INTEGRAL IDENTITIES AND MINKOWSKI TYPE INEQUALITIES INVOLVING SCHOUTEN TENSOR

SUN-YUNG A. CHANG AND FENGBO HANG

Abstract. New formulas for the integration of the k-th elementary symmetric functions of the Shouten tensor are derived and applied to deduce some Minkowski type inequalities.

1. Introduction

The purpose of this note is to derive Minkowski type integral inequalities for the elementary symmetric functions of the Schouten tensor arising from the considerations of conformal geometry. Let (M^n, g) be a smooth Riemannian manifold with dimension $n \geq 3$. The Riemann curvature tensor decomposes as (see [B, p48, (1.116)])

$$Rm = W + A \bigcirc\!\!\!\!\wedge\, g.$$

Here W is the Weyl tensor, A is the Schouten tensor given by

$$A = \frac{1}{n-2}\left(Rc - \frac{R}{2(n-1)} g\right)$$

and $\bigcirc\!\!\!\!\wedge$ is the Kulkarni-Nomizu product (see [B, p47, Definition 1.110]).

For a $n \times n$ matrix B,

$$\det(\lambda I + B) = \sum_{k=0}^{n} \sigma_k(B) \lambda^{n-k}.$$

$\sigma_k(B)$ is equal to the kth elementary symmetric polynomial function of the eigenvalues of B. Since $\sigma_k(B)$ is a degree k homogeneous polynomial in B, we may find a unique symmetric k-linear functional Σ_k such that $\sigma_k(B) = \Sigma_k(B, \cdots, B)$. If $\alpha \in \mathbb{Z}_+^m$, $\mathbb{Z}_+$ is the set of all nonnegative integers, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\alpha! = \alpha_1! \cdots \alpha_m!$. Assume $|\alpha| = k$, then we use

$$\Sigma_k(B_1, \alpha_1; B_2, \alpha_2; \cdots; B_m, \alpha_m)$$

to mean B_i appears α_i times. If we omit α_i, it means $\alpha_i = 1$. Similar notations apply when the matrix is replaced by a linear map or a $(1,1)$ tensor.

Let A be the Schouten tensor, then $g^{-1}A$ is a $(1,1)$ tensor and we write $\sigma_k(A) = \sigma_k(g^{-1}A)$. More generally, if $A_1, \cdots, A_k$ are $(0,2)$ tensors, then we write

$$\Sigma_k(A_1, \cdots, A_k) = \Sigma_k\left(g^{-1}A_1, \cdots, g^{-1}A_k\right).$$

When confusion could happen, we will point out explicitly which metric g we are using or simply use the $(1,1)$ tensor notation.

Theorem 1.1. *Let (M^n, g) be a smooth compact Riemannian manifold with $n \geq 3$. Assume either $k \leq 2$ or $3 \leq k \leq n$ but (M, g) is locally conformally flat. Then for $\widetilde{g} = v^{-2}g$, $v \in C^{\infty}(M)$, $v > 0$,*

$$\begin{aligned}(1.1)\quad & \int_M \sigma_k\left(\widetilde{A}\right) d\widetilde{\mu} \\ &= \int_M \sigma_k(A)\, v^{2k-n} d\mu + \sum_{\substack{\alpha\in\mathbb{Z}_+^3 \\ |\alpha|=k-1}} \frac{n-2k}{2^{\alpha_3}} \frac{k!}{\alpha!} \frac{(\alpha_1+\alpha_3)!\,(\alpha_2+\alpha_3+1)!}{(\alpha_1+\alpha_2+2\alpha_3+2)!} \cdot \\ & \int_M \Sigma_k\left(g^{-1}\widetilde{A}, \alpha_1; g^{-1}A, \alpha_2; I, \alpha_3; g^{-1}(dv\otimes dv)\right) |\nabla v|^{2\alpha_3} v^{2k-n-2-2\alpha_3} d\mu.\end{aligned}$$

Here I means the identity map, μ is the measure associated with metric g, $\widetilde{\mu}$ is the measure associated with $\widetilde{g}$ and $\widetilde{A}$ is the Schouten tensor of $\widetilde{g}$.

A benefit of the formula is when A and $\widetilde{A}$ both lie in suitable positive cone, then all the terms behind the integral sign will be nonnegative according to Garding's theory on hyperbolic polynomials (see [G]). As a consequence we have the following inequalities between $\int_M \sigma_k\left(\widetilde{A}\right) d\widetilde{\mu}$ which resembles the classical Minkowski inequalities in convex geometry (see [S, chapter 6]). For convenience, denote

$$\mathcal{F}_k(g) = \frac{\int_M \sigma_k(A)\, d\mu}{\mu(M)^{\frac{n-2k}{n}}} \tag{1.2}$$

for $k = 0, 1, \cdots, n$. Note that $\mathcal{F}_0(g) = 1$.

Theorem 1.2. *Let (M^n, g) be a smooth compact Riemannian manifold with $n > 4$. Assume $\sigma_1(A) > 0, \sigma_2(A) > 0$, then for every smooth metric $\widetilde{g}$ conformal to g with $\sigma_1\left(\widetilde{A}\right) \geq 0, \sigma_2\left(\widetilde{A}\right) \geq 0$, we have*

$$\left[\int_M \sigma_1\left(\widetilde{A}\right) d\widetilde{\mu}\right]^2 \leq c(M, g)\, \widetilde{\mu}(M) \int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu}. \tag{1.3}$$

In another way, it is

$$\mathcal{F}_1(\widetilde{g}) \leq c(M, g)\, \mathcal{F}_2(\widetilde{g})^{\frac{1}{2}}. \tag{1.4}$$

Note here $c(M, g)$'s are different in different inequalities. For convenience we will use this convention in future inequalities. Under the assumption of Theorem 1.2, the standard Sobolev inequality tells us $\mathcal{F}_1(\widetilde{g}) \geq c(M, g) > 0$, as a consequence we see $\mathcal{F}_2(\widetilde{g}) \geq c(M, g) > 0$ and

$$\mathcal{F}_1(\widetilde{g})^{\theta} \leq c(M, g, \theta)\, \mathcal{F}_2(\widetilde{g})$$

for $\theta \leq 2$.

