CONFORMAL INVARIANTS ASSOCIATED TO A MEASURE

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ABSTRACT. In this note we study some conformal invariants of a Riemannanian manifold (M^n, g) equipped with a smooth measure m. In particular, we show that there is a natural definition of the Ricci and scalar curvatures associated to such a space, both of which are conformally invariant. We also adapt the methods of Fefferman-Graham [3] and Graham-Jenne-Mason-Sparling [5] to construct families of conformally covariant operators defined on these spaces. Certain variational problems in this setting are considered, including a generalization of the Einstein-Hilbert action.

1. INTRODUCTION

By a *Riemannian measure space* (or *RM-space*) we will mean a triple (M^n, g, m) consisting of a smooth oriented manifold M^n, a Riemannian metric g, and a smooth measure m defined on M^n. In this paper we introduce some conformal invariants of an RM-space; that is, local quantities which depend on g and m but which are insensitive to conformal changes of the metric. In particular, we define conformally invariant notions of the Riemannian, Ricci, and scalar curvature associated to (M^n, g, m). When m is the Riemannian measure of g the conformally invariant curvatures agree with their classical counterparts (see Section 2).

The dependence of our invariants on the measure m is mediated by the density function f, defined by

$$dm = e^{-f} dVol(g). \tag{1.1}$$

In particular the conformally invariant curvatures are local expressions in g and f. For example, the conformally invariant scalar curvature is defined by

$$R_n^m(g) = R(g) + \frac{2(n-1)}{n}\Delta_g f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2,$$

where $R(g)$ is the scalar curvature of g. It is invariant in the sense that $R_n^m(e^{2w}g) = e^{-2w}R_n^m(g)$.

Geometric quantities associated a metric and measure are of course not new, and go back at least to the work of Bakry-Emery [1], who introduced a notion of the Ricci curvature in this setting. In Perelman's recent work on the Ricci flow ([14]) he defined the scalar curvature associated to the BE-Ricci tensor. Previosuly, Gromov [6] defined a notion of mean curvature in the presence of a measure. This was recently

Date: August 31, 2005.
Research of the first and third authors was partially supported by NSF Grant DMS-0245266.
Research of the second author was partially supported by NSF Grant DMS-0500538.

expanded on by Morgan [11], who considered generalized isoperimetric inequalities. The work of Perelman, Lott ([8], [9]), and Lott-Villani ([10]) have amply demonstrated the importance of metric/measure spaces in various mathematical contexts, from the Ricci flow to optimal transport to collapsing theory.

As we explain below, the Ricci curvature of Bakry-Emery and the associated scalar curvature defined by Perelman can be viewed as the infinite-dimensional versions of the conformally invariant Ricci and scalar curvature:

$$\lim_{n\to\infty} R_n^m(g) = R^m(g),$$

where $R^m(g)$ is the scalar curvature introduced by Perelman. The meaning of this formula is described in Section 4.

The diffeomorphism group $\mathcal{D}$ of the manifold acts on an RM-space via pull-back: $\varphi \in \mathcal{D} \Longrightarrow \varphi^*(M^n, g, m) = (M^n, \varphi^* g, \varphi^* m)$. This action gives rise to a notion of *RM-invariance*: a quantity $q = q(g,m)$ is an *RM-invariant* if $q(\varphi^* g, \varphi^* m) = \varphi^* q(g,m)$. The conformally invariant curvatures are RM-invariants, as are the Ricci curvature of Bakry-Emery and the scalar curvature of Perelman. However, if we allow $\mathcal{D}$ to act on the metric alone (keeping the measure m fixed), then in general $q(\varphi^* g, m) \neq \varphi^* q(g,m)$. This dependence on the diffeo-class of a metric is in stark contrast to the Riemannian setting, and has some interesting consequences.

We also make some preliminary observations about the variational theory of the conformally invariant curvature. For example, we introduce a generalization of the Einstein-Hilbert action, defined by

$$\mathcal{S}^m(g) = \int R_n^m(g) e^{-\frac{(n-2)}{n} f} \, dVol(g).$$

Viewed as a functional on the space of metrics (with m fixed), critical points of $\mathcal{S}^m$ are conformal to an Einstein metric. Though $\mathcal{S}^m$ is conformally invariant, it is not invariant under pull-backs of the metric by a diffeomorphism. This fact naturally leads to the constrained problem of restricting $\mathcal{S}^m$ to the orbit of g under the action of $\mathcal{D}$. Somewhat surprisingly, this is intimately related to the Yamabe problem.

