformly converges to $G(x,p)$ for $|x-p_k| = \delta_0$ for any fixed small $\delta_0 > 0$, the difference $|\tilde{u}_k(x) - \tilde{u}_k(y)|$ is uniformly bounded for any two points x, y on $\partial B_{\delta_0}(p_k)$. Thus, we can apply Li's theorem to $\tilde{u}_k$. Hereafter, $B_{\delta_0}(p)$ denote the ball of center p and radius δ_0. In [19], Li proved

Lemma 3.2. *There exists constants c and δ_0 such that*

$$|\tilde{u}_k(x) - \tilde{\lambda}_k + 2\log(1 + e_k e^{\tilde{\lambda}_k}|x - p_k|^2)| \le c \tag{3.6}$$

for $|x - p_k| \le \delta_0$, where $\tilde{\lambda}_k = \max_\Omega \tilde{u}_k = \lambda_k - d_k$ and $e_k = \frac{\rho_k h(p_k)}{8}$.

For $|x - p_k| = \delta_0$, (3.6) implies

$$|\tilde{u}_k(x) + \tilde{\lambda}_k| \le c. \tag{3.7}$$

Since $u_k(x)$ uniformly converges to $G(x, p)$ for $|x - p_k| \ge \delta_0$, we have $|\tilde{u}_k(x) + d_k| = |u_k(x)| \le c_1$ for $|x - p_k| \ge \delta_0$. Thus by (3.7),

$$|d_k - \tilde{\lambda}_k| \le c_1 \quad \text{and} \quad |\lambda_k - 2\tilde{\lambda}_k| \le c_1. \tag{3.8}$$

As in [11], the estimate (3.6) is not enough for us to derive (1.12). Therefore, we have to estimate the profile of u_k more precisely in $B_{\delta_0}(p)$. In order to achieve it, we introduce the error term η_k by

$$\eta_k(x) = \tilde{u}_k(x) - U_k(x) - \rho_k(\tilde{G}_k(x, p_k) - \gamma(p_k)), \tag{3.9}$$

for $x \in \bar{\Omega}$, where

$$U_k(x) = \tilde{\lambda}_k - 2\log\left[1 + e_k e^{\tilde{\lambda}_k}|x - q_k|^2\right], \tag{3.10}$$

and q_k satisfies

$$\bigtriangledown U_k(p_k) = \bigtriangledown \log h(p_k). \tag{3.11}$$

Note that $\bigtriangledown U_k(x) = \frac{-4e_k e^{\tilde{\lambda}_k}(x - q_k)}{1 + e_k e^{\tilde{\lambda}_k}|x - q_k|^2}$. From it, q_k satisfies

$$|p_k - q_k| \le c\, e^{-\tilde{\lambda}_k}. \tag{3.12}$$

Clearly from (3.3), we expect that $\bigtriangledown \eta_k(p_k) = -\bigtriangledown U_k(p_k) - \rho_k \bigtriangledown \tilde{G}(p_k, p_k) = -[\bigtriangledown \log h(p_k) + \rho_k \bigtriangledown \tilde{G}(p_k, p_k)]$ should be small. In fact, we have

$$|\bigtriangledown \eta_k(p_k)| = O(\tilde{\lambda}_k e^{-\tilde{\lambda}_k}). \tag{3.13}$$

See Lemma 5.4 of [11]. Recall that $U_k(x)$ is an entire solution of

$$\Delta U_k + \rho_k h(p_k) e^{U_k(x)} = 0 \quad \text{in} \quad \mathbf{R}^2.$$

For the rest of this section, we denote $G_k(x) = G(x, p_k)$ and $\tilde{G}_k(x) = \tilde{G}(x, p_k)$ to simplify our notation. First, we derive the equation for η_k.

Lemma 3.3. *The error term η_k satisfies*

$$\begin{cases} \Delta\eta_k(y) + \rho_k h(p_k) e^{U_k(y)} H_k(y, \eta_k) = 0 & \text{for} \ \ y \in \bar{\Omega}, \\ \eta_k(y) = O(\tilde{\lambda}_k e^{-\tilde{\lambda}_k}) & \text{on} \ \ \partial\bar{\Omega}, \end{cases} \tag{3.14}$$

where

$$H_k(y, t) = \frac{h(y)}{h(p_k)} e^{t + \rho_k(\tilde{G}_k(y) - \tilde{G}_k(p_k))} - 1.$$

Proof. (3.14) is already proved in [11]. See Theorem 1.4 of [11]. For the convenience of readers, we give a scatch of the proof. Since $\tilde{G}_k(x)$ is harmonic, (3.9) yields

$$\begin{aligned} \Delta\eta_k &= \rho_k h(p_k) e^{U_k(x)} - \rho_k h(x) e^{\tilde{u}_k} \\ &= -\rho_k h(p_k) e^{U_k(x)} \left\{ \frac{h(x)}{h(p_k)} e^{\tilde{u}_k - U_k(x)} - 1 \right\} \\ &= -\rho_k h(p_k) e^{U_k(x)} \left\{ \frac{h(x)}{h(p_k)} e^{\eta_k + \rho_k(\tilde{G}_k(x) - \tilde{G}_k(p_k))} - 1 \right\} \end{aligned}$$

by applying $\tilde{u}_k - U_k(x) = \eta_k + \rho_k(\tilde{G}_k(x) - \tilde{G}_k(p_k))$.

For $y \in \partial\Omega$, we have

$$\begin{aligned} \eta_k(y) &= -d_k - U_k(y) + \frac{\rho_k}{2\pi} \log|y - p_k| + \rho_k \tilde{G}_k(p_k) \\ &= -d_k + \rho_k \tilde{G}_k(p_k) + \tilde{\lambda}_k + 2\log\left(\frac{\rho_k h(p_k)}{8}\right) + \frac{\rho_k}{2\pi} \log\left(\frac{|y - p_k|}{|y - q_k|}\right) \\ &\quad + 2\log\left(1 + \frac{8}{\rho_k h(p_k)} |y - q_k|^{-2} e^{-\tilde{\lambda}_k}\right) + O(|\rho_k - 8\pi|) \\ &= -d_k + \rho_k \tilde{G}_k(p_k) + \tilde{\lambda}_k + 2\log\left(\frac{\rho_k h(p_k)}{8}\right) + O(\tilde{\lambda}_k e^{-\tilde{\lambda}_k}). \end{aligned}$$

Here, we use (3.12) and $|\rho_k - 8\pi| = O(\tilde{\lambda}_k e^{-\tilde{\lambda}_k})$. For the proof of the latter, see Theorem 1.1 of [11]. In [11], we denote

$$I_k = -d_k + \tilde{\lambda}_k + \rho_k \tilde{G}_k(p_k) + 2\log\left(\frac{\rho_k h(p_k)}{8}\right).$$

By using Theorem 7.2 in [11],

$$|I_k| \le c\ \tilde{\lambda}_k e^{-\tilde{\lambda}_k}. \tag{3.15}$$

Thus, $\eta_k(y) = O(\tilde{\lambda}_k e^{-\tilde{\lambda}_k})$ for $y \in \partial\Omega$. Q.E.D.

The estimate (3.15) yields the following.

Lemma 3.4. $e^{-\tilde{\lambda}_k} = \pi^2 h^2(p) e^{8\pi\gamma(p)} e^{-d_k}[1 + o(1)]$.

Proof of Theorem 1.3. By the definition of $H_k(x,t)$, we have

$$\rho_k h(x) e^{\tilde{u}_k(x)} = \rho_k h(p_k) e^{U_k(x)} + \rho_k h(p_k) e^{U_k(x)} H_k(x, \eta_k(x)) \tag{3.16}$$

for $x \in \Omega$. Substituting (3.16) into (1.11),

$$\begin{aligned} \rho_k &= \int_\Omega \rho_k h e^{\tilde{u}_k(x)} dx \\ &= \int_\Omega \rho_k h(p_k) e^{U_k(x)} dx + \int_\Omega \rho_k h(p_k) e^{U_k(y)} H_k(y, \eta_k) dy \\ &= 8\pi - \int_{\Omega^c} \rho_k h(p_k) e^{U_k(y)} dy + \int_\Omega \rho_k h(p_k) e^{U_k(y)} H_k(y, \eta_k) dy. \end{aligned} \tag{3.17}$$

Clearly

$$\begin{aligned} &\int_{\Omega^c} \rho_h h(p_k) e^{U_k(y)} dy \\ =\ & \left(\frac{\rho_k h(p_k)}{e_k^2} \int_{\Omega^c} \frac{dy}{|y - p_k|^4} + o(1)\right) e^{-\tilde{\lambda}_k} \\ =\ & \left(\frac{8}{\pi h(p)} \int_\Omega \frac{dy}{|y - p|^4} + o(1)\right) e^{-\tilde{\lambda}_k}. \end{aligned} \tag{3.18}$$

The integrand of the last term of (3.17) can be expressed, by the Taylor expansion, as

$$
\begin{aligned}
(3.19) \qquad & \rho_k h(p_k)e^{U_k(y)}H_k(y,\eta_k) \\
= \ & \rho_k h(p_k)e^{U_k(y)}H_k(y,0) + \rho_k h(p_k)e^{U_k(y)}(H_k(y,0)+1)\eta_k \\
& +\rho_k h(p_k)e^{U_k(y)}O(|\eta_k|^2).
\end{aligned}
$$

Except for the first item of (3.19), heuristically the integration of all other terms in (3.19) should bounded by $o(\varepsilon_k)$. However, this $o(\varepsilon_k)$-estimate of the integral of $\rho_k h(p_k)e^{U_k(y)}\eta_k$ is not obvious at all. In fact, its smallness is due to the effect of cancellation. This effect of cancellation can be made clear only when η_k is explicitly solved in the order of ε_k. To avoid the complexity necessary for this delicate estimates, we apply an useful trick from [11]. Set

$$
(3.20) \qquad \psi(x) = \frac{1-a|x-q_k|^2}{1+a|x-q_k|^2},
$$

where $a = \frac{\rho_k h(p_k)}{8}e^{\tilde{\lambda}_k}$. The function $\psi(x)$ is chosen as a comparison function because it satisfies

$$
(3.21) \qquad \Delta\psi(x) + \rho_k h(p_k)e^{U_k(y)}\psi = 0 \ \text{ in } \ \mathbf{R}^2.
$$

Since η_k satisfies

$$
\Delta\eta_k + \rho_k h(p_k)e^{U_k(y)}H_k(y,\eta_k) = 0 \ \text{ in } \ \Omega.
$$

by Lemma 3.3, we have

$$
\begin{aligned}
(3.22) \qquad & \int_\Omega \rho_k h(p_k)e^{U_k}H_k(y,\eta_k)dy \ = \ -\int_{\partial\Omega}\frac{\partial\eta_k}{\partial\nu}d\sigma \\
= \ & \int_{\partial\Omega}\left(\psi\frac{\partial\eta_k}{\partial\nu} - \eta_k\frac{\partial\psi}{\partial\nu}\right)d\sigma + \int_{\partial\Omega}\left(\eta_k\frac{\partial\psi}{\partial\nu} - (1+\psi)\frac{\partial\eta_k}{\partial\nu}\right)d\sigma.
\end{aligned}
$$