On locally conformally flat manifolds, one has more Minkowski type inequalities as follows.

Theorem 1.3. *Let (M^n, g) be a smooth compact locally conformally flat Riemannian manifold. Assume $1 \leq k < \frac{n}{2} - 1$, $\sigma_1(A) > 0, \cdots, \sigma_{k+1}(A) > 0$, then for every smooth metric $\widetilde{g}$ conformal to g with $\sigma_1\left(\widetilde{A}\right) \geq 0, \cdots, \sigma_{k+1}\left(\widetilde{A}\right) \geq 0$, we have*

$$\left[\int_M \sigma_k\left(\widetilde{A}\right) d\widetilde{\mu}\right]^2 \leq c(M, g, k) \int_M \sigma_{k-1}\left(\widetilde{A}\right) d\widetilde{\mu} \int_M \sigma_{k+1}\left(\widetilde{A}\right) d\widetilde{\mu}. \tag{1.5}$$

In another way, it is

$$\mathcal{F}_k(\widetilde{g})^2 \leq c(M,g,k)\,\mathcal{F}_{k-1}(\widetilde{g})\,\mathcal{F}_{k+1}(\widetilde{g}). \tag{1.6}$$

By induction we get

Corollary 1.1. *Let (M^n, g) be a smooth compact locally conformally flat Riemannian manifold. Assume $1 \leq k < \frac{n}{2} - 1$, $\sigma_1(A) > 0, \cdots, \sigma_{k+1}(A) > 0$, then for every smooth metric $\widetilde{g}$ conformal to g with $\sigma_1\left(\widetilde{A}\right) \geq 0, \cdots, \sigma_{k+1}\left(\widetilde{A}\right) \geq 0$, we have*

$$\mathcal{F}_k(\widetilde{g})^{\frac{1}{k}} \leq c(M,g,k)\,\mathcal{F}_{k+1}(\widetilde{g})^{\frac{1}{k+1}}. \tag{1.7}$$

Under the assumption of Corollary 1.1, the standard Sobolev inequality gives us $\mathcal{F}_1(\widetilde{g}) \geq c(M,g) > 0$, as a consequence we see $\mathcal{F}_k(\widetilde{g}) \geq c(M,g,k) > 0$ and

$$\mathcal{F}_k(\widetilde{g})^{\theta} \leq c(M,g,k,\theta)\,\mathcal{F}_{k+1}(\widetilde{g})$$

for $\theta \leq \frac{k+1}{k}$. In particular,

$$\mathcal{F}_k(\widetilde{g})^{\frac{n-2k-2}{n-2k}} \leq c(M,g,k)\,\mathcal{F}_{k+1}(\widetilde{g}) \tag{1.8}$$

In [GW, part (A) of theorem 1], based on careful study of a nonlinear parabolic flow and the compactness of solutions of the corresponding nonlinear elliptic equations, it was shown that for locally conformally flat manifolds, the inequality (1.8) is true, moreover the best constant is achieved by some particular metrics. Our approach is different from [GW]. The main point of our argument is systematically applying the symmetric k-linear functional Σ_k associated with σ_k to rewrite $\int_M \sigma_k(A)\,d\mu$ in suitable forms and applying the Garding's theory of hyperbolic polynomials [G]. At the end of the article we discuss some evidences that in general the best constant in (1.3) does not seem to be achieved. We note that [GW, Theorem 1] proved some sharp inequalities for other range of k's too.

We organize the note as follows. In Section 2 we list some algebraic identities which will be needed later. Section 3 is devoted to the proof of the integral identity in Theorem 1.1. In Section 4 we use the integral identity to derive the Minkowski type inequalities (1.3) and (1.5).

2. Some algebraic preparations

Here we will describe some basic algebraic facts which will be used freely later. For a $n \times n$ matrix, since $\det A$ is a degree n homogeneous polynomial in A, there exists a unique symmetric n-linear functional D such that $\det A = D(A, \cdots, A)$. If we let

$$\delta^{i_1\cdots i_k}_{j_1\cdots j_k} = \begin{cases} \text{sign of the permutation}, & \text{if } i_1\cdots i_k \text{ are mutually different and } j_1\cdots j_k \text{ is a permutation of } i_1\cdots i_k; \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

then

$$D(A_1, \cdots, A_n) = \frac{1}{n!}\delta^{i_1\cdots i_n}_{j_1\cdots j_n} A_{1,i_1j_1}\cdots A_{n,i_nj_n}. \tag{2.2}$$

Here $i_1, \cdots, i_n$ runs from 1 to n. Next we will describe some basic properties of D.

$$D(A_1B, \cdots, A_nB) = D(A_1, \cdots, A_n) \det B. \tag{2.3}$$

$$D(BA_1, \cdots, BA_n) = \det B \cdot D(A_1, \cdots, A_n). \tag{2.4}$$

$$D\left(A_1^T, \cdots, A_n^T\right) = D(A_1, \cdots, A_n). \tag{2.5}$$

These equalities follow from polarizing the variable A in $\det AB = \det A \det B$, $\det BA = \det B \det A$ and $\det A^T = \det A$. On the other hand, a simple calculation shows

$$D(A; I, n-1) = \frac{1}{n} \operatorname{tr} A. \tag{2.6}$$

Here $D(A; I, n-1) = D(A, I, \cdots, I)$, I is repeated $n-1$ times. For convenience we will apply similar notations below. We also have

(2.7)
$$\begin{aligned} &D(A_1B, A_2, \cdots, A_n) + D(A_1, A_2B, \cdots, A_n) + \cdots + D(A_1, A_2, \cdots, A_nB) \\ &= D(BA_1, A_2, \cdots, A_n) + D(A_1, BA_2, \cdots, A_n) + \cdots + D(A_1, A_2, \cdots, BA_n) \\ &= D(A_1, \cdots, A_n) \operatorname{tr} B. \end{aligned}$$

This follows from polarizing the variable A in $D(AB; A, n-1) = \frac{1}{n} \det A \operatorname{tr} B$ and $D(BA; A, n-1) = \frac{1}{n} \det A \operatorname{tr} B$.