In subsequent papers we will provide detailed proofs of the results announced below. We will also present some geometric applications of these ideas, and give a more systematic treatment of the conformal invariants of RM-spaces by the methods of Fefferman-Graham [3].

Acknowledgements. The authors are indebted to the referees for several excellent suggestions. The first and third author would also like to thank Robin Graham for an informative discussion about the ambient metric construction.

2. Conformally invariant curvatures

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$. We denote the Ricci tensor by $Ric(g)$, the scalar curvature by $R(g)$, and the volume form by $dVol(g)$.

Given a measure m, we define the density function f by

$$dm = e^{-f} dVol(g). \tag{2.1}$$

That is, f is the logarithm of the Radon-Nikodym derivative of $dVol(g)$ with respect to dm.

Consider a conformal change of metric $\hat{g} = e^{2w} g$. Let $\hat{f}$ denote the density function associated to $\hat{g}$; i.e.,

$$dm = e^{-\hat{f}} dVol(\hat{g}). \tag{2.2}$$

Since $dVol(\hat{g}) = e^{nw}\, dVol(g)$, it follows that

$$\hat{f} = f + nw. \tag{2.3}$$

In particular, if we define $g_m = e^{-(2/n)f} g$, then for this conformal metric we have

$$\begin{aligned} dm &= dVol(g_m), \\ \hat{f} &= 0. \end{aligned} \tag{2.4}$$

We call g_m the *canonical base metric* in [g]. It is obviously the unique metric in $[g]$ with the properties (2.4).

We now introduce conformally invariant versions of the curvature associated to an RM-space (M^n, g, m). There are several possible approaches to this, which differ only by perspective. The first is perhaps the least elegant, but we include it for two reasons: it shows why the existence of these invariants is a non-trivial fact, and it gives some insight into our initial approach to their construction.

The approach begins with a simple question: Does there exist a tensor with the same structure as the BE-Ricci tensor (or the scalar curvature of Perelman–linear in the second derivatives of f, quadratic in the first), but with the additional property of conformal invariance? The answer is 'yes', and the formula for such a tensor is given by the following Proposition:

Proposition 2.1. *Let λ be any real number. Then the tensor*

(2.5)

$$\begin{aligned} T = T(g) = Ric(g) + (\frac{n-2}{n})\nabla_g^2 f + \frac{1}{n}(\Delta_g f)g + (\frac{n-2}{n^2})df \otimes df - \frac{(n-2)}{n^2}|\nabla f|^2 g \\ + \lambda\Big[R(g) + \frac{2(n-1)}{n}\Delta_g f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2\Big]g \end{aligned}$$

is a pointwise conformal invariant. More precisely,

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad T(\hat{g}) = T(g). \tag{2.6}$$

Proof. To prove (2.5), and to understand why the existence of T is somewhat surprising, we begin by considering a tensor of the form

$$T = T(g) = Ric(g) + c_1 R(g) g + c_2 \nabla^2 f + c_3 (\Delta_g f) g + c_4 df \otimes df + c_5 |\nabla f|^2 g, \tag{2.7}$$

where $c_1, \ldots, c_5$ are arbitrary constants. Using the formulas for the curvature, Hessian, and laplacian under a conformal change of metric (see [2], p. 58), and the identity (2.3), we want to find $c_1, \ldots, c_5$ so that

$$T(\hat{g}) = T(g). \tag{2.8}$$

It turns out that the condition (2.8) imposes six linear equations on the five unknowns $c_1, \ldots, c_5$, raising the possibility that the system could be over-determined. Surprisingly, this system turns out to be *under-determined*: if we let $\lambda = c_1$ be the free parameter and solve for the remaining constants in terms of c_1, we get (2.5). □

In view of (2.5) it is natural to make the following two definitions:

Definition 2.1. *Let (M^n, g, m) be an RM-space. The* conformally invariant Ricci curvature *of (M^n, g, m) is the symmetric $(0,2)$-tensor*

(2.9)
$$Ric_n^m(g) = Ric(g) + (\frac{n-2}{n})\nabla_g^2 f + \frac{1}{n}(\Delta_g f)g + (\frac{n-2}{n^2})df \otimes df - \frac{(n-2)}{n^2}|\nabla f|^2 g,$$

where f is given by (2.1).