Applying (3.22) and (3.21) together, the Green Theorem yields

$$
\int_{\partial\Omega}\left(\psi\frac{\partial\eta_k}{\partial\nu} - \eta_k\frac{\partial\psi}{\partial\nu}\right)d\sigma = -\int_\Omega \rho_k h(p_k)e^{U_k(y)}(H_k(y,\eta_k)-\eta_k)\psi dy.
$$

Note that

$$
\psi = -1 + \frac{2}{1+a|x-q_k|^2}.
$$

$$\int_\Omega \rho_k h(p_k) e^{U_k(y)} H_k(y,0)(-\psi)dy$$
$$= \int_\Omega \rho_k h(p_k) e^{U_k(y)} H_k(y,0)dy - 2\int_\Omega \frac{\rho_k h(p_k) e^{U_k(y)}}{(1+a|x-q_k|^2)} H_k(y,0)dy.$$

By Lemma 3.3,

$$\begin{aligned}
&h(p_k)H_k(y,0) \\
= \ & \exp\{\log(h) + \rho_k(\tilde{G}_k(y) - \tilde{G}_k(p_k))\} - h(p_k) \\
= \ & \sum_{j=1}^{2} a_j(y_j - p_{k,j}) + \frac{B_{11}}{2}(y_1 - p_{k,1})^2 + \frac{B_{22}}{2}(y_2 - p_{k,2})^2 \\
& + \frac{1}{2}\left(\sum_{j=1}^{2} a_j(y_j - p_{k,j})\right)^2 + O(|y-p_k|^3),
\end{aligned} \tag{3.23}$$

where $a = (a_1, a_2) = \bigtriangledown[\log h(y) + \rho_k \tilde{G}_k(y)]$ at $y = p_k$, and $\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$ is the Hessian of $\log h + \rho_k \tilde{G}_k$ at $y = p_k$, after orthogonal transformation of the coordinate. By (3.13), we have

$$|a| = |\bigtriangledown \log h(y) + \rho_k \tilde{G}_k(y)| \le \tilde{\lambda}_k e^{-\tilde{\lambda}_k}$$

for $y = p_k$. Since $|p_k - q_k| = O(e^{-\tilde{\lambda}_k})$ by (3.12), (3.23) yields

$$\begin{aligned}
h(p_k)H_k(y,0) = \ & \sum_{j=1}^{2} b_j(y_j - q_{k,j}) + \frac{1}{2}\sum_{j=1}^{2} B_{jj}(y_j - q_{k,j})^2 \\
& + O(\tilde{\lambda}_k e^{-2\tilde{\lambda}_k} + \tilde{\lambda}_k^2 e^{-2\tilde{\lambda}_k}|y-q_k|^2 + |y-q_k|^3).
\end{aligned} \tag{3.24}$$

Note
$$B_{11} + B_{22} = 0.$$
because $\log h$ is harmonic. Recall that $U_k(y)$ is radially symmetric with respect to q_k. Thus,

$$\rho_k \int_{B_{\delta(q_k)}} e^{U_k(y)} \left\{ \sum_{j=1}^{2} b_j(y_j - q_{k,j}) + \frac{1}{2}\sum_{j=1}^{2} B_{jj}(y_j - q_{k,j})^2 \right\} dy = 0.$$

Therefore, by (3.24) we have

$$
\begin{aligned}
(3.25)\qquad & \left|\rho_k \int_{B_\delta(q_k)} h(p_k) e^{U_k(y)} H_k(y,0)\,dy\right| \\
\le\ & c\left\{\int_{B_{\delta(q_k)}} e^{U_k(y)} |y-q_k|^3 dy + \tilde\lambda_k e^{-2\tilde\lambda_k}\right\} \\
\le\ & c\left\{\delta e^{-\tilde\lambda_k} + \tilde\lambda_k e^{-2\tilde\lambda_k}\right\},
\end{aligned}
$$

where c is a constant independent of δ and k.

For any arbitrary small $\delta > 0$,

$$
\begin{aligned}
& \rho_k \int_{\Omega\setminus B_\delta(q_k)} h(p_k) e^{U_k(y)} H_k(y,0)\,dy \\
=\ & \rho_k e_k^{-2} e^{-\tilde\lambda_k} h(p_k) \int_{\Omega\setminus B_\delta(q_k)} \frac{H_k(y,0)}{|y-q_k|^4} dy + O(1) e^{-2\tilde\lambda_k} \int_{\Omega\setminus B_\delta(q_k)} \frac{H_k(y,0)}{|y-q_k|^6} dy \\
=\ & \frac{8}{\pi h(p)} e^{-\tilde\lambda_k}\left(\int_{\Omega\setminus B_\delta(p)} \frac{H(y,0)}{|y-p|^4} dy + o(1)\right),
\end{aligned}
$$

because

$$|e^{U_k(y)} - e_k^{-2}|y-q_k|^{-4} e^{-\tilde\lambda_k}| \le c\,|y-q_k|^{-6} e^{-2\tilde\lambda_k}$$

for $|y-q_k| >> e^{-\frac{1}{2}\tilde\lambda_k}$. With (3.25), it yields

$$
(3.26)\quad \rho_k \int_\Omega h(p_k) e^{U_k(y)} H_k(y,0)\,dy = \frac{8}{\pi h(p)} e^{-\tilde\lambda_k}\left(\int_\Omega \frac{H(y,0)}{|y-p|^4} dy + o(1)\right),
$$

where

$$\int_\Omega \frac{H(y,0)}{|y-p|^4} dy := \lim_{\delta\downarrow 0} \int_{\Omega\setminus B_\delta(p)} \frac{H(y,0)}{|y-p|^4} dy.$$

By the scaling $y' = e^{\frac{\tilde\lambda_k}{2}} y$, it is easy to obtain

$$
(3.27)\qquad \left|\int_\Omega \frac{e^{U_k(y)}}{1+a|y-q_k|^2} H_k(y,0)\,dy\right| = O(e^{-\frac{3}{2}\tilde\lambda_k})
$$

by (3.24). By Lemma 3.3, $|\eta_k(y)| + |\bigtriangledown \eta_k(y)| = O(\tilde\lambda_k e^{-\lambda_k})$ for $y \in \partial\Omega$. Hence

$$
(3.28)\qquad \int_{\partial\Omega}\left(\eta_k \frac{\partial\psi}{\partial\nu} - (1+\psi)\frac{\partial\eta_k}{\partial\nu}\right) d\sigma = O(\tilde\lambda_k e^{-2\tilde\lambda_k}),
$$

and (1.12) follows readily from (3.18), (3.26) – (3.28). This proves Theorem 1.3. Q.E.D.

Proof of Theorem 1.4. To derive (1.17), we use a conformal mapping $z = f(w)$ from the unit ball $B_1 \to \Omega$ such that

$$f(0) = p.$$

Since p is a critical point of γ, we have

$$f''(0) = 0.$$

To simplify our notation, we denote the function $u_k(f(w))$ by $u_k(w)$. Then u_k satisfies

$$\begin{cases} \Delta u_k + \rho_k \dfrac{h(w)e^{u_k}}{\displaystyle\int_{B_1} h(w)e^{u_k} dw} = 0 \ \text{ in } \ B_1, \\ u_k = 0 \ \text{ on } \ \partial B_1, \end{cases}$$

where $h(w) = |f'(w)|^2$ and dw stands for the volume form in B_1. By the choice of f, the blowup point of u_k is the origin and $\tilde{G}_k(x) \to 0$ uniformly in $\bar{B}_1$. Thus

(3.29) $h(0)H(w,0) = |f'(w)|^2 - |f'(0)|^2, \ \text{ and } \ h(0) = |f'(0)|^2 = |a_1|^2.$

Let $f(w) = f(0) + \sum_{n=1}^{\infty} a_n w^n$. Then (3.29) implies

$$\begin{aligned} h(0)\int_{B_1} \frac{H(w,0)}{|w|^4} dw &= 2\pi \int_0^1 r^{-3} \sum_{n=3}^{\infty} |a_n|^2 n^2 r^{2(n-1)} dr \\ &= 2\pi \sum_{n=3} |a_n|^2 n^2 \int_0^1 r^{2n-5} dr \\ &= \pi \sum_{n=3}^{\infty} \frac{n^2}{n-2} |a_n|^2. \end{aligned} \tag{3.30}$$

On the other hand,

$$\int_{B_1^c} \frac{dy}{|y|^4} = 2\pi \int_1^{\infty} \frac{dr}{r^3} = \pi.$$

Therefore, we have by (1.12)

$$\rho_k - 8\pi = \frac{8}{h^2(0)} e^{-\tilde{\lambda}_k} \left(\sum_{n=3}^{\infty} \frac{n^2}{n-2} |a_n|^2 - |a_1|^2 \right). \tag{3.31}$$

By Lemma 3.4, we have

$$e^{-d_k} = \frac{1}{h^2(0)\pi^2} e^{-\tilde{\lambda}_k}(1 + o(1)),$$

and

$$\varepsilon_k = \rho_k e^{-d_k} = \frac{8}{\pi h(0)^2} e^{-\tilde{\lambda}_k}. \tag{3.32}$$

Substituting (3.32) into (3.31), we have (1.17). Q.E.D.

4 Uniqueness Theorem

The main purpose of this section is to prove Theorem 1.6. First, we start with the beautiful theorem of uniqueness of solution (1.1) with $\rho < 8\pi$, which was due to T. Suzuki [29].

Theorem 4.1. *Suppose that Ω is a bounded piecewise C^2 simply-connected domain. Then for each $\rho \in [0, 8\pi)$, there exists an unique solution for (1.1).*

By applying the classical Bol inequality, Suzuki [29] proved Theorem 4.1 for smooth domains by virture of a special technique of symmetrization, which was initiated by Bandle [2] first. Without employing Bol's inequality, this method of symmetrization has been successfully applied to other problems related to equation (1.1) in recent years, e.g., see [9], [13]. Our proof of Theorem 1.6 also employs the method of symmetrization, which could allow the original problem to be reduced to the radial case. We treat the radial case in more elementary and straightforward way than in [29] for $\rho < 8\pi$. However, our complete proof becomes lengthy because non-smooth domains are considered here. We would like to give a proof for non-smooth domain, because it has its own interest. We begin with the following isoperimetric inequality, which was due to Bandle [2] when ω is a simply-connected domain.