If we write

$$\det(\lambda I + A) = \sigma_0(A)\lambda^n + \sigma_1(A)\lambda^{n-1} + \cdots + \sigma_n(A), \tag{2.8}$$

then

$$\sigma_k(A) = \binom{n}{k} D(A, k; I, n-k) = \frac{1}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} A_{i_1 j_1} \cdots A_{i_k j_k}. \tag{2.9}$$

Let Σ_k be the symmetric k-linear functional associated with σ_k, then

$$\begin{aligned} \Sigma_k(A_1, \cdots, A_k) &= \binom{n}{k} D(A_1; \cdots; A_k; I, n-k) \\ &= \frac{1}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} A_{1, i_1 j_1} \cdots A_{k, i_k j_k}. \end{aligned} \tag{2.10}$$

Using this equation and the properties of D listed above we see

$$\Sigma_k\left(A_1^T, \cdots, A_k^T\right) = \Sigma_k(A_1, \cdots, A_k). \tag{2.11}$$

$$\Sigma_k(A_1, \cdots, A_{k-1}, I) = \frac{n-k+1}{k} \Sigma_{k-1}(A_1, \cdots, A_{k-1}). \tag{2.12}$$

and

$$\begin{aligned} &(n-k+1)\Sigma_k(A_1, \cdots, A_{k-1}, B) + \Sigma_k(A_1B, \cdots, A_{k-1}, I) + \\ &\cdots + \Sigma_k(A_1, \cdots, A_{k-1}B, I) \\ &= (n-k+1)\Sigma_k(A_1, \cdots, A_{k-1}, B) + \Sigma_k(BA_1, \cdots, A_{k-1}, I) + \\ &\cdots + \Sigma_k(A_1, \cdots, BA_{k-1}, I) \\ &= \Sigma_k(A_1, \cdots, A_{k-1}, I) \operatorname{tr} B. \end{aligned} \tag{2.13}$$

In particular,

$$k\Sigma_k(A,k-1;B) = \sigma_{k-1}(A)\operatorname{tr}B-(k-1)\Sigma_{k-1}(A,k-2;AB) = \sigma_{k-1}(A)\operatorname{tr}B-(k-1)\Sigma_{k-1}(A,k-2;BA). \tag{2.14}$$

Let $\mathcal{S}_n$ be the space of all real symmetric $n\times n$ matrix. We recall a basic fact about σ_k proven in [G]. On $\mathcal{S}_n$, σ_k is hyperbolic I, the associated convex cone is

$$\Gamma_k=\{A\in\mathcal{S}_n:\sigma_1(A)>0,\cdots,\sigma_k(A)>0\}, \tag{2.15}$$

the closure of Γ_k is given by

$$\overline{\Gamma}_k=\{A\in\mathcal{S}_n:\sigma_1(A)\geq 0,\cdots,\sigma_k(A)\geq 0\}.$$

It was shown in [G] that

$$\Sigma_k(A_1,\cdots,A_k)>0\quad\text{for } A_1,\cdots,A_k\in\Gamma_k. \tag{2.16}$$

In particular, if $A_1,\cdots,A_{k-1}\in\Gamma_k$, $B-C\in\overline{\Gamma}_k$, then

$$\Sigma_k(A_1,\cdots,A_{k-1},B)\geq\Sigma_k(A_1,\cdots,A_{k-1},C). \tag{2.17}$$

3. An integral identity

The aim of this section is to prove Theorem 1.1. If $\widetilde{g}=v^{-2}g$, then

$$\widetilde{A}=A+v^{-1}D^2v-\frac{|\nabla v|^2}{2v^2}g.$$

This implies

$$\widetilde{g}^{-1}\widetilde{A}=v^2g^{-1}A+vg^{-1}D^2v-\frac{|\nabla v|^2}{2}I.$$

When we write $\sigma_k\left(\widetilde{A}\right)$, we mean $\sigma_k\left(\widetilde{g}^{-1}\widetilde{A}\right)$. Denote

$$\Theta=A+v^{-1}D^2v-\frac{|\nabla v|^2}{2v^2}g,$$

when we write $\sigma_k(\Theta)$ etc, we mean $\sigma_k\left(g^{-1}\Theta\right)$. It is clear that $\sigma_k\left(\widetilde{A}\right)=v^{2k}\sigma_k(\Theta)$, hence $\int_M\sigma_k\left(\widetilde{A}\right)d\widetilde{\mu}=\int_M\sigma_k(\Theta)v^{2k-n}d\mu$. For convenience, we write $\mathcal{A}\sim\mathcal{B}$ to mean expressions $\mathcal{A}$ and $\mathcal{B}$ differ only by divergence terms, in particular, they have the same integral over M (with respect to $d\mu$, the measure of g). When $k=1$, we have

$$\sigma_1(\Theta)v^{2-n}=\sigma_1(A)v^{2-n}+\Delta v\cdot v^{1-n}-\frac{n}{2}|\nabla v|^2v^{-n} \sim\sigma_1(A)v^{2-n}+\frac{n-2}{2}|\nabla v|^2v^{-n}.$$

This gives the integral identity (1.1).

Assume $k=2$. We have

$$\begin{aligned}&\sigma_2(\Theta)v^{4-n}\\ &=\Sigma_2(\Theta,\Theta)v^{4-n}\\ &=\sigma_2(A)v^{4-n}+\sigma_2\left(D^2v\right)v^{2-n}+\frac{n}{4}\Sigma_2(g,dv\otimes dv)|\nabla v|^2v^{-n}\\ &+2\Sigma_2\left(A,D^2v\right)v^{3-n}-\Sigma_2(A,g)|\nabla v|^2v^{2-n}-\Sigma_2\left(D^2v,g\right)|\nabla v|^2v^{1-n}.\end{aligned} \tag{3.1}$$

Fix a local orthonormal frame on M with respect to g, then

$$v_{ijk}-v_{ikj}=-R_{iljk}v_l=-W_{iljk}v_l-A_{ij}v_k-g_{ij}A_{kl}v_l+A_{ik}v_j+g_{ik}A_{jl}v_l.$$