Definition 2.2. *Let (M^n, g, m) be an RM-space. The* conformally invariant scalar curvature *is the trace of Ric_n^m; i.e., the function*

$$R_n^m(g) = R(g) + \frac{2(n-1)}{n}\Delta_g f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2, \tag{2.10}$$

where f is given by (2.1).

By Proposition 2.1, the conformally invariant Ricci and scalar curvatures satisfy the following identities:

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad Ric_n^m(\hat{g}) = Ric_n^m(g), \tag{2.11}$$

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad R_n^m(\hat{g}) = e^{-2w} R_n^m(g). \tag{2.12}$$

It follows from the proof of Proposition 2.1 that Ric_n^m is the *unique* tensor of the form (2.7) which is conformally invariant and which reduces to the usual Ricci tensor when $f \equiv 0$. A similar uniqueness statement holds for R_n^m.

Because the scaling properties of Ric_n^m and R_n^m differ, it will be convenient to define a scale-invariant version of R_n^m. To this end we define

$$\tau^m(g) = e^{\frac{2}{n}f} R_n^m(g). \tag{2.13}$$

From (2.3) and (2.12) we see that

$$\tau^m(e^{2w} g) = \tau^m(g). \tag{2.14}$$

It is clear from the definitions that for any metric g,

$$\begin{aligned} Ric_n^{dVol(g)}(g) &= Ric(g), \\ R_n^{dVol(g)}(g) &= R(g), \\ \tau^{dVol(g)}(g) &= R(g). \end{aligned} \tag{2.15}$$

In particular, for the canonical base metric we have

$$
\begin{aligned}
Ric^m_n(g_m) &= Ric(g_m),\\
R^m_n(g_m) &= R(g_m),\\
\tau^m(g_m) &= R(g_m).
\end{aligned}
\tag{2.16}
$$

The connection between the scalar curvature and the conformal laplacian suggests a different construction for R^m_n and τ^m. To explain this, it will be helpful if we use different notation. If we write the density function as

$$dm = e^{nv} dVol(g), \tag{2.17}$$

then

$$R^m_n(g) = -c_n^{-1} e^{-\frac{n-2}{2}v} L_g(e^{\frac{n-2}{2}v}), \qquad \tau^m(g) = -c_n^{-1} e^{\frac{n+2}{2}v} L_g(e^{\frac{n-2}{2}v}), \tag{2.18}$$

where $c_n = \frac{(n-2)}{4(n-1)}$ and $L_g = \Delta_g - c_n R(g)$ is the conformal laplacian. The conformal invariance of R^m_n and τ^m can be seen as a consequence of the conformal invariance of L. Let $\hat{g} = e^{2w} g$; since

$$dm = e^{n\hat{v}} dVol(\hat{g}) = e^{n\hat{v}} e^{nw} dVol(g),$$

it follows that the density functions associated associated to g and $\hat{g}$ are related by

$$\hat{v} = v - w. \tag{2.19}$$

Then, for example,

$$
\begin{aligned}
R^m_n(\hat{g}) &= -c_n^{-1} e^{-\frac{n-2}{2}\hat{v}} L_{\hat{g}}(e^{\frac{n-2}{2}\hat{v}})\\
&= -c_n^{-1} e^{-\frac{n-2}{2}(v-w)} L_{e^{2w}g}(e^{\frac{n-2}{2}(v-w)})\\
&= -c_n^{-1} e^{-\frac{n-2}{2}v} e^{\frac{n-2}{2}w} e^{-\frac{n+2}{2}w} L_g(e^{\frac{n-2}{2}w} e^{\frac{n-2}{2}(v-w)})\\
&= -c_n^{-1} e^{-2w} e^{-\frac{n-2}{2}v} L_g(e^{\frac{n-2}{2}v})\\
&= e^{-2w} R^m_n(g).
\end{aligned}
$$

This approach can be generalized to other curvatures.

Finally, following a suggestion of the referee, we present a simple method for constructing general conformal invariants of an RM-space. Let $I = I(g)$ be a Riemannian invariant (scalar curvature, Ricci curvature, etc.). Suppose we wish to construct its conformally invariant counterpart $\mathcal{I}(g, f)$, in the sense that we want $\mathcal{I}(g, 0) = I(g)$ and $\mathcal{I}(\hat{g}, \hat{f}) = \mathcal{I}(g, f)$ for any conformal metric $\hat{g} \in [g]$. If we define

$$
\begin{aligned}
\mathcal{I}(g, f) &= I(e^{-(2/n)f} g)\\
&= I(g_m),
\end{aligned}
\tag{2.20}
$$

where g_m is the canonical base metric in $[g]$, it is clear from (2.3) that $\mathcal{I}$ satisfies the desired properties. Indeed, it follows that this is the *unique* such invariant satisfying these conditions.