Lemma 4.2. *Suppose v satisfies $\Delta v + e^v = 0$ in Ω and satisfies*

$$\int_\Omega e^{v(x)} dx \leq 8\pi, \tag{4.1}$$

where Ω is a simply-connected domain. Set

$$m(\omega) = \int_\omega e^v dx \quad \text{and} \quad l(\partial\omega) = \int_{\partial\omega} e^{\frac{v}{2}} d\sigma$$

for any $\omega \subset\subset \Omega$. Then the inequality holds

$$2l^2(\partial\omega) \geq [(8\pi - m(\omega)]m(\omega). \tag{4.2}$$

Furthermore, if ω is not simply-connected, then the inequality (4.2) is always strict.

Proof. When ω is simply-connected, (4.2) was proved by Bandle [2]. For the general case, the proof can be given by induction on the number of "holes" of ω. For the simplicity, we assume $\partial\omega = \Gamma_1 + \Gamma_2$ and Γ_2 bounds another simply connected domain $\tilde{\Omega}$. Then by (4.2) for simply-connected domains, we have

$$\begin{aligned}
2l^2(\partial\omega) &= 2[l(\Gamma_1) + l(\Gamma_2)]^2 \\
&= 2l^2(\Gamma_1) + 2l^2(\Gamma_2) + 4l(\Gamma_1)l(\Gamma_2) \\
&> [8\pi - m(\omega \cup \tilde{\Omega})]m(\omega \cup \tilde{\Omega}) + [8\pi - m(\tilde{\Omega})]m(\tilde{\Omega}) \\
&= [8\pi - m(\omega) - m(\tilde{\Omega})][m(\omega) + m(\tilde{\Omega})] + [8\pi - m(\tilde{\Omega})]m(\tilde{\Omega}) \\
&= [8\pi - m(\omega)]m(\omega) + 2m(\tilde{\Omega})[8\pi - m(\tilde{\Omega}) - m(\omega)] \\
&\geq [8\pi - m(\omega)]m(\omega),
\end{aligned}$$

because $m(\tilde{\Omega}) + m(\omega) \leq \int_\Omega e^v dx \leq 8\pi$. Q.E.D.

Let Ω be a piecewisely C^2 simply-connected domain, and $\partial\Omega = \bigcup_{l=1}^N \bar{\Gamma}_l$, where Γ_l is a connected component of C^2 portion of the boundary $\partial\Omega$.

Lemma 4.3. *Let $\mu_2(\Omega)$ be the second eigenvalue of $\Delta + e^v$ for the Dirichlet problem. Then*

$$\mu_2(\Omega) > 0,$$

provided that $\int_\Omega e^v dx \leq 8\pi$.

Proof. We prove Lemma 4.3 by contradiction. Suppose $\mu_2(\Omega) \leq 0$. Then there exists a positive constant $K \leq 1$ and a second eigenfunciton φ satisfying

$$\begin{cases} \Delta\varphi + Ke^v\varphi = 0 \quad \text{in} \quad \Omega, \\ \varphi = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.3}$$

Before continuing to give a proof, we want to make a remark concerning the continuity of φ. It is natural that φ is assumed to be in $H_0^1(\Omega)$. By composing with a conformal map as done in Lemma 2.1, we can prove $\varphi \in C^\infty(\Omega) \cap C(\bar{\Omega})$. Since it is standard, we skip the proof here.

Let $\Omega^+ = \{x \in \Omega \mid \varphi(x) > 0\}$. Set $U(x) = -2\log(1 + \frac{1}{8}|x|^2)$ which satisfies $\Delta U + e^U = 0$ in $\mathbf{R}^2$. Without loss of generality, we may assume

$$\int_{\Omega^+} e^{v(x)} dx \le 4\pi, \tag{4.4}$$

because the total measure $\int_\Omega e^{v(x)} dx \le 8\pi$. For any $t > 0$, the set $\Omega_t = \{x \in \Omega | \varphi(x) > t\}$ is compactly contained in Ω because φ is continuous on $\bar{\Omega}$. We set $r(t) \ge 0$ such that

$$\int_{B_{r(t)}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx, \tag{4.5}$$

where $B_{r(t)}$ is the open ball of center 0 and radius $r(t)$. Note that by the regularity of φ, for any $t < s < \max_{\bar{\Omega}} \varphi$, the set $\{x \mid t \le \varphi(x) < s\}$ has a positive measure. Therefore $r(t)$ is strictly decreasing in t and is continuous in t for $t \in (0, \max_{\bar{\Omega}} \varphi)$. Denote $\varphi^*(r)$ to be the symmetrization of φ with respect to the measures $e^{U(x)}dx$ and $e^{v(x)}dx$ respectively, that is,

$$\varphi^*(r) = \sup\{t \mid r < r(t)\}. \tag{4.6}$$

Then $\varphi^*(x)$ has the same distribution of φ with respect to measures $e^{U(x)}dx$ and $e^{v(x)}dx$ respectively, that is, for $t > 0$,

$$\int_{\{\varphi^* > t\}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx. \tag{4.7}$$

Therefore,

$$\int_{B_{R_0}} e^{U(x)} (\varphi^*)^2 dx = \int_{\Omega^+} e^{v(x)} \varphi^2 dx \tag{4.8}$$

holds where $R_0 = r(0+) = \lim_{t \downarrow 0} r(t)$. To derive a contradiction, we use the coarea formulas

$$\begin{aligned} -\frac{d}{dt} \int_{\Omega^t} |\bigtriangledown \varphi|^2 dx &= \int_{\partial\Omega_t} |\bigtriangledown \varphi| ds, \quad \text{and} \\ -\frac{d}{dt} \int_{\Omega_t} e^{v(x)} dx &= \int_{\partial\Omega_t} \frac{e^v}{|\bigtriangledown \varphi|} ds \end{aligned} \tag{4.9}$$

hold for almost everywhere t. Since $\int_\Omega e^v dx \le 8\pi$, we have by the co-area formulas (4.9) and by Lemma 4.2,

$$
\begin{aligned}
(4.10)\qquad -\frac{d}{dt}\int_{\Omega_t} |\bigtriangledown \varphi|^2 dx \ &= \ \int_{\{\varphi=t\}} |\bigtriangledown \varphi| ds \\
&\ge \left(\int_{\{\varphi=t\}} e^{\frac{v}{2}} ds\right)^2 \left(\int_{\{\varphi=t\}} \frac{e^v}{|\bigtriangledown \varphi|} ds\right)^{-1} \\
&= -\left[\frac{d}{dt}\left(\int_{\Omega_t} e^{v(x)} dx\right)\right]^{-1} l^2(\{\varphi=t\}) \\
&\ge \frac{1}{2}\left(8\pi - \int_{\Omega_t} e^{v(t)} dx\right)\left(\int_{\Omega_t} e^{v(x)} dx\right)\left[-\frac{d}{dt}\int_{\Omega_t} e^{v(x)} dx\right]^{-1} \\
&= \frac{1}{2}\left(8\pi - \int_{\Omega_t^*} e^{U(x)} dx\right)\left(\int_{\Omega_t^*} e^{U(x)} dx\right)\left[-\frac{d}{dt}\int_{\Omega_t^*} e^{U(x)} dx\right]^{-1},
\end{aligned}
$$

for almost everywhere t where $\Omega_t^* = B_{r(t)} = \{x \mid \varphi^*(x) > t\}$. Since $U(r)$ is a radial function, (4.2) becomes an equality for any ball $B_{r(t)}$. Thus, the coarea formulas (4.9) yields

$$
\begin{aligned}
(4.11)\qquad & -\frac{d}{dt}\left(\int_{\Omega_t^*} |\bigtriangledown \varphi^*|^2 dx\right) \\
&= \frac{1}{2}\left(8\pi - \int_{\Omega_t^*} e^{U(x)} dx\right)\left(\int_{\Omega_t^t} e^{U(x)} dx\right)\left[-\frac{d}{dt}\int_{\Omega_t^*} e^{U(x)} dx\right]^{-1}.
\end{aligned}
$$

Together with (4.10), (4.11) implies

$$
(4.12)\qquad -\frac{d}{dt}\int_{\Omega_t} |\bigtriangledown \varphi|^2 dx \ge -\frac{d}{dt}\int_{\Omega_t^*} |\bigtriangledown \varphi^*|^2 dx
$$

for almost everywhere t. By integrating (4.12) with respect to t, we obtain then

$$
(4.13)\qquad \int_{B_{R_0}} |\bigtriangledown \varphi^*|^2 dx \le \int_{\Omega_+} |\bigtriangledown \varphi|^2 dx.
$$

Together with (4.8) and (4.13), the equation (4.3) yields

$$
\begin{aligned}
(4.14)\qquad 0 \ &\ge \ (K-1)\int_{\Omega^+} e^{v(x)} \varphi^2(x) dx \\
&= \int_{\Omega^+} |\bigtriangledown \varphi|^2 dx - \int_{\Omega^+} e^{v(x)} \varphi^2(x) dx \\
&\ge \int_{B_{R_0}} |\bigtriangledown \varphi^*|^2 dx - \int_{B_{R_0}} e^{U(x)} |\varphi^*|^2(x) dx.
\end{aligned}
$$

Therefore, the first eigenvalue λ_1 of $\Delta + e^U$ on B_{R_0} is nonpositive.

Since $U(x) = U(|x|)$ is radial, the first eigenfunction of $\Delta + e^{U(r)}$ is also radial. By a straightforward computation, the function $\psi(r) = \frac{8-r^2}{8+r^2}$ satisfies

$$\Delta\psi(r) + e^{U(r)}\psi(r) = 0 \quad \text{in} \quad \mathbf{R}^2. \tag{4.15}$$

Therefore, the nonpositive first eigenvalue λ_1 of $\Delta + e^U$ implies $R_0 \geq \sqrt{8}$. On the other hand, since

$$\int_{B_{R_0}} e^{U(x)} dx = \int_{\Omega^+} e^{v(x)} dx \leq 4\pi,$$

we deduce $R_0 = \sqrt{8}$, $\lambda_1 = 0$ and

$$\int_{B_{R_0}} e^U dx = 4\pi. \tag{4.16}$$

Consequently, the inequality of (4.12) and then all the inequalities of (4.10), turn out to be equalities. Particularly, by Lemma 4.2, Ω_t is simply connected for almost every t. Due to the regularity of φ, Ω_t and then Ω_+ are all simply-connected domains for all $t > 0$. Similarly, the Schwartz inequality on each level set $\{\varphi = t\}$ becomes an equality and it implies

$$e^{v(x)} = \Phi^+(\varphi(x))|\bigtriangledown\varphi(x)|^2 \tag{4.17}$$

holds for almost everywhere $t = \varphi(x)$ and for some function Φ^+ of t.