Here we have used the decomposition $Rm=W+A \owedge g$. To derive the formula (1.1) we try to reorganize terms in (3.1) into linear combinations of $\sigma_2(A)v^{4-n}$, $\Sigma_2(g,dv\otimes dv)|\nabla v|^2 v^{-n}$, $\Sigma_2(A,dv\otimes dv)v^{2-n}$ and $\Sigma_2(\Theta,dv\otimes dv)v^{2-n}$. We have

$$\begin{aligned}
&\sigma_2\left(D^2v\right)v^{2-n} \\
&\sim (n-2)\Sigma_2(\Theta,dv\otimes dv)v^{2-n}-2(n-2)\Sigma_2(A,dv\otimes dv)v^{2-n} \\
&+\frac{n-2}{2}\Sigma_2(g,dv\otimes dv)|\nabla v|^2 v^{-n}+\Sigma_2(A,g)|\nabla v|^2 v^{2-n}.
\end{aligned}\tag{3.2}$$

Indeed,

$$\begin{aligned}
&\sigma_2\left(D^2v\right)v^{2-n} \\
&=\frac{1}{2}\delta^{i_1i_2}_{j_1j_2}v_{i_1j_1}v_{i_2j_2}v^{2-n} \\
&\sim -\frac{1}{2}\delta^{i_1i_2}_{j_1j_2}v_{i_1j_1j_2}v_{i_2}v^{2-n}+\frac{n-2}{2}\delta^{i_1i_2}_{j_1j_2}v_{i_1j_1}v_{i_2}v_{j_2}v^{1-n} \\
&=-\frac{1}{4}\delta^{i_1i_2}_{j_1j_2}\left(v_{i_1j_1j_2}-v_{i_1j_2j_1}\right)v_{i_2}v^{2-n}+(n-2)\Sigma_2\left(D^2v,dv\otimes dv\right)v^{1-n} \\
&=\frac{1}{4}\delta^{i_1i_2}_{j_1j_2}W_{i_1lj_1j_2}v_lv_{i_2}v^{2-n}+\Sigma_2(A,dv\otimes dv)v^{2-n}+\Sigma_2(A(dv\otimes dv),g)v^{2-n} \\
&+(n-2)\Sigma_2\left(D^2v,dv\otimes dv\right)v^{1-n}.
\end{aligned}$$

It follows from (2.13) that

$$(n-1)\Sigma_2(A,dv\otimes dv)+\Sigma_2(A(dv\otimes dv),g)=\Sigma_2(A,g)|\nabla v|^2.$$

In addition

$$\delta^{i_1i_2}_{j_1j_2}W_{i_1lj_1j_2}v_lv_{i_2}=2\sum_{i_1\neq i_2}W_{i_1li_1i_2}v_lv_{i_2}=2W_{i_1li_1i_2}v_lv_{i_2}=0,$$

hence

$$\begin{aligned}
&\sigma_2\left(D^2v\right)v^{2-n} \\
&\sim (n-2)\Sigma_2\left(D^2v,dv\otimes dv\right)v^{1-n}-(n-2)\Sigma_2(A,dv\otimes dv)v^{2-n} \\
&+\Sigma_2(A,g)|\nabla v|^2 v^{2-n}.
\end{aligned}$$

Considering $D^2v=v\Theta-vA+\frac{|\nabla v|^2}{2v}g$, (3.2) follows.

Next we claim

$$\Sigma_2\left(A,D^2v\right)v^{3-n}\sim(n-3)\Sigma_2(A,dv\otimes dv)v^{2-n}.\tag{3.3}$$

Indeed by (2.14) we have

$$\begin{aligned}
\Sigma_2\left(A,D^2v\right)v^{3-n}&=\frac{1}{2}\left(\sigma_1(A)g_{ij}-A_{ij}\right)v_{ij}v^{3-n} \\
&\sim\frac{n-3}{2}\left(\sigma_1(A)g_{ij}-A_{ij}\right)v_iv_jv^{2-n} \\
&=(n-3)\Sigma_2(A,dv\otimes dv)v^{2-n}.
\end{aligned}$$

Here we have used the fact $\sigma_1(A)_i=A_{ijj}$, which follows from the Bianchi's identity.

We also have

$$\begin{aligned}
&\Sigma_2\left(D^2 v, g\right)|\nabla v|^2 v^{1-n} \\
&\sim \frac{2}{3}(n-1)\,\Sigma_2(\Theta, dv\otimes dv)\,v^{2-n}-\frac{2}{3}(n-1)\,\Sigma_2(A, dv\otimes dv)\,v^{2-n} \\
&+\frac{2}{3}(n-1)\,\Sigma_2(g, dv\otimes dv)|\nabla v|^2 v^{-n}.
\end{aligned} \tag{3.4}$$

Indeed,

$$\begin{aligned}
&\Sigma_2\left(D^2 v, g\right)|\nabla v|^2 v^{1-n} \\
&=\frac{1}{2}\delta_{j_1 j_2}^{i_1 i_2} v_{i_1 j_1} g_{i_2 j_2}|\nabla v|^2 v^{1-n} \\
&\sim-\frac{1}{2}\delta_{j_1 j_2}^{i_1 i_2} v_{i_1} g_{i_2 j_2}\cdot 2 v_l v_{l j_1} v^{1-n}+\frac{n-1}{2}\delta_{j_1 j_2}^{i_1 i_2} v_{i_1} v_{j_1} g_{i_2 j_2}|\nabla v|^2 v^{-n} \\
&=-2\Sigma_2\left(D^2 v\cdot(dv\otimes dv), g\right)v^{1-n}+(n-1)\,\Sigma_2(g, dv\otimes dv)|\nabla v|^2 v^{-n} \\
&=-2\Sigma_2\left(D^2 v, g\right)|\nabla v|^2 v^{1-n}+2(n-1)\,\Sigma_2\left(D^2 v, dv\otimes dv\right)v^{1-n} \\
&+(n-1)\,\Sigma_2(g, dv\otimes dv)|\nabla v|^2 v^{-n},
\end{aligned}$$

here we have used (2.13). It follows that

$$\begin{aligned}
&\Sigma_2\left(D^2 v, g\right)|\nabla v|^2 v^{1-n} \\
&\sim\frac{2(n-1)}{3}\Sigma_2\left(D^2 v, dv\otimes dv\right)v^{1-n}+\frac{n-1}{3}\Sigma_2(g, dv\otimes dv)|\nabla v|^2 v^{-n}.
\end{aligned}$$

Using $D^2 v = v\Theta - vA + \frac{|\nabla v|^2}{2v} g$, we get (3.4). Combine (3.1)-(3.4), we get

$$\begin{aligned}
&\sigma_2(\Theta)\,v^{4-n} \\
&\sim\sigma_2(A)\,v^{4-n}+\frac{n-4}{3}\Sigma_2(\Theta, dv\otimes dv)\,v^{2-n}+\frac{2(n-4)}{3}\Sigma_2(A, dv\otimes dv)\,v^{2-n} \\
&+\frac{n-4}{12}\Sigma_2(g, dv\otimes dv)|\nabla v|^2 v^{-n}
\end{aligned}$$

and this proves (1.1) for the case $k=2$.