At first glance it is difficult to see why the conformally invariant notions of Ricci and scalar should be natural to consider, aside from the transformation laws (2.11)-(2.12). However, we shall see that some important properties of the usual Ricci and scalar curvatures are retained by their conformally invariant counterparts.

Proposition 2.2. *Let (M^n, g, m) be an RM-space.*

(i) If $Ric_n^m(g) > 0$ (respectively, ≥ 0, $= 0$, ≤ 0, < 0), then $[g]$ contains a metric whose Ricci curvature is positive (resp., non-negative, zero, non-positive, negative).

(ii) If $R_n^m(g) > 0$ (resp., ≥ 0, $= 0$, ≤ 0, < 0), then $[g]$ contains a metric whose scalar curvature is positive (resp., non-negative, zero, non-positive, negative).

Proof. Both statements are immediate consequences of (2.16). □

The converse of Proposition 2.2 is false. For example, suppose we are given a metric g of positive Ricci curvature; then it is easy to construct a measure m so that $Ric_n^m(g)$ has negative eigenvalues on an open set.

2.1. **The Einstein condition.** Finally, let us define

$$\begin{aligned} E_n^m(g) &= Ric_n^m(g) - \frac{1}{n} R_n^m(g) g \\ &= Ric(g) - \frac{1}{n} R(g) g + (\frac{n-2}{n}) \nabla_g^2 f - (\frac{n-2}{n^2})(\Delta_g f) g \\ &\quad + (\frac{n-2}{n^2}) df \otimes df - (\frac{n-2}{n^3}) |\nabla f|^2 g, \end{aligned} \tag{2.21}$$

the *trace-free conformally invariant Ricci curvature*. Obviously E_n^m enjoys the same invariance properties as Ric_n^m. In the same way that the sign of the conformally invariant Ricci curvature detects the existence of a conformal metric with the same sign, the tensor E_n^m detects the existence of a conformal Einstein metric:

Proposition 2.3. *Let (M^n, g) be a smooth Riemannian manifold of dimension $n \geq 3$.*

(i) If m is a measure for which $E_n^m(g) = 0$, then $[g]$ contains an Einstein metric, namely, the canonical base metric.

(ii) If g is an Einstein metric and m is a measure for which $E_n^m(g) = 0$, then either $dm = \lambda dVol(g)$ for some constant $\lambda > 0$, or (M^n, g) is homothetically isometric to (S^n, g_c), where g_c denotes the round metric, and $dm = \lambda \varphi^* dVol(g_c)$ for some conformal map φ and constant $\lambda > 0$.

The proof of part *(ii)* follows from Obata's uniqueness theorem for conformally Einstein metrics of constant scalar curvature (see [13]).

One consequence of the classical second Bianchi identity is that the scalar curvature of an Einstein metric must be constant. An analogous result holds in the setting of RM-spaces: an RM-space with $n \geq 3$ and $E_n^m(g) = 0$ has constant τ^m-curvature. Note that the condition $\tau^m(g) = const.$ is conformally invariant, while by (2.12) the condition $R_n^m(g) = const.$ is not.

3. Conformally covariant operators on RM-spaces

Using the constructions of Fefferman-Graham [3] and Graham-Jenne-Mason-Sparling [5] one may construct families of conformally covariant differential operators associated to an RM-space. Moreover, the conformally invariant scalar and Ricci arise naturally in these constructions.

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$. A metrically defined differential operator $\mathcal{A} = \mathcal{A}_g$ is said to be *conformally covariant of bi-degree* (a, b) if it obeys the following transformation under a conformal change of metric $\hat{g} = e^{2w} g$:

$$\mathcal{A}_{\hat{g}}(\psi) = e^{-bw} \mathcal{A}_g(e^{aw}\psi) \tag{3.1}$$

for some constants a, b and all $\psi \in C^\infty(M^n)$. For example, when $n = 2$, $\mathcal{A}_g = \Delta_g$ is conformally covariant with $a = 0$ and $b = 2$. More generally, when $n \geq 3$ the conformal laplacian $\mathcal{A}_g = L_g = \Delta_g - \frac{(n-2)}{4(n-1)} R(g)$ is conformally covariant with $a = (n-2)/2$ and $b = (n+2)/2$.