By (4.16), we have

$$\int_{\Omega^-} e^{v(x)} dx \leq 4\pi,$$

where $\Omega^- = \{x \mid \varphi(x) < 0\}$. By the same argument of symmetrization, we have

$$\int_{\Omega^-} e^{v(x)} dx = 4\pi, \tag{4.18}$$

Ω^- is also a simply connected domain and

$$e^{v(x)} = \Phi^-(\varphi(x))|\bigtriangledown\varphi(x)|^2 \tag{4.19}$$

for almost everywhere $s = \varphi(x) < 0$ and a function Φ^- of s.

Since both Ω^+ and Ω^- are simply connected, the nodal line, that is, the closure of $\{\varphi(x) = 0 \mid x \in \Omega\}$, must intersect with $\partial\Omega$ and at least, one of

$\partial\Omega^{\pm} \cap \partial\Omega$ contains an arc of positive length. Without loss of generality, one may assume $\partial\Omega^{+} \cap \partial\Omega$ contains an arc of positive length. Furthermore, for any two points $y_k \in \partial\Omega \cap \partial\Omega^{+}$, $k = 1, 2$, which are not on the nodal line, we choose a sequence of $x_{i,k} \in \Omega \to y_k$ for $k = 1, 2$ such that $\varphi(x_{i,k}) = t_i \downarrow 0$ where (4.17) holds for such t_i. Since $\Phi^{+}(\varphi(x_{i,1})) = \Phi^{+}(\varphi(x_{i,2}))$ and $e^{v(x_{i,k})} \to 1$ as $i \to +\infty$ for $k = 1, 2$, (4.17) implies $| \bigtriangledown \varphi(y_1)| = | \bigtriangledown \varphi(y_2)|$, i.e., $| \bigtriangledown \varphi(y)| =$ constant for all y which are contained in a connected component of $\Gamma_l \cap \partial\Omega^{+}$, $l = 1, 2, \ldots, N$. We recall Γ_l is a C^2 portion of the boundary $\partial\Omega$.

Since the nodal line has a non-empty intersection with $\partial\Omega$, we let x_0 denote any point of the intersection (Note that the nodal line is continuous up to the boundary because it has a finite length.) Two cases are discussed seperately.

Cases 1. Suppose $x_0 \in \Gamma_l$ for some $l \in \{1, 2, \ldots, N\}$. Then by the L^p elliptic regularity, $\varphi(x)$ is $W^{2,p}$ in a neighborhood of x_0 with any $p > 1$. Thus $\varphi(x) \in C^{1,\alpha}$ at x_0 for any $\alpha \in (0, 1)$. Since x_0 is the intersection of the boundary and the nodal line of φ, we have $\bigtriangledown\varphi(x_0) = 0$. Then $| \bigtriangledown \varphi(y)| = 0$ for all $y \in \Gamma_l \cap \partial\Omega^{+}$, which yields a contradiction to the Hopf boundary point lemma, unless $\varphi(x)$ is identically to be zero. This proves Lemma 4.3 for case 1.

Cases 2. Suppose x_0 is not a smooth point of the boundary, say, $x_0 \in \Gamma_1 \cap \Gamma_2$. Let θ denote the inner angle of Γ_1 and Γ_2 at x_0. For the simplicity of notations, we assume $x_0 = 0$. By case 1, we conclude that one of Γ_1 and Γ_2 must be contained in $\partial\Omega^{+}$. Without loss of generality, we may assume $\Gamma_1 \subseteq \partial\Omega^{+}$. Let $w = g(z)$ be a conformal mapping which maps $\Omega \cap B_{\delta_0}(0)$ into $\tilde{\Omega}$ such that one part of the boundary $\tilde{\Omega}$, which is the image of $\partial\Omega \cap B_{\delta_0}(0)$ under g, is C^2 at the image of $g(0)$. Let $g(0) = 0$ and $f(w)$ denote the inverse map of g. Set $\tilde{v}(w) = v(f(w))$ and $\tilde{\varphi}(w) = \varphi(f(w))$. Then $\tilde{\varphi}$ satisfies

$$\begin{cases} \Delta\tilde{\varphi} + |f'(w)|^2 e^{\tilde{\varphi}}\tilde{v} = 0 \ \text{ in } \ \tilde{\Omega}, \\ \tilde{\varphi} = 0 \ \text{ in } \ \partial'\tilde{\Omega}, \end{cases} \tag{4.20}$$

where $\partial'\tilde{\Omega} = \partial\tilde{\Omega} \cap g(\partial\Omega \cap B_{\delta_0})$. Since θ is the inner angle of Γ_1 and Γ_2 at 0, we have

$$|f'(w)|^2 = |w|^{2(\frac{\theta}{\pi}-1)}(1 + o(1)), \tag{4.21}$$

where $o(1) \to 0$ as $|w| \to 0$.

If $\pi < \theta < 2\pi$, then $|f'(w)|^2$ is a C^{α_0} function for some $\alpha_0 > 0$. By the regularity of elliptic equation, $\tilde{\varphi}$ is $C^{1,\alpha}$ near 0 for any $\alpha \in (0,1)$. Thus,

$$\bigtriangledown \tilde{\varphi}(0) = 0 \quad \text{and} \quad |\bigtriangledown \tilde{\varphi}(w)| \leq c_\alpha |w|^\alpha \tag{4.22}$$

for any $0 < \alpha < 1$ because the nodal line of $\tilde{\varphi}$ touch $\partial'\tilde{\Omega}$ at 0. On the other hand, for $w = w(z)$, $z \in \Gamma_1$, we have

$$|\bigtriangledown_w \tilde{\varphi}(w)| = |\bigtriangledown_z \tilde{\varphi}(z)||\frac{\partial z}{\partial w}| = c\,|f'(w)|, \tag{4.23}$$

where $|\bigtriangledown \varphi(z)| = c$ for all $z \in \Gamma_1$. Choose $\frac{\theta}{\pi} - 1 < \alpha < 1$ and $w = w(z)$ where $z \in \Gamma_1 \subseteq \partial\Omega^+$. Then by (4.22) and (4.21), we have

$$c\,|w|^{\frac{\theta}{\pi}-1} \leq c_\alpha |w|^\alpha,$$

which implies $c = 0$. Hence, we obtain a contradition again.

Suppose $0 < \theta < \pi$. Then $\tilde{\varphi}$ is H^1 in a neighborhood of 0. Since $|f'(w)|^2 = O(|w|^{\frac{\theta}{\pi}-1})$, we have $|f'(w)|^2 e^{\tilde{v}}\tilde{\varphi} \in L^p$ locally near 0 for any $1 < p < \frac{1}{1-\frac{\theta}{\pi}}$. By the elliptic regularity and Sobolev's embedding theorem, $\tilde{\varphi} \in C^{\alpha_0}$ locally for some $\alpha_0 < 1$. We want to prove $\tilde{\varphi} \in C^\alpha$ locally for any $\alpha \in (0,1)$ by iteration. To see it, we assume $\tilde{\varphi} \in C^{\alpha_0}$. Then

$$|\tilde{\varphi}(w)| \leq c_0 |w|^{\alpha_0}. \tag{4.24}$$

Substituting (4.24) into the equation, we have

$$|f'(w)|^2 e^{\tilde{v}} |\tilde{\varphi}(w)| \leq c_1 |w|^{(\frac{2\theta}{\pi}+\alpha_0-2)}.$$

By the elliptic regularity, $\tilde{\varphi} \in W^{2,p}$ locally near 0 for

$$1 < p < \left(\frac{2}{2-(\frac{2\theta}{\pi}+\alpha_0)}\right)_+ = \begin{cases} +\infty & \text{if } 2 \leq \frac{2\theta}{\pi}+\alpha_0, \\ \frac{2}{2-(\frac{2\theta}{\pi}+\alpha_0)} & \text{if } 2 > \frac{2\theta}{\pi}+\alpha_0. \end{cases}$$

If $2 \leq \frac{2\theta}{\pi} + \alpha_0$, then $\tilde{\varphi} \in C^\alpha$ locally near 0 for any $\alpha \in (0,1)$. So, we assume $2 > \frac{2\theta}{\pi} + \alpha_0$. Then by the Sobolev embedding, $\tilde{\varphi} \in C^\alpha$ for any $\alpha \in (0,1)$ if $\frac{2\theta}{\pi} + \alpha_0 > 1$, and $\tilde{\varphi} \in C^\alpha$ for any $\alpha < \frac{2\theta}{\pi} + \alpha_0$ if $\frac{2\theta}{\pi} + \alpha_0 \leq 1$. In particular, $\tilde{\varphi} \in C^{\alpha_0+\frac{\theta}{\pi}}$. By repeating the process a finite time, we have established $\tilde{\varphi} \in C^\alpha$ locally near 0 for any α. Then $|\tilde{\varphi}(w)| \leq c_\alpha |w|^\alpha$ for $\alpha \in (0,1)$. Substituting it into equation (4.20) again, by noting $|f'(w)|^2 e^{\tilde{v}}\tilde{\varphi}(w) = O(|w|^{-\beta})$ for some

$\beta < 1$, we have $|f'(w)|^2 e^{\tilde{v}} \tilde{\varphi}(w) \in L^p$ locally for some $p > 2$. Applying the regularity theorem and the Sobolev embedding once again, we conclude that $\tilde{\varphi} \in C^{\alpha_1}$ locally near 0 for some $\alpha_1 > 0$.

Since the nodal line of $\tilde{\varphi}$ touch the boundary at 0, $\bigtriangledown \tilde{\varphi}(0) = 0$. In particular, for $w = w(z)$, $0 \neq z \in \Gamma_1$ and $z \to 0$,

$$\begin{aligned} 0 &= |\bigtriangledown \tilde{\varphi}(0)| = \lim_{w \to 0} |\bigtriangledown \tilde{\varphi}(w)| \\ &= \lim_{z \to 0} |\bigtriangledown \varphi(z)||w|^{\frac{\theta}{\pi}-1}(1 + o(1)). \end{aligned}$$

Recall that $|\bigtriangledown \varphi(z)| =$ constant for $z \in \Gamma_1$. By the identity above, we deduce $|\bigtriangledown \varphi(z)| = 0$ for $z \in \Gamma_1$, which yields a contradiction to the Hopf boundary point lemma. This contradiction finishes the proof of Lemma 4.3 Q.E.D.

Remark 4.4. If $\int_\Omega e^v dx < 8\pi$, then a contradiction can be obtained by using (4.16) and (4.18) only, that is, no assumption of smoothness of $\partial\Omega$ is required for Lemma 4.3 when $\rho < 8\pi$. However, in order to apply the symmetrization, the continuity of φ on $\bar{\Omega}$ is needed. For example, this continuity of φ is guaranteed if Ω is a Lipschitz domain. Even for a domain such as $\Omega_0 = \{z \mid z = \frac{1}{3}w^3 + w, |w| > 1\}$, this holds true. Note that $\partial\Omega_0$ has a cusp at $\pm\frac{2}{3}i$ and the inner angles at $\pm\frac{2}{3}i$ are equal to 2π. Hence Theorem 4.1 holds for Ω_0. However, for $\rho = 8\pi$, it is still an open problem, because the nodal line of φ might touch with the boundary $\partial\Omega_0$ at $\pm\frac{2}{3}i$, where our method simply fails.