From now on we assume $k\geq 3$ and (M,g) is locally conformally flat, then we have the Weyl tensor $W=0$ and $A_{ijk}=A_{ikj}$. We have

$$\begin{aligned}
&\sigma_k(\Theta)\,v^{2k-n} \\
&=\Sigma_k\left(v^{-1}D^2 v + A - \frac{|\nabla v|^2}{2v^2} g, k\right)v^{2k-n} \\
&=\sum_{\substack{\alpha\in\mathbb{Z}_+^3\\ |\alpha|=k}}\frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3}\Sigma_k\left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3\right)|\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}.
\end{aligned} \tag{3.5}$$

Again we try to reorganize terms in (3.5) into a linear combination of $\sigma_k(A)\,v^{2k-n}$ and $\Sigma_k(\Theta, \alpha_1; A, \alpha_2; g, \alpha_3; dv\otimes dv)|\nabla v|^{2\alpha_3} v^{2k-n-2-2\alpha_3}$ with $|\alpha|=k-1$.

For $\alpha_1 \geq 1$,

$$\Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$= \frac{1}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} v_{i_1 j_1} \cdots v_{i_{\alpha_1} j_{\alpha_1}} A_{i_{\alpha_1+1} j_{\alpha_1+1}} \cdots A_{i_{\alpha_1+\alpha_2} j_{\alpha_1+\alpha_2}} g_{i_{\alpha_1+\alpha_2+1} j_{\alpha_1+\alpha_2+1}} \cdots g_{i_k j_k} \cdot$$
$$|\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$\sim -\frac{\alpha_1 - 1}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} v_{i_1 j_1} \cdots v_{i_{\alpha_1-2} j_{\alpha_1-2}} v_{i_{\alpha_1-1} j_{\alpha_1-1} j_{\alpha_1}} v_{i_{\alpha_1}} A_{i_{\alpha_1+1} j_{\alpha_1+1}} \cdots A_{i_{\alpha_1+\alpha_2} j_{\alpha_1+\alpha_2}} \cdot$$
$$g_{i_{\alpha_1+\alpha_2+1} j_{\alpha_1+\alpha_2+1}} \cdots g_{i_k j_k} |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$- \frac{2\alpha_3}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} v_{i_1 j_1} \cdots v_{i_{\alpha_1-1} j_{\alpha_1-1}} v_{i_{\alpha_1}} v_l v_{l j_{\alpha_1}} A_{i_{\alpha_1+1} j_{\alpha_1+1}} \cdots A_{i_{\alpha_1+\alpha_2} j_{\alpha_1+\alpha_2}} \cdot$$
$$g_{i_{\alpha_1+\alpha_2+1} j_{\alpha_1+\alpha_2+1}} \cdots g_{i_k j_k} |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{n - 2k + \alpha_1 + 2\alpha_3}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} v_{i_1 j_1} \cdots v_{i_{\alpha_1-1} j_{\alpha_1-1}} v_{i_{\alpha_1}} v_{j_{\alpha_1}} A_{i_{\alpha_1+1} j_{\alpha_1+1}} \cdots A_{i_{\alpha_1+\alpha_2} j_{\alpha_1+\alpha_2}} \cdot$$
$$g_{i_{\alpha_1+\alpha_2+1} j_{\alpha_1+\alpha_2+1}} \cdots g_{i_k j_k} |\nabla v|^{2\alpha_3} v^{2k-n-1-\alpha_1-2\alpha_3}$$
$$= (\alpha_1 - 1) \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2 + 1; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ (\alpha_1 - 1) \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2; g, \alpha_3 + 1; A \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$- 2\alpha_3 \Sigma_k \left(D^2 v, \alpha_1 - 1; A, \alpha_2; g, \alpha_3; D^2 v \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ (n - 2k + \alpha_1 + 2\alpha_3) \Sigma_k \left(D^2 v, \alpha_1 - 1; A, \alpha_2; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-1-\alpha_1-2\alpha_3}$$
$$= (\alpha_1 - 1) \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2 + 1; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ (\alpha_1 - 1) \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2; g, \alpha_3 + 1; A \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$- \frac{2\alpha_3}{\alpha_1} \Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{2\alpha_3 (n - k + \alpha_3)}{\alpha_1} \Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3 - 1; dv \otimes dv\right) |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{2\alpha_2 \alpha_3}{\alpha_1} \Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2 - 1; g, \alpha_3; A \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ (n - 2k + \alpha_1 + 2\alpha_3) \Sigma_k \left(D^2 v, \alpha_1 - 1; A, \alpha_2; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-1-\alpha_1-2\alpha_3}.$$

In between we have used the fact $v_{ijk} - v_{ikj} = -A_{ij} v_k - g_{ij} A_{kl} v_l + A_{ik} v_j + g_{ik} A_{jl} v_l$ and (2.13). Hence

$$\Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$\sim \frac{\alpha_1 (\alpha_1 - 1)}{\alpha_1 + 2\alpha_3} \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2 + 1; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{\alpha_1 (\alpha_1 - 1)}{\alpha_1 + 2\alpha_3} \Sigma_k \left(D^2 v, \alpha_1 - 2; A, \alpha_2; g, \alpha_3 + 1; A \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{2\alpha_3 (n - k + \alpha_3)}{\alpha_1 + 2\alpha_3} \Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2; g, \alpha_3 - 1; dv \otimes dv\right) |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{2\alpha_2 \alpha_3}{\alpha_1 + 2\alpha_3} \Sigma_k \left(D^2 v, \alpha_1; A, \alpha_2 - 1; g, \alpha_3; A \cdot (dv \otimes dv)\right) |\nabla v|^{2\alpha_3-2} v^{2k-n-\alpha_1-2\alpha_3}$$
$$+ \frac{\alpha_1 (n - 2k + \alpha_1 + 2\alpha_3)}{\alpha_1 + 2\alpha_3} \Sigma_k \left(D^2 v, \alpha_1 - 1; A, \alpha_2; g, \alpha_3; dv \otimes dv\right) |\nabla v|^{2\alpha_3} v^{2k-n-1-\alpha_1-2\alpha_3}.$$