In [5], Graham-Jenne-Mason-Sparling constructed conformally covariant operators P_k for all positive integers k when n is odd, and for $1 \leq k \leq n/2$ when n is even, with $a = (n-2k)/2$ and $b = (n+2k)/2$. The principal part of P_k is given by $(\Delta)^k$; when $k = 1$ then P_1 is just the conformal laplacian. These operators were derived from the ambient metric construction of Fefferman-Graham which is briefly described below. Given an RM-space (M^n, g, m), we can modify the method of [5] to derive a family of operators $\mathcal{A}^m_g$ satisfying

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad \mathcal{A}^m_{\hat{g}}(\psi) = e^{-bw} \mathcal{A}^m_g(e^{aw}\psi), \tag{3.2}$$

for some constants a, b and all $\psi \in C^\infty(M^n)$.

Theorem 3.1. *Let (M^n, g, m) be an RM-space with $n \geq 3$. Let k be a positive integer; if n is even we assume in addition that $1 \leq k \leq n/2$. For $\alpha \in \mathbf{R}$, denote $\beta_k(\alpha) = (n\alpha - n + 2k)/2$. Then, given any $\alpha \in \mathbf{R}$ there is an operator $P^m_{\alpha,k}$ satisfying (3.2) with $a = -\beta_k(\alpha)$ and $b = 2k - \beta_k(\alpha)$, the leading term of which is given by*

$$P^m_{\alpha,k} = \big(\Delta_g - \alpha\langle \nabla f, \nabla \,\cdot\, \rangle\big)^k + \cdots$$

When $\alpha = 0$ the operator $P^m_{\alpha,k}$ coincides with P_k. For $k = 1$ we have the formula

$$P^m_{\alpha,k}(\psi) = \Delta_g \psi - \alpha \langle \nabla f, \nabla \psi \rangle + \frac{n\alpha - n + 2}{2(n-2)} \big(\alpha \Delta_g f + \frac{n\alpha + n - 2}{2(n-1)} R(g)\big)\psi. \tag{3.3}$$

As in [5], our operators are constructed by an inductive algorithm; when k becomes large the formulas become increasingly complicated. In fact, Graham-Jenne-Mason-Sparling presented two (equivalent) ways of deriving their operators. We will briefly describe one of their methods, indicating the modifications necessary to produce the measure-dependent operators $P^m_{\alpha,k}$.

To begin, given a Riemannian manifold (M^n, g), let $\mathcal{G} \subset S^2 T^* M^n$ denote the ray bundle consisting of metrics in the conformal class of g. Fixing a representative $g \in [g]$ determines a fiber variable t on $\mathcal{G}$, by writing a general point in $\mathcal{G}$ in the form $(x, t^2 g(x))$. If $\{x^i\}$ are local coordinates on M^n, the coordinate system (t, x^i) on $\mathcal{G}$

extends to a coordinate system (t, x^i, ρ) on $\tilde{\mathcal{G}} = \mathcal{G} \times (-1, 1)$, where ρ is a defining function for $\mathcal{G}$, homogeneous of degree 0 (see [3] for details). Using these coordinates we can define the *ambient metric* $\tilde{g}$ on $\tilde{\mathcal{G}}$ by

$$\tilde{g} = 2\rho dt^2 + 2t dt\, d\rho + t^2 g_{ij}(x,\rho) dx^i dx^j, \tag{3.4}$$

where $g_{ij}(x, 0) = g_{ij}(x)$ is the given representative of $[g]$. For $\rho \neq 0$ the Taylor expansion of $g_{ij}(x, \rho)$ is determined by formally solving the Einstein equation

$$Ric(\tilde{g}) = 0.$$

Let $\delta_s : \mathcal{G} \to \mathcal{G}$ denote the dilations $\delta_s(g) = s^2 g$, with $s > 0$. Functions on $\mathcal{G}$ which are homogeneous of degree β with respect to δ_s are known as *conformal densities of weight* β. Given a density ϕ of weight β, consider the problem of extending ϕ to a harmonic function on $\tilde{\mathcal{G}}$ with the same homogeneity. That is, we want to find the formal power series solution of

$$\tilde{\Delta}(t^\beta \phi) = 0. \tag{3.5}$$

The operators of [5] arise as the obstruction to formally solving (3.5) with $\beta + \frac{1}{2}n = k = 1, 2, 3, \ldots$

Given an RM-space (M^n, g, m) we can also construct the ambient metric $\tilde{g}$, but we need to extend the density function f associated to m as well.