Proof of Theorem 1.6. We first prove that the linearized equation of (1.1) at u has no null eigenfunctions in $H_0^1(\Omega)$. Suppose φ is a solution of the linearized equation

$$\text{(4.25)} \qquad \begin{cases} \Delta\varphi + \dfrac{\rho e^u \varphi}{\int_\Omega e^u dx} - \dfrac{\rho(\int_\Omega e^u \varphi dx)e^u}{(\int_\Omega e^u dx)^2} = 0 \ \text{ in } \ \Omega, \\ \varphi = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\rho = 8\pi$ and u is a solution of (1.1) with $\rho = 8\pi$. For $\rho < 8\pi$, we refere the proof to [29].

By the remark in the proof of Lemma 4.3, φ is continuous on $\bar{\Omega}$. We want to prove $\varphi \equiv 0$ in $\bar{\Omega}$. Let

$$\text{(4.26)} \qquad \tilde{\varphi} = \varphi - \frac{\int_\Omega e^u \varphi dx}{\int_\Omega e^u dx}, \quad \text{and} \quad v = u + \log\rho - \log(\int_\Omega e^u dx)$$

Then $\tilde{\varphi}$ satisfies

$$(4.27) \qquad \begin{cases} \Delta\tilde{\varphi} + e^v\tilde{\varphi} = 0 \text{ in } \Omega, \\ \tilde{\varphi} \mid_{\partial\Omega} = \text{constant on } \partial\Omega. \end{cases}$$

By (4.26), $\tilde{\varphi}$ also satisfies

$$(4.28) \qquad \int_\Omega e^v\tilde{\varphi}(x)dx = 0$$

By (4.28), $\tilde{\varphi}$ must change sign provided that $\tilde{\varphi} \not\equiv 0$ in Ω. If $\tilde{\varphi} \mid_{\partial\Omega} \equiv c_1$ and $c_1 = 0$, $\tilde{\varphi} \equiv 0$ by to Lemma 4.3 and consequently $\varphi \equiv 0$. So, we may assume $c_1 < 0$. Hence, the nodal line $\{\tilde{\varphi} = 0\}$ is away from $\partial\Omega$ and divides Ω into two components $\Omega^+ = \{x \mid \tilde{\varphi}(x) > 0\}$ and $\Omega^- = \{x \mid \tilde{\varphi}(x) < 0\}$. Furthermore, by Lemma 4.3, Ω^+ is simply-connected and satisfies

$$(4.29) \qquad \int_{\Omega^+} e^{v(x)}dx \geq 4\pi \geq \int_{\Omega^-} e^{v(x)}dx.$$

Set $\Omega^+ \subset \tilde{\Omega}^+ = \{x \mid \tilde{\varphi}(x) > c_1\}$ and $\tilde{\Omega}^- = \{x \mid \tilde{\varphi}(x) < c_1\}$. We discuss two cases separately.

Cases 1. Assume $\tilde{\Omega}^-$ is an empty set.

In this case we can do the symmetrization for $\tilde{\varphi}$ in Ω as in the proof of Lemma 4.3, that is, for any $t > c_1$, we set $B_{r(t)}$ to be the open ball of center 0 and radius $r(t)$ such that

$$\int_{B_r(t)} e^{v(x)}dx = \int_{\{\tilde{\varphi} > t\}} e^{U(x)}dx,$$

where $U(x)$ is defined as in Lemma 4.3. As before, $r(t)$ is continuous, strictly decreasing in t and $\lim_{t\downarrow c_1} R(t) = +\infty$, because $\int_\Omega e^{v(x)}dx = 8\pi$. The symmetrization $\tilde{\varphi}^*(x) = \tilde{\varphi}^*(|x|)$ is defined as the same as before. By Lemma 4.2 and the coarea formulas, we obtain as before

$$(4.30) \quad \int_{\mathbf{R}^2} |\bigtriangledown \tilde{\varphi}^*|^2 dx \leq \int_\Omega |\bigtriangledown \tilde{\varphi}|^2 dx = \int_\Omega e^v\tilde{\varphi}^2(x)dx = \int_{\mathbf{R}^2} e^U {\tilde{\varphi}_*}^2 dx,$$

and

$$(4.31) \qquad \int_{\mathbf{R}^2} e^U\tilde{\varphi}^*(x)dx = \int_\Omega e^{v(x)}\tilde{\varphi}(x)dx = 0.$$

Set

$$(4.32) \qquad K^* = \inf\left\{\int_{\mathbf{R}^2} |\bigtriangledown \psi|^2 dx \mid \psi(x) \text{ is radially symmetric,} \right.$$
$$\left. \int_{\mathbf{R}^2} e^{U(x)}\psi(x)dx = 0 \text{ and } \int_{\mathbf{R}^2} e^{U(x)}\psi^2(x)dx = 1\right\}.$$

Note that

$$\left|\int_{\mathbf{R}^2} e^{U(x)}\psi(x)dx\right| \le \left(\int_{\mathbf{R}^2} e^{U(x)}\psi^2(x)dx\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^2} e^{U(x)}dx\right)^{\frac{1}{2}}.$$

Hence, condition (4.31) also holds for any minimizer of (4.32). Set ψ^* to be the minimizer of (4.32) and ψ^* satisfies

$$\begin{cases} \Delta\psi^* + K^* e^{U(x)}\psi^* = 0 \quad \text{in} \quad \mathbf{R}^2, \\ \int_{\mathbf{R}^2} e^{U(x)}\psi^* dx = 0. \end{cases} \tag{4.33}$$

By (4.30), $K^* \le 1$. Therefore, ψ^* changes the sign once and only once, otherwise it yields a contradiction to Lemma 4.3. Let ξ_0 be the zero of ψ^* and we may assume $\psi^*(r) > 0$ if $r < \xi_0$ and $\psi^*(r) < 0$ if $r > \xi_0$. By (4.31),

$$\begin{aligned} r\psi^{*'}(r) &= \int_0^r e^{U(s)}\psi^*(s)sds \\ &= \int_r^\infty e^{U(s)}\psi^*(s)sds < 0 \end{aligned} \tag{4.34}$$

for $r \in (\xi_0, \infty)$. Therefore, $\psi^{*'}(r)$ is decreasing for $r \ge \xi_0$ and $r\psi^{*'}(r) \to 0$ as $r \to +\infty$. Clearly, (4.33) yields

$$|r\psi^{*'}(r)| \le \left(\int_r^\infty e^{U(s)}\psi^{*^2}(s)sds\right)^{\frac{1}{2}} \left(\int_r^\infty e^{U(s)}sds\right)^{\frac{1}{2}} \le c\, r^{-1}$$

for large r. Hence $\lim_{r\to+\infty}\psi^*(r)$ exists and

$$\lim_{r\to+\infty}\psi^*(r) < 0 \tag{4.35}$$

Let $\psi(r) = \frac{8-|x|^2}{8+|x|^2}$. Then ψ satisfies

$$\Delta\psi + e^U\psi(r) = 0. \tag{4.36}$$

Together with (4.33), we have

$$r\left(\frac{\psi^*}{\psi(r)}\right)' = \frac{(1-K^*)}{\psi^2(r)}\int_0^r e^{U(s)}\psi^*(s)\psi(s)sds.$$

If $\xi_0 < \sqrt{8}$ where $\psi(\sqrt{8}) = 0$, then $\frac{\psi^*(r)}{\psi(r)}$ is increasing for $r \in [0, \xi_0]$. Clearly, it yields

$$0 < \frac{\psi^*(0)}{\psi(0)} < \frac{\psi^*(\xi_0)}{\psi(\xi_0)} = 0,$$

a contradiction. Thus, $\xi_0 \geq \sqrt{8}$. Similarly, we have

$$\lim_{R \to +\infty} R\left(\frac{\psi^*}{\psi}\right)'(R)\psi^2(R) - r\left(\frac{\psi^*}{\psi}\right)'\psi^2(r) = (1 - K^*)\int_r^\infty e^{U(s)}\psi^*(s)\psi(s)sds.$$

Since

$$\lim_{R \to +\infty} R[\psi^{*'}(R)\psi(R) - \psi'(R)\psi^*(R)] = 0,$$

We have

$$-\left(\frac{\psi^*}{\psi}\right)'\psi^2(r) = (1 - K^*)\int_r^\infty e^{U(s)}\psi^*(s)\psi(s)sds.$$

If $\xi_0 > \sqrt{8}$, then $\frac{\psi^*}{\psi}(r)$ is decreasing for $r \geq \xi_0$. Then it yields

$$0 = \frac{\psi^*(\xi_0)}{\psi'(\xi_0)} > \lim_{r \to +\infty} \frac{\psi^*(r)}{\psi(r)} = -\lim_{r \to +\infty} \psi^*(r),$$

a contradiction to (4.35). Therefore, we conclude $\xi_0 = \sqrt{8}$ and $\psi^*(r)\psi(r) > 0$ for all $r \neq \sqrt{8}$. Again,

$$\begin{aligned} 0 &= \lim_{r \to +\infty} (\psi^*(r)'\psi(r) - \psi'(r)\psi^*(r))r \\ &= (1 - K^*)\int_0^\infty e^{U(s)}\psi^*(s)\psi(s)sds \end{aligned}$$

yields $K^* = 1$, and (4.30) becomes an equality. Therefore, all the inequalities of (4.10) turn out to be equalities. In particular,

$$l^2\{\tilde{\varphi} = t\} = \frac{1}{2}\left(8\pi - \int_{\Omega_t} e^{v(x)}dx\right)\int_{\Omega_t} e^{v(x)}dx \tag{4.37}$$

holds for almost everywhere t. Since $\tilde{\varphi}$ is continuous on $\bar{\Omega}$, we have

$$l^2\{\varphi = t\} \geq c > 0$$

for some constant $c > 0$ and for t is close to c_1. On the other hand, for any $\varepsilon > 0$,

$$8\pi - \int_{\Omega_t} e^{v(x)} dx \leq \varepsilon$$

holds if t is sufficiently close to c_1. Thus, it yields a contradiction to (4.37) when t is close to c_1. This contradiction proves that $\tilde{\Omega}^-$ is non-empty.
Cases 2. $\tilde{\Omega}^-$ is not empty. In this case, we still do the symmetrization for $\tilde{\Omega}^+$ as before. Set $\tilde{R}_0$ be

$$\int_{\{\varphi > c_1\}} e^{v(x)} dx = \int_{B_{\tilde{R}_0}} e^{U(x)} dx \tag{4.38}$$