This formula still remains true for $\alpha_1 = 0$, $\alpha_3 \geq 1$. Hence it is true for $\alpha_1 + \alpha_3 \geq 1$. If we sum up, we get

$$
\begin{aligned}
&\sigma_k(\Theta)\, v^{2k-n} \\
&\sim \sigma_k(A)\, v^{2k-n} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3, |\alpha|=k \\ \alpha_1+\alpha_3\geq 1}} \frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3} \frac{\alpha_1(\alpha_1-1)}{\alpha_1+2\alpha_3} \Sigma_k\left(D^2v, \alpha_1-2; A, \alpha_2+1; g, \alpha_3; dv\otimes dv\right)\cdot \\
&|\nabla v|^{2\alpha_3}\, v^{2k-n-\alpha_1-2\alpha_3} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3, |\alpha|=k \\ \alpha_1+\alpha_3\geq 1}} \frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3} \frac{\alpha_1(\alpha_1-1)}{\alpha_1+2\alpha_3} \Sigma_k\left(D^2v, \alpha_1-2; A, \alpha_2; g, \alpha_3+1; A\cdot(dv\otimes dv)\right)\cdot \\
&|\nabla v|^{2\alpha_3}\, v^{2k-n-\alpha_1-2\alpha_3} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3, |\alpha|=k \\ \alpha_1+\alpha_3\geq 1}} \frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3} \frac{2\alpha_3(n-k+\alpha_3)}{\alpha_1+2\alpha_3} \Sigma_k\left(D^2v, \alpha_1; A, \alpha_2; g, \alpha_3-1; dv\otimes dv\right)\cdot \\
&|\nabla v|^{2\alpha_3-2}\, v^{2k-n-\alpha_1-2\alpha_3} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3, |\alpha|=k \\ \alpha_1+\alpha_3\geq 1}} \frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3} \frac{2\alpha_2\alpha_3}{\alpha_1+2\alpha_3} \Sigma_k\left(D^2v, \alpha_1; A, \alpha_2-1; g, \alpha_3; A\cdot(dv\otimes dv)\right)\cdot \\
&|\nabla v|^{2\alpha_3-2}\, v^{2k-n-\alpha_1-2\alpha_3} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3, |\alpha|=k \\ \alpha_1+\alpha_3\geq 1}} \frac{k!}{\alpha!}\left(-\frac{1}{2}\right)^{\alpha_3} \frac{\alpha_1(n-2k+\alpha_1+2\alpha_3)}{\alpha_1+2\alpha_3} \Sigma_k\left(D^2v, \alpha_1-1; A, \alpha_2; g, \alpha_3; dv\otimes dv\right)\cdot \\
&|\nabla v|^{2\alpha_3}\, v^{2k-n-1-\alpha_1-2\alpha_3} \\
&= \sigma_k(A)\, v^{2k-n} \\
&+ \sum_{\substack{\alpha\in\mathbb{Z}_+^3 \\ |\alpha|=k-1}} \frac{k!}{\alpha!} \frac{n-2k}{(\alpha_1+2\alpha_3+1)(\alpha_1+2\alpha_3+2)} \left(-\frac{1}{2}\right)^{\alpha_3}\cdot \\
&\Sigma_k\left(D^2v, \alpha_1; A, \alpha_2; g, \alpha_3; dv\otimes dv\right) |\nabla v|^{2\alpha_3}\, v^{2k-n-2-\alpha_1-2\alpha_3}.
\end{aligned}
$$

Using

$$D^2v = v\Theta - vA + \frac{|\nabla v|^2}{2v} g,$$

we see

$$
\begin{aligned}
&\sigma_k(\Theta)\, v^{2k-n} \\
&\sim \sigma_k(A)\, v^{2k-n} + \sum_{\substack{\alpha\in\mathbb{Z}_+^3 \\ |\alpha|=k-1}} \sum_{\substack{\beta\in\mathbb{Z}_+^3 \\ |\beta|=\alpha_1}} \frac{k!}{\alpha_2!\alpha_3!\beta!} \frac{n-2k}{(\alpha_1+2\alpha_3+1)(\alpha_1+2\alpha_3+2)} (-1)^{\alpha_3+\beta_2} \cdot \\
&\frac{1}{2^{\alpha_3+\beta_3}} \Sigma_k(\Theta,\beta_1;A,\alpha_2+\beta_2;g,\alpha_3+\beta_3;dv\otimes dv)\, |\nabla v|^{2(\alpha_3+\beta_3)}\, v^{2k-n-2-2\alpha_3-2\beta_3} \\
&= \sigma_k(A)\, v^{2k-n} + (n-2k) \sum_{\substack{\gamma\in\mathbb{Z}_+^3 \\ |\gamma|=k-1}} \sum_{\substack{\beta_1=\gamma_1 \\ \alpha_2+\beta_2=\gamma_2 \\ \alpha_3+\beta_3=\gamma_3}} \frac{k!}{\alpha_2!\alpha_3!\beta!} \frac{1}{(|\beta|+2\alpha_3+1)(|\beta|+2\alpha_3+2)} \cdot \\
&(-1)^{\alpha_3+\beta_2} \frac{1}{2^{\gamma_3}} \Sigma_k(\Theta,\gamma_1;A,\gamma_2;g,\gamma_3;dv\otimes dv)\, |\nabla v|^{2\gamma_3}\, v^{2k-n-2-2\gamma_3} \\
&= \sigma_k(A)\, v^{2k-n} + (n-2k) \sum_{\substack{\gamma\in\mathbb{Z}_+^3 \\ |\gamma|=k-1}} \frac{k!}{\gamma!} \frac{1}{2^{\gamma_3}} \sum_{\substack{0\le p\le\gamma_2 \\ 0\le q\le\gamma_3}} \frac{\binom{\gamma_2}{p}\binom{\gamma_3}{q}(-1)^{p+q}}{(p+q+\gamma_1+\gamma_3+1)(p+q+\gamma_1+\gamma_3+2)} \cdot \\
&\Sigma_k(\Theta,\gamma_1;A,\gamma_2;g,\gamma_3;dv\otimes dv)\, |\nabla v|^{2\gamma_3}\, v^{2k-n-2-2\gamma_3}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{\substack{0\le p\le\gamma_2 \\ 0\le q\le\gamma_3}} \frac{\binom{\gamma_2}{p}\binom{\gamma_3}{q}(-1)^{p+q}}{(p+q+\gamma_1+\gamma_3+1)(p+q+\gamma_1+\gamma_3+2)} \\
&= \sum_{\substack{0\le p\le\gamma_2 \\ 0\le q\le\gamma_3}} \binom{\gamma_2}{p}\binom{\gamma_3}{q}(-1)^{p+q} \int_0^1 \left(t^{p+q+\gamma_1+\gamma_3} - t^{p+q+\gamma_1+\gamma_3+1}\right) dt \\
&= \int_0^1 t^{\gamma_1+\gamma_3}(1-t)^{\gamma_2+\gamma_3+1}\, dt \\
&= \frac{(\gamma_1+\gamma_3)!\,(\gamma_2+\gamma_3+1)!}{(\gamma_1+\gamma_2+2\gamma_3+2)!}.
\end{aligned}
$$

The conclusion of Theorem 1.1 follows.

4. Some applications of the integral identities

Here we will apply Theorem 1.1 to prove some inequalities between $\int_M \sigma_k(A)\, d\mu$. Note that if $k<\frac{n}{2}$ and both A and $\widetilde{A}$ belong to $\overline{\Gamma}_k$, then it follows from Garding's theory that all the terms on the right hand side of (1.1) are nonnegative. This serves as the main point in proving Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. We write $\widetilde{g} = v^{-2}g$ and still use the notation Θ as in Section 3 to avoid confusion. It follows from Theorem 1.1 that

$$\begin{aligned}
(4.1) \qquad & \int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu} \\
&= \int_M \sigma_2(A)\, v^{4-n} d\mu + \frac{n-4}{3} \int_M \Sigma_2(\Theta, dv \otimes dv)\, v^{2-n} d\mu \\
&+ \frac{2(n-4)}{3} \int_M \Sigma_2(A, dv \otimes dv)\, v^{2-n} d\mu \\
&+ \frac{n-4}{12} \int_M \Sigma_2(g, dv \otimes dv) |\nabla v|^2 v^{-n} d\mu,
\end{aligned}$$

Using the positivity assumption on A, $\widetilde{A}$ we get

$$\int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu} \geq c(M,g) \int_M \left(v^{4-n} + |\nabla v|^4 v^{-n}\right) d\mu.$$

It follows that

$$\begin{aligned}
\int_M \sigma_1\left(\widetilde{A}\right) d\widetilde{\mu} &= \int_M \sigma_1(A)\, v^{2-n} d\mu + \frac{n-2}{2} \int_M |\nabla v|^2 v^{-n} d\mu \\
&\leq c(M,g) \int_M \left(v^{2-n} + |\nabla v|^2 v^{-n}\right) d\mu \\
&\leq c(M,g) \left[\int_M v^{-n} d\mu\right]^{\frac{1}{2}} \left[\int_M \left(v^{4-n} + |\nabla v|^4 v^{-n}\right) d\mu\right]^{\frac{1}{2}} \\
&\leq c(M,g)\, \widetilde{\mu}(M)^{\frac{1}{2}} \left[\int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu}\right]^{\frac{1}{2}}.
\end{aligned}$$

□

As we see from the argument above, Theorem 1.2 follows from the integral formula (1.1) and a straightforward application of Holder's inequality. To get Theorem 1.3 we need more efforts to deal with the middle terms. For example if we want to prove (1.5) for $k = 3$ i.e.

$$\left[\int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu}\right]^2 \leq c(M,g) \int_M \sigma_1\left(\widetilde{A}\right) d\widetilde{\mu} \int_M \sigma_3\left(\widetilde{A}\right) d\widetilde{\mu}.$$

In view of (4.1), we need to bound the middle terms $\int_M \Sigma_2(\Theta, dv \otimes dv)\, v^{2-n} d\mu$ and $\int_M \Sigma_2(A, dv \otimes dv)\, v^{2-n} d\mu$ and this will be handled below.

Proof of Theorem 1.3. We may find $c(M,g,k) > 0$ such that

$$c(M,g,k)\, g - A,\ c(M,g,k)\, A - g \in \Gamma_{k+1}.$$

Using the same notation Θ as in Section 3 we have

$$\int_M \sigma_k\left(\widetilde{A}\right) d\widetilde{\mu} = \int_M \sigma_k(\Theta)\, v^{2k-n} d\mu.$$

It follows from Theorem 1.1 and (2.17) that $\int_M \sigma_k\left(\widetilde{A}\right) d\widetilde{\mu}$ is equivalent to

$$\int_M v^{2k-n} d\mu + \sum_{\substack{\alpha \in \mathbb{Z}_+^3 \\ |\alpha| = k-1}} \int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_2 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-2-2\alpha_3} d\mu.$$

Here being equivalent means the two quantities can bound each other by a positive constant $c(M,g,k)$. To continue we first observe that

$$\begin{aligned}
&\int_M v^{2k-n} d\mu \\
&\leq \left[\int_M v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \left[\int_M v^{2k-n+2} d\mu\right]^{\frac{1}{2}} \\
&\leq c(M,g,k) \left[\int_M \sigma_{k-1}(\Theta) v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \left[\int_M \sigma_{k+1}(\Theta) v^{2k-n+2} d\mu\right]^{\frac{1}{2}}.
\end{aligned}$$