But for $t < c_1$, we symmetrize $\{\varphi < t\}$ as

$$\int_{\mathbf{R}^2 \setminus \bar{B}_{\tilde{r}(t)}} e^{U(x)} dx = \int_{\{\varphi(x) < t\}} e^{v(x)} dx \tag{4.39}$$

Since $\int_\Omega e^v dx = 8\pi$, we have $\lim_{t \uparrow c_1} \tilde{r}(t) = \tilde{R}_0$. Set

$$\tilde{\varphi}^{**}(r) = \inf\{t \mid x \in \mathbf{R}^2 \setminus \bar{B}_{\tilde{r}(t)}\}.$$

for $r > \tilde{R}_0$ and

$$\lim_{r \downarrow \tilde{R}_0} \tilde{\varphi}^{**}(r) = \lim_{s \uparrow \tilde{R}_0} \tilde{\varphi}^*(s) \tag{4.40}$$

By the isoperimetric inequality (4.2), the symmetrization yields

$$\int_{B_{\tilde{R}_0}} |\bigtriangledown \tilde{\varphi}^*(x)|^2 dx \leq \int_{\tilde{\Omega}^+} |\bigtriangledown \tilde{\varphi}|^2 dx \tag{4.41}$$

as (4.30). For $t < c_1$, we note that $\{\varphi = t\}$ bounds the domain $\{x \mid \varphi < t\}$ which obiviously is not a simply-connected domain. Thus, by (4.2)

$$\left(\int_{\{\varphi = t\}} e^{\frac{v}{2}}\right)^2 > \frac{1}{2}\left(8\pi - \int_{\{\varphi < t\}} e^v dx\right)\left(\int_{\{\varphi < t\}} e^v dx\right)$$

for $t < c_1$. Then (4.10) yields for $t < c_1$

$$-\frac{d}{dt}\int_{\Omega_t} |\bigtriangledown \tilde{\varphi}|^2 dx = \int_{\{\varphi = t\}} |\bigtriangledown \tilde{\varphi}| ds$$

$$
\begin{aligned}
&> \frac{1}{2}\left(8\pi - \int_{\{\varphi<t\}} e^{v}dx\right)\left(\int_{\{\varphi<t\}} e^{v}dx\right)\left[-\frac{d}{dt}\int_{\Omega_t} e^{v(x)}dx\right]^{-1} \\
&= \frac{1}{2}\left(8\pi - \int_{\mathbf{R}^2\setminus B_{\tilde{r}(t)}} e^{U}dx\right)\left(\int_{\mathbf{R}^2\setminus B_{\tilde{r}(t)}} e^{U}dx\right)\left[-\frac{d}{dt}\int_{\mathbf{R}^2\setminus B_{\tilde{r}(t)}} e^{U(x)}dx\right]^{-1} \\
&= -\frac{d}{dt}\int_{\mathbf{R}^2\setminus B_{\tilde{r}(t)}} |\bigtriangledown \tilde{\varphi}^{**}|^2 dx.
\end{aligned}
$$

Integrating from c_1 to $\inf_{\bar{\Omega}} \tilde{\varphi}$, we obtain

$$
(4.42) \qquad \int_{\Omega} |\bigtriangledown \tilde{\varphi}|^2 dx - \int_{\tilde{\Omega}^+} |\bigtriangledown \tilde{\varphi}|^2 dx \geq \int_{\mathbf{R}^2\setminus B_{\tilde{R}_0}} |\bigtriangledown \tilde{\varphi}^{**}|^2 dx.
$$

By (4.40), $\hat{\varphi}(r)$ is continuous at $\tilde{R}_0$, where $\hat{\varphi}$ is defined by

$$
\hat{\varphi}(r) = \begin{cases} \tilde{\varphi}^{*} & \text{if } r < \tilde{R}_0, \\ \tilde{\varphi}^{**} & \text{if } r > \tilde{R}_0. \end{cases}
$$

Together with (4.41) and (4.42), it yields

$$
(4.43) \qquad \int_{\mathbf{R}^2} |\bigtriangledown \hat{\varphi}|^2 dx < \int_{\Omega} |\bigtriangledown \tilde{\varphi}|^2 dx = \int_{\Omega} e^{v}\tilde{\varphi}^2 dx = \int_{\mathbf{R}^2} e^{U}(\hat{\varphi})^2 dx.
$$

Clearly, $\hat{\varphi}$ also satisfies

$$
\int_{\mathbf{R}^2} e^{U}\hat{\varphi}(x)dx = \int_{\Omega} e^{v}\tilde{\varphi}(x)dx = 0.
$$

As before, we set

$$
\begin{aligned}
K^{*} = & \left\{\int_{\mathbf{R}^2} |\bigtriangledown \psi|^2 dx \,\middle|\, \psi(x) \text{ is radially symmetric,} \right. \\
& \left. \int_{\mathbf{R}^2} e^{U}\psi(x)dx = 0 \text{ and } \int_{\mathbf{R}^2} e^{U(x)}\psi^2(x) = 1 \right\}.
\end{aligned}
$$

The minimizing is exactly the same one as (4.32). By the previous argument, we have $K^* = 1$. On the other hand, $K^* < 1$ by (4.43). This contradiction shows that there are no null eigenfunctions of the linearized equation except the trivial one.

Since the linearized equation has only the trivial solution, it is not hard to prove the estimate for the inverse operator. For example, we can show that

$$\|\varphi\|_{H_0^1(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \leq c\, \|\mathcal{L}(\varphi)\|_{L^\infty(\Omega)}, \tag{4.44}$$

where $\mathcal{L}$ denotes the linearized operator of (4.25). Note that both $\|\varphi\|_{H_0^1(\Omega)}$ and $\|\varphi\|_{L^\infty(\Omega)}$ is invariant under the conformal transformation. By using the implicit function theorem and (4.44), we can prove that for any solution u of (1.1) with $\rho = 8\pi$, there is a solution $u_\rho(x)$ of (1.1) with small $|\rho - 8\pi|$ and $\lim_{\rho\to 8\pi} u_\rho(x) = u(x)$. Thus, the uniqueness of (1.1) with $\rho = 8\pi$ follows readily from Theorem 4.1. Q.E.D.

5 Domains with $D(p) = 0$

In this section, we discuss critical points of γ with $D(p) \leq 0$. Throughout this section, Ω is always assumed to be simply-connected. Now we are in the position to prove Theorem 1.7.

Proof of Theorem 1.7. We first note that if the nondegenerate condition is not satisfied, then

$$D(p) = \sum_{n\geq 3}^{\infty} \frac{n^2|a_n|^2}{n-2} - |a_1|^2 > 0$$

unless $a_n = 0$ for $n \geq 4$. But, if $a_n = 0$ for $n \geq 4$, $f(z) = z + \frac{1}{3}z^3$ after a rotation and a scaling. Clearly, $f'(\pm i) = 0$, and it implies that Ω has a cusp at its boundary point $\pm\frac{2}{3}i$ and its inner angle at $\pm\frac{2}{3}i$ are 2π. Call Ω_0 to be this particular domain. Note that Ω_0 is not a piecewisely C^2 domain discussed in this paper. Hence we conclude that if $D(p) \leq 0$ then p is a nondegenerate critical point.

Suppose $D(p) < 0$. Since p is a nondegenerate critical point, there exists a sequence of solutions u_k of (1.1) with $\rho = \rho_k$ such that u_k blows up at p. By Theorem 1.4, we know $\rho_k < 8\pi$. Thus, by the uniqueness theorem of Suzuki, u_k is the minimizer of the nonlinear functional J_{ρ_k}. Clearly, the blowup point p must be a maximum point of γ. Now suppose that there exists another maximum point $q \neq p$. If $D(q) < 0$, then there exists another sequence of solutions v_k with $\rho_k < 8\pi$, which blows up at q. For k large, we have $v_k \neq u_k$. Then we obtain a contradiction to the uniqueness theorem of Suzuki. If $D(q) = 0$, we can perturb Ω slightly to Ω' such that q is also a critical point of $\gamma_{\Omega'}$ and $D_{\Omega'}(q) < 0$. We can perturb the domain Ω by using the conformal map. Set $f(w)$ be the conformal mapping from the unit ball to Ω with $f(0) = q$. Denote $f_\eta(w) = q + \sum_{n\geq 3}^{\infty} a_n \eta^n w^n + (a_1\eta)w$, for $0 < \eta < 1$. Let Ω_η be the image of the unit B_1 under the conformal f_η. Clearly, Ω_η is a smooth domain $\Omega_\eta \to \Omega$ as $\eta \to 1$. By (1.16), $D_{\Omega_\eta}(q) < D(q) = 0$. So, we choose Ω' to be one of Ω_η for small $1 - \eta > 0$. Note that because $a_2 = 0$, q is also a critical point of Ω'. Since Ω' is a small perturbation of Ω, there exists another critical point p' of $\gamma_{\Omega'}$ near by p such that $D_{\Omega'}(p') < 0$ (because p is a nondegenerate critical point of γ_Ω). Since $p' \neq q$, it violates the conclusion of the previous step. The last case is $D(q) > 0$. Then by the computation in the next section, we have $I_{8\pi}(\Omega) > 1 + \sup_{\bar\Omega} \gamma(x) + \log \frac{|B_1|}{|\Omega|}$. Then $I_{8\pi}(\Omega)$ can be attained by some extremal function v. by Theorem 1.6 and the implicit function theorem, there exists a sequence of solutions $v(x;\rho)$ of (1.1) with $v(x;\rho) \to v(x)$ as $\rho \uparrow 8\pi$. Thus, $v(x;\rho_k) \neq u_k(x)$ because u_k blows up at p. This violates the uniqueness theorem for $\rho < 8\pi$. This finishes the proof of the uniqueness of the maximum point p with $D(p) < 0$.