Next, if $\alpha_2 \geq 1$, then

$$\begin{aligned}
&\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_2 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-2-2\alpha_3} d\mu \\
&\leq \left[\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_2 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-4-2\alpha_3} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_2 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-2\alpha_3} d\mu\right]^{\frac{1}{2}} \\
&\leq c(n,k) \left[\int_M \Sigma_{k-1}(\Theta, \alpha_1; g, \alpha_2 - 1 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-4-2\alpha_3} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_M \Sigma_{k+1}(\Theta, \alpha_1; g, \alpha_2 + 1 + \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-2\alpha_3} d\mu\right]^{\frac{1}{2}} \\
&\leq c(M,g,k) \left[\int_M \sigma_{k-1}(\Theta) v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \left[\int_M \sigma_{k+1}(\Theta) v^{2k-n+2} d\mu\right]^{\frac{1}{2}}.
\end{aligned}$$

If $\alpha_2 = 0$ and $\alpha_3 \geq 1$, then

$$\begin{aligned}
&\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3} v^{2k-n-2-2\alpha_3} d\mu \\
&\leq \left[\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3-2} v^{2k-n-2-2\alpha_3} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_M \Sigma_k(\Theta, \alpha_1; g, \alpha_3; dv \otimes dv) |\nabla v|^{2\alpha_3+2} v^{2k-n-2-2\alpha_3} d\mu\right]^{\frac{1}{2}} \\
&\leq c(n,k) \left[\int_M \Sigma_{k-1}(\Theta, \alpha_1; g, \alpha_3 - 1; dv \otimes dv) |\nabla v|^{2\alpha_3-2} v^{2k-n-2-2\alpha_3} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_M \Sigma_{k+1}(\Theta, \alpha_1; g, \alpha_3 + 1; dv \otimes dv) |\nabla v|^{2\alpha_3+2} v^{2k-n-2-2\alpha_3} d\mu\right]^{\frac{1}{2}} \\
&\leq c(M,g,k) \left[\int_M \sigma_{k-1}(\Theta) v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \left[\int_M \sigma_{k+1}(\Theta) v^{2k-n+2} d\mu\right]^{\frac{1}{2}}.
\end{aligned}$$

At last, if $\alpha_2 = \alpha_3 = 0$, then

$$\begin{aligned}
&\int_M \Sigma_k(\Theta, k-1; dv \otimes dv)\, v^{2k-n-2} d\mu \\
&= \int_{\nabla v \neq 0} \Sigma_k(\Theta, k-1; dv \otimes dv)\, v^{2k-n-2} d\mu \\
&\leq \left[\int_{\nabla v \neq 0} \Sigma_k(\Theta, k-1; dv \otimes dv)\, |\nabla v|^{-2} v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_{\nabla v \neq 0} \Sigma_k(\Theta, k-1; dv \otimes dv)\, |\nabla v|^{2} v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \\
&\leq c(n,k)\left[\int_{\nabla v \neq 0} \Sigma_k(\Theta, k-1; g)\, v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \cdot \\
&\left[\int_{\nabla v \neq 0} \Sigma_{k+1}(\Theta, k-1; g; dv \otimes dv)\, |\nabla v|^{2} v^{2k-n-2} d\mu\right]^{\frac{1}{2}} \\
&\leq c(M,g,k)\left[\int_M \sigma_{k-1}(\Theta)\, v^{2k-n-2} d\mu\right]^{\frac{1}{2}}\left[\int_M \sigma_{k+1}(\Theta)\, v^{2k-n+2} d\mu\right]^{\frac{1}{2}}.
\end{aligned}$$

Here we have used (2.17) and the fact $|\nabla v|^2 g - dv \otimes dv$ is nonnegatively definite. Theorem 1.3 follows. □

Theorem 1.3 together with an induction process imply Corollary 1.1.

It follows from Theorem 1.2 that for $n > 4$, $g \in [g_{S^n}]$ with $\sigma_1(A) > 0, \sigma_2(A) > 0$, then

$$c(n)\left[\int_{S^n} \sigma_1(A)\, d\mu\right]^2 \leq \mu(S^n)\int_{S^n} \sigma_2(A)\, d\mu. \tag{4.2}$$

Define a functional

$$I(g) = \frac{\mu(S^n)\int_{S^n} \sigma_2(A)\, d\mu}{\left[\int_{S^n} \sigma_1(A)\, d\mu\right]^2}. \tag{4.3}$$

Then (4.2) says $I(g) \geq c(n) > 0$ for all metrics $g \in [g_{S^n}]$ with $\sigma_1(A) > 0, \sigma_2(A) > 0$. A surprising fact is that g_{S^n} is a critical point of I with nonpositive second variation. To see this, we write $g = v^{-2} g_{S^n}$, $I(v) = I\left(v^{-2} g_{S^n}\right)$, then calculation shows for $h \in C^\infty(S^n, \mathbb{R})$,

$$\begin{aligned}
&I''(1)(h,h) \\
&= -\frac{4(n-1)}{n^2|S^n|}\left[\int_{S^n} |\nabla h|^2 d\mu_{S^n} - n\int_{S^n} h^2 d\mu_{S^n} + \frac{n}{|S^n|}\left(\int_{S^n} h\, d\mu_{S^n}\right)^2\right]
\end{aligned}$$

and it is nonpositive definite. In particular g_{S^n} can not be a minimizer for I and the best $c(n)$ in (4.2) satisfies

$$c(n) < I(g_{S^n}) = \frac{n-1}{2n}.$$

This suggests that the best $c(n)$ in (4.2) may not be achieved.

Acknowledgments. Research of the first author was supported in part by NSF grant DMS-0245266. Research of the second author was supported in part by NSF grant DMS-0647010 and a Sloan Research Fellowship.

References

[B] A. L. Besse. Einstein manifolds. Results in Mathematics and Related Areas, 10. Springer-Verlag, Berlin, 1987.

[G] L. Garding. An inequality for hyperbolic polynomials. *J. Math. Mech.* **8** (1959), 957–965.

[GW] P. F. Guan, G. F. Wang. Geometric inequalities on locally conformally flat manifolds. *Duke Math. J.* **124** (2004), no. 1, 177–212.

[S] R. Schneider. Convex bodies: the Brunn-Minkowski theory. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993.

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544

E-mail address: chang@math.princeton.edu

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544

E-mail address: fhang@math.princeton.edu