Now suppose $D(p) = 0$. We claim that *there exists a blowup sequence of solutions u_k of (1.1) with $\rho_k \uparrow 8\pi$ such that p is the blowup point.* The claim is proved by using Theorem 1.6, the uniqueness theorem for $\rho = 8\pi$. To see it , we can perturb the domain Ω to obtain a sequence of domains $\Omega_j \to \Omega$ such that for each j, $p \in \Omega_j$ and p is a critical point of $\gamma_j = \gamma_{\Omega_j}$ with $D_j(p) = D_{\Omega_j}(p) < 0$. By the previous step, we know p is the unique maximum point of γ_j and for each j and any $\rho \in [0, 8\pi)$, there exists an unique solution $u_j(x;\rho)$ of (1.1). Since $u_j(x;\rho)$ is unique, $u_j(x;\rho)$ is smoothly depending on the parameter ρ for each j. Since $D_j(p) < 0$, $\sup_{\bar\Omega_j} u_j(x;\rho) \to +\infty$ as $\rho \uparrow 8\pi$ for each j and the blowup point $u_j(x,\rho)$ is p only, as $\rho \uparrow 8\pi$. Thus, for any large constant $C > 0$ and for any j, there exists a $\rho_j \in [0, 8\pi)$ such that

$$\sup_{\bar\Omega_j} u_j(x;\rho_j) = C, \quad \text{and} \tag{5.1}$$

$$\sup_{B_{\frac{\delta_0}{2}}(p)} u_j(x;\rho_j) = C > 2 \sup_{\bar\Omega_j \setminus B_{\delta_0}(p)} u_j(x;\rho_j), \tag{5.2}$$

where δ_0 is a small positive number such that $4\delta_0 <$ dist (p,q) for any critical point $q \neq p$ of γ. By passing to a subsequence of j, $u_j(x;\rho_j)$ uniformly converges to a solution $u(x,C)$ of (1.1) with $\rho(C) = \lim_{j\to+\infty}\rho_j \le 8\pi$. By the uniqueness theorem of (1.1) with $\rho = 8\pi$, equation (1.1) for Ω possesses one solution at most. Hence if C is chosen to be large, then $\rho(C) < 8\pi$. By letting $C = k \to +\infty$, we then construct a sequence of blowup solutions u_k of (1.1) with $\rho = \rho_k \uparrow 8\pi$ which blows up at p. By (5.2), the blowup point of u_k must be p. Hence the claim is proved. Since p is a maximum point of $\gamma_j(x)$, by passing to the limit, p is a maximum point of $\gamma(x)$.

We remain to prove p is the unique maximum point. suppose $q \ne p$ is another maximum point. Then by the previous step, $D(q) \ge 0$. The case $D(q) > 0$ is also excluded by the same argument before. So, we assume $D(q) = 0$. Then by the claim, we have two different sequence of blowup solutions of (1.1) with $\rho \uparrow 8\pi$. This violates Theorem 1.6 again. Hence Theorem 1.7 is proved. Q.E.D.

6 Extremal functions

Proposition 6.1. *Suppose that Ω is bounded piecewisely C^2 simply connected domain and $u(x;\rho)$ is the minimizer of J_ρ with $\rho \in [0, 8\pi)$. Then the followings are equivalent.*

(i) $u(x;\rho)$ is uniformly bounded for $x \in \bar\Omega$ and $\rho \in [0, 8\pi)$,

(ii) Ω is of type C, and

(iii) Equation (1.1) with $\rho = 8\pi$ possesses a solution.

Proof. Obviously, (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). Now suppose (1.1) possesses a solution u for $\rho = 8\pi$. Then by Theorem 1.6 and the implicit function theorem, for $|\rho - 8\pi|$ small, equation (1.1) possesses a solution $u(x;\rho)$ which tends to u as $\rho \uparrow 8\pi$ By the uniqueness theorem, $u(x;\rho)$ is the minimizer of J_ρ for $\rho < 8\pi$. Hence (iii) $\Rightarrow$ (i) is proved.

Lemma 6.2. *Let Ω be a bounded domain of $\mathbf{R}^2$ and p be a critical point of γ. If $D(p) > 0$, then $I_{8\pi}(\Omega) > 1 + \max_{\bar\Omega}\gamma(x) + \log(\frac{|B_1|}{|\Omega|})$.*

Proof. Let f be the conformal map from the unit ball B_1 onto Ω, with $f(0) = p$. For any $\varepsilon > 0$, we set

$$v_\varepsilon(z) = 2\log\left(\frac{1+\varepsilon}{\varepsilon+|z|^2}\right) \tag{6.1}$$

for $|z|<1$, and denote u_ε as a test function by

$$u_\varepsilon(y) = v_\varepsilon(f^{-1}(y))$$

for $y\in\Omega$. Then

$$\begin{aligned} \frac{1}{16\pi}\int_\Omega |\bigtriangledown u_\varepsilon(y)|^2 dy &= \frac{1}{16\pi}\int_{B_1} |\bigtriangledown v_\varepsilon|^2 dz \\ &= 2\int_0^1 \frac{r^3}{(\varepsilon+r^2)^2}dr \\ &= \log\left(\frac{1+\varepsilon}{\varepsilon}\right) - \frac{1}{1+\varepsilon} \end{aligned} \tag{6.2}$$

and,

$$\begin{aligned} \int_\Omega e^{u_\varepsilon(y)}dy &= \int_{B_1} |f'(z)|^2 \frac{(1+\varepsilon)^2}{(\varepsilon+|z|^2)^2}dz \\ &= 2\pi\left(|a_1|^2\int_0^1 \frac{(1+\varepsilon)^2 r}{(\varepsilon+r^2)^2}dr + \sum_{n=3}^\infty |a_n|^2 n^2(1+\varepsilon^2)\int_0^1 \frac{r^{2n-1}}{(\varepsilon+r^2)^2}dr\right) \\ &= \pi\left(\frac{|a_1|^2(1+\varepsilon)}{\varepsilon} + \sum_{n=3}^\infty \frac{n^2}{n-2}|a_n|^2 + O(\varepsilon)\right). \end{aligned}$$

Thus,

$$\begin{aligned} J_{8\pi}(u_\varepsilon) &= \log\frac{|B_1|}{|\Omega|} + \log\left(\frac{1}{\pi}\int_{B_1} |f'(z)|^2 e^{v_\varepsilon(z)}dx\right) - \log\frac{(\varepsilon+1)^2}{\varepsilon} \\ &= \log\frac{|B_1|}{|\Omega|} + \log\left[\frac{|a_1|^2}{\varepsilon} + \left(|a_1|^2 + \sum_{n=3}^\infty \frac{n^2}{n-2}|a_n|^2\right) + O(|\varepsilon|)\right] \\ &\quad + \log\frac{\varepsilon}{(\varepsilon+1)} + \frac{1}{1+\varepsilon} \\ &= \log\frac{|B_1|}{|\Omega|} + \log\left[|a_1|^2 + \left(\sum_{n=3}^\infty \frac{n^2}{n-2}|a_n|^2\right)\varepsilon + O(|\varepsilon|^2)\right] + 1 - \varepsilon + O(\varepsilon^2) \\ &= \log\frac{|B_1|}{|\Omega|} + 1 + \log|a_1|^2 + |a_1|^{-2}D(p)\varepsilon + O(|\varepsilon|^2). \end{aligned}$$

Since $\gamma(p) = \frac{1}{2\pi}\log|a_1|$, we have

$$\begin{aligned} J_{8\pi}(u_\varepsilon) &= 4\pi\gamma(p) + 1 + \log\frac{|B_1|}{|\Omega|} + |a_1|^{-2}D(p)\varepsilon + O(|\varepsilon|^2) \\ &> 1 + 4\pi\gamma(p) + \log\frac{|B_1|}{|\Omega|}, \end{aligned}$$

provided that ε is sufficiently small and $D(p) > 0$. Q.E.D.

Now we can give proofs of Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. By lemma 6.2, if there is a maximum point p of $\gamma(x)$ with $D(p) > 0$, then $I_{8\pi}(\Omega) > 1 + 4\pi\max_{\bar{\Omega}}\gamma(x) + \log\frac{|B_1|}{|\Omega|}$. Then by the "if" part of Theorem 1.1 (which was already proved in [5,Theorem 7.1]), the supreman of $J_{8\pi}$ can be attained.

Now suppose all the maximum points p of $\gamma(p)$ satisfy $D(p) \le 0$. Then by Theorem 1.7, there is only one unique maximum point. By Theorem 1.8, there exists a sequence of solutions $u(x;\rho)$ of (1.1) with $\rho < 8\pi$ such that $u(x;\rho)$ blows up at p exactly. Then by (i) of Proposition 6.1, the supreman of $J_{8\pi}$ can not be attained. Q.E.D.

Proof of Theorem 1.6. Now suppose that the supreman of $J_{8\pi}$ can be attained. Then by Theorem 1.5, $D(p) > 0$ for all the maximum points p of γ. Applying Lemma 6.2, we then have $I_{8\pi}(\Omega) > 1 + 4\pi\max_{\bar{\Omega}}\gamma + \log\frac{|B_1|}{|\Omega|}$. Q.E.D.

7 Final discussions

In this section, we will give several examples to interprete our theorems. First, we would like to apply Theorem 1.5 to the domain Ω of regular n-polygon.

Proposition 7.1. *Let Ω be the regular n-polygon. Then Ω is not of type C.*

Proof. Let the origin be the center of Ω. Since Ω is convex, 0 is the unique critical point of γ. By the Schwartz-Christoffel formulas, we have

$$f'(z) = (1 - z^n)^{-\frac{2}{n}} = \sum_{k=0}^{\infty} \frac{\frac{2}{n}(\frac{2}{n}+1)\dots(\frac{2}{n}+k-1)}{k!} z^{nk}.$$

See [1]. Then

$$D_n(0) = \sum_{k=1}^{\infty} \left(\frac{\frac{2}{n}(\frac{2}{n}+1)\dots(\frac{2}{n}+k-1)}{k!} \right)^2 \left(\frac{1}{nk-1} \right) - 1.$$

Clearly, $D_n(0) \le D_3(0)$. Let $A_k = \left(\frac{\frac{2}{3}(\frac{2}{3}+1)\dots(\frac{2}{3}+k-1)}{k!} \right)^2 \frac{1}{3k-1}$. Since $\frac{A_{k+1}}{A_k} = 1 - \frac{15k+11}{9(k+1)^2} \le (\frac{k+1}{k+2})^{\frac{5}{3}}$, we have $A_k \le C(k+1)^{-\frac{5}{3}}$ for $k = 2, \dots$, where $C = \frac{5}{27} 3^{\frac{2}{3}}$. Therefore,

$$\begin{aligned} D_3(0) &\le \frac{2}{9} + \frac{5}{27} 3^{\frac{2}{3}} \sum_{k=2}^{\infty} (k+1)^{-\frac{5}{3}} - 1 \\ &= \frac{5}{18} (\frac{3}{2})^{\frac{2}{3}} - \frac{7}{9} < 0, \end{aligned}$$

where $\sum_{k=3}^{\infty} k^{-\frac{5}{3}} \le 3(\frac{1}{2})^{\frac{5}{3}}$ is used. By Theorem 1.5, Ω is not of type C.Q.E.D.

Now suppose Ω to be a rectangle with sides $0 < a \le b$. Proposition 7.1 tells us that if $a = b$, then Ω is not of type C. On the other hand, if $\frac{b}{a}$ is sufficiently small, then Ω is of type C. In general, we have the following result.

Proposition 7.2. *There exists a constant* $0 < \eta_0 < 1$ *such that* Ω *is of type C if and only if* $\frac{b}{a} < \eta_0$.

Proof. Let

$$f'_\theta(z) = (1 - z^2)^{-\frac{1}{2}} (1 - e^{-2i\theta} z^2)^{-\frac{1}{2}}$$

for $|z| < 1$, where θ is a constant with $0 < \theta < \frac{\pi}{2}$. By the Christoffel-Schwartz formulas, f_θ maps the unit ball onto a rectangle $\Omega(\theta)$. Let $a(\theta)$ and $b(\theta)$ denotes the two sides of $\Omega(\theta)$. We claim

$$\text{(7.1)} \qquad \frac{a(\theta)}{b(\theta)} \text{ is an increasing function of } \theta.$$

Note that

$$\begin{aligned} a(\theta) &= \int_0^\theta \frac{d\varphi}{|1-e^{2i\varphi}|^{\frac{1}{2}}|1-e^{2i(\varphi-\theta)}|^{\frac{1}{2}}} &= \frac{1}{4}\int_0^\theta \frac{d\varphi}{\sqrt{\sin\varphi\sin(\theta-\varphi)}} \\ &= \frac{1}{4}\int_{\frac{-\theta}{2}}^{\frac{\theta}{2}} \frac{d\varphi}{\sqrt{\sin(\varphi+\frac{\theta}{2})\sin(\frac{\theta}{2}-\varphi)}} &= \frac{1}{4}\int_{\frac{-\theta}{2}}^{\frac{\theta}{2}} \frac{d\varphi}{\sqrt{\sin^2\frac{\theta}{2}-\sin^2\varphi}}. \end{aligned}$$

By changing the variable $\tau = \sin\varphi$, we have

$$a(\theta) = \frac{1}{4}\int_{-\tau_0}^{\tau_0} \frac{d\tau}{\sqrt{(1-\tau^2)(\tau_0^2-\tau^2)}} = \frac{1}{4}\int_{-1}^{1} \frac{d\tau}{\sqrt{(1-\tau_0^2\tau^2)(1-\tau^2)}},$$

where $\tau_0 = \sin\frac{\theta}{2}$. Obviously, $a(\theta)$ is increasing in θ.

The area of the rectangle $\Omega(\theta)$ is

$$A(\theta) = \int_{B_1} \frac{dxdy}{|1-z^2||1-e^{-2i\theta}z^2|} = \int_{B_1} \frac{dxdy}{|1-z^2||e^{2i\theta}-z^2|}.$$

We want to prove $A(\theta)$ is decreasing in θ. Set $0 < \theta_1 < \theta_2 \le \frac{\pi}{2}$, the line $L = \{z = te^{i(\theta_1+\theta_2)} \mid |t| \le 1\}$ decomposes the unit ball into two regions Ω_1 and Ω_2, where $e^{2i\theta_1} \in \Omega_1$ and $e^{2i\theta_2} \in \Omega_2$. Then

$$\begin{aligned} (7.2) \qquad & A(\theta_1) - A(\theta_2) \\ =\ & \int_{B_1} (|1-z^2||e^{2i\theta_1}-z^2|)^{-1} - (|1-z^2||e^{2i\theta_2}-z^2|)^{-1} dxdy \\ =\ & \frac{1}{2}\int_{B_1} \frac{1}{|z|}\Big\{[(1-z)|e^{2i\theta_1}-z|]^{-1} - [|1-z||e^{2i\theta_2}-z|]^{-1}\Big\} dxdy \\ =\ & \frac{1}{2}\int_{\Omega_1} |z|^{-1}(|1-z|)^{-1}\Big\{(|e^{2i\theta_1}-z|)^{-1} - (|e^{2i\theta_2}-z|)^{-1}\Big\} dxdy \\ & -\frac{1}{2}\int_{\Omega_2} |z|^{-1}(|1-z|)^{-1}\Big\{|e^{2i\theta_2}-z|)^{-1} - (|e^{2i\theta_1}-z|)^{-1}\Big\} dxdy. \end{aligned}$$

Let z^* be the reflection point of z with respect to the line L. Then

$$\begin{aligned} |e^{2i\theta_2}-z| &= |e^{2i\theta_1}-z^*|, \quad \text{and,} \\ |e^{2i\theta_1}-z| &= |e^{2i\theta_2}-z^*| \end{aligned}$$

for $z \in \Omega_2$. Therefore

$$\begin{aligned} & A(\theta_1) - A(\theta_2) \\ = \; & \frac{1}{2}\int_{\Omega_1} \frac{1}{|z|}\left\{\frac{1}{|1-z|} - \frac{1}{|1-z^*|}\right\}\left\{(|e^{2i\theta_1} - z|)^{-1} - (|e^{2i\theta_2} - z|)^{-1}\right\}dxdy > 0, \end{aligned}$$

where $1 \in \Omega_1$ and $|1-z| < 1 - |z^*|$ for $z \in \Omega_1$.

Write $\frac{a(\theta)}{b(\theta)} = \frac{a^2(\theta)}{A(\theta)}$. Then (7.1) is proved. Obviously, $\lim_{\theta\to\frac{\pi}{2}} \frac{a(\theta)}{b(\theta)} = 1$ and $\lim_{\theta\to 0} \frac{a(\theta)}{b(\theta)} = 0$.

Set

$$D(\theta) = \int_{B_1}\left(\frac{|f_\theta'(z)|^2}{|z|^4} - \frac{1}{|z|^4}\right)dxdy - \int_{B_1^c}\frac{1}{|z|^4}dxdy$$

For $0 < \theta_1 < \theta_2$, we have

$$D(\theta_1) - D(\theta_2) = \int_{B_1}\frac{1}{|z|^4}\left\{(|1-z^2||e^{2i\theta_1} - z^2|)^{-1} - (|1-z^2||e^{2i\theta_2} - z^2|)^{-1}dxdy.\right.$$

By using the same argument of (7.2), we can prove $D(\theta_1) > D(\theta_2)$. Hence $D(\theta)$ is decreasing in θ. For $\theta = \frac{\pi}{2}$, $\Omega(\frac{\pi}{2})$ is a cube. Hence $D(\frac{\pi}{2}) < 0$. On the other hand, it is known that $D(\theta) > 0$ if θ is small. Therefore, there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that $D(\theta) > 0$ if and only if $\theta < \theta_0$. Let $\eta_0 = \frac{a(\theta_0)}{b(\theta_0)}$. Proposition 7.2 follows readily from Theorem 1.5. Q.E.D.

Next, we consider the domain Ω_h to be a dumbell which consists of two disjoint balls B_1 and B_2 connected with a tube of small width $h > 0$. Let $r_1 \le r_2$ be the radius of B_1 and B_2 respectively.

Proposition 7.3. *Let Ω_h be the domain described above.*

(i) If $r_1 < r_2$, then for small $h > 0$, Ω_h is not of type C, and

(ii) If $r_1 = r_2$ and Ω_h is assumed to be symmetric with respect to x_2-axis, then Ω_h is of type C.

Proof. Let p_1 and p_2 are the center of B_1 and B_2 respectively. G_h and γ_h denote the Green function and the regular part of G_h. Obviously, $\gamma_h(x)$ converges to the regular part of B_1 (and of B_2) uniformly in any compact set x of B_1 (and of B_2). Thus, there exists a local maximum of γ_h at $p_{1,h}$ and

$p_{2,h}$ with $\lim_{h\to 0} p_{i,h} = p_i$ for $i = 1, 2$. We use the formulas (1.12), instead of the series sum (1.16), to compute $D(p_{i,h})$ for $i = 1, 2$. For example at $p_{1,h}$, the associated function H_h converges to

$$H(y,0) = \begin{cases} 0 & \text{if } y \in B_1 \\ r_1^{-4}|y-p_1|^4 - 1 & \text{if } y \in B_2 \end{cases}$$

as $h \downarrow 0$. Thus,

$$\begin{aligned} \int_{B_1\cup B_2} \frac{H(y,0)}{|y-p_1|^4}dy - \int_{(B_1\cup B_2)^c} \frac{dy}{|y-p_1|^4} &= \pi r_1^{-4} r_2^2 - \int_{B_1^c} \frac{dy}{|y-p_1|^4} \\ &= \pi(r_2^2 - r_1^2) r_1^{-4} > 0. \end{aligned}$$

Hence $D(p_{1,h}) > 0$ for Ω_h. Similarly, we can compute

$$\lim_{h\downarrow 0} D(p_{2,h}) = \pi(r_1^2 - r_2^2) r_2^{-4} < 0.$$

By Theorem 1.5, $p_{2,h}$ must be the maximum point and Ω_h is not of type C provided that h is sufficiently small. This proves (i).

For $r_2 = r_1$, the shape of the connection tube might affect the type of domains. We consider only the symmetric one. Obviously, $p_{2,h}$ and $p_{1,h}$ are the maximum points of γ_h for small h where $p_{1,h}$ and $p_{2,h}$ tends to the centers of B_1 and B_2 respectively as $h \downarrow 0$. Since those maximum point p of γ with $D(p) \le 0$ is unique, we see $D(p_{1,h}) = D(p_{2,h})$ must be positive. By Theorem 1.5, Ω_h is of type C. This proves (ii). Q.E.D.

In this paper, we have already seen that Theorem 1.6 plays an important role. In general, we do not expect it holds for non-simply connected domains. Nevertheless, it still holds for an annulus domain. Note for the domain $\Omega_a = \{x \mid a < |x| < 1\}$. We have proved that $I_{8\pi}(\Omega_a)$ is attained by some extremal functions. See [10].

Proposition 7.4. *Let $\Omega_a = \{x \mid a < |x| < 1\}$. Then any solution u of (1.1) is radially symmetric for any $\beta \in (0, 8\pi]$. Moreover, the solution is unique.*

Proof. Set $\varphi(x) = \frac{\partial u}{\partial \theta}$. Then φ satisfies

$$\begin{cases} \Delta\varphi + \frac{\rho e^u \varphi}{\int_{\Omega_a} e^u dx} = 0 \text{ in } \Omega_a, \\ \varphi = 0 \text{ on } \partial\Omega_a. \end{cases}$$

If $\rho \le 8\pi$, we will apply the isoperimatic inequality to show that $\varphi \equiv 0$. Assume $\varphi \not\equiv 0$. Our proof is based on the simple observation: for any $r \in (a, 1)$, there exists at least two zeros of φ on the cricle $\{x \mid |x| = r\}$. Thus $\{\varphi = 0\}$ must intersect with each connected component of the boundary $\partial\Omega_a$ and φ has at least two critical points. Now suppose that φ has a critical point at some point $p \in \{\varphi = 0\}$, then by the simple geometry of the plane, the nodal line $\{\varphi = 0\}$ must enclose two simply connected domains. Then we can follows the symmetrization argument of Lemma 4.3 to obtain a contradiction if $\rho \le 8\pi$. This proves $\varphi \equiv 0$ and then u must be radially symmetric. The uniqueness of radial solution (1.1) for any $\beta \in \mathbf{R}$ has been proved in [5]. Q.E.D.

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