Lemma 3.1. *Let (M^n, g, m) be a Riemannian measure space with $dm = e^{-f} dVol(g)$. Let k be a positive integer; if n is even we assume in addition that $1 \leq k \leq n/2$. Then there is an extension $\tilde{f} : \tilde{\mathcal{G}} \to \mathbf{R}$ with $\tilde{f}(t, x, \rho) = f(x, \rho) + n \log t$, such that $f(x, 0) = f(x)$ for all $x \in M^n$, and $\tilde{f}$ satisfies*

$$\tilde{\Delta}\tilde{f} = O(\rho^k)$$

near $\mathcal{G}$ on $\tilde{\mathcal{G}}$.

To derive the operators $P^m_{\alpha,k}$ we replace (3.5) with

$$\tilde{\Delta}(t^\beta \phi) - \alpha \langle \tilde{\nabla} \tilde{f}, \tilde{\nabla}(t^\beta \phi) \rangle = 0. \tag{3.6}$$

Again, the operators arise as the obstruction to writing a formal power series solution of (3.6).

Remarks.

(1) The conformally invariant curvatures can also be defined in terms of the extension $\tilde{f}$. For example, $R^m_n(g)$ is given by

$$R^m_n(g) = -\frac{(n-1)(n-2)}{n^2} |\tilde{\nabla}\tilde{f}|^2 \Big|_{M^n}. \tag{3.7}$$

(2) The referee pointed out another possible construction of conformally covariant operators on RM-spaces, by using the operators P_k of [5]. Letting

$$G^m_{\alpha,k}(\phi) = e^{\frac{\alpha f}{2}} P_k(e^{-\frac{\alpha f}{2}} \phi),$$

it is easy to see that these operators satisfy the same conformal covariance as the operators $P^m_{\alpha,k}$ in Theorem 3.1. Interestingly, in general $G^m_{\alpha,k}$ and $P^m_{\alpha,k}$ do not agree. For example, when $k = 1$ they differ by a multiple $C_{\alpha,n}$ of $R^m_n(g)$.

4. Relation to the curvatures of Bakry-Emery and Perelman

There is a surprising relationship between the conformally invariant Ricci and scalar curvatures, and the curvatures defined by Bakry-Emery [1] and Perelman [14]. To recall the definition of these latter invariants, once again we let (M^n, g, m) denote an RM-space, and let f be defined by (2.1):

$$dm = e^{-f} dVol(g).$$

The Bakry-Emery Ricci curvature is defined by

$$Ric^m(g) = Ric(g) + \nabla^2_g f. \tag{4.1}$$

Later, Perelman introduced a notion of the scalar curvature associated to the BE-Ricci tensor:

$$R^m(g) = R(g) + 2\Delta f - |\nabla f|^2. \tag{4.2}$$

In particular, R^m is *not* the trace of Ric^m. The definition of Ric^m arose in the analysis of infinite-dimensional diffusion processes. Subsequently Perelman pointed out the interpretation of both Ric^m and R^m as the curvature terms in various "weighted" Weitzenböck formulas (see [14], [8]).

As we now explain, these quantities can also be viewed as the infinite-dimensional limit of their conformally invariant counterparts. Arguing informally, notice if we let the dimension $n \to \infty$ in the formula (2.9), in the limit we obtain the BE-Ricci curvature:

$$\begin{aligned} \lim_{n\to\infty} Ric^m_n(g) \text{ " = " } & Ric(g) + \nabla^2_g f \\ &= Ric^m(g). \end{aligned}$$

Similarly, letting $n \to \infty$ in the definition of R^m_n gives the scalar curvature defined by Perelman:

$$\begin{aligned} \lim_{n\to\infty} R^m_n(g) \text{ " = " } & R(g) + 2\Delta_g f - |\nabla f|^2 \\ &= R^m(g). \end{aligned}$$

In these formulas we used quotation marks to emphasize the fact that the process of letting the dimension go to infinity is not meant to be taken literally. However, we can make these formal observations more concrete by a routine construction.

Let $(\mathbf{T}^{d-n}, ds^2)$ denote the flat $(d-n)$-dimensional torus, and consider the product manifold $N^d = M^n \times \mathbf{T}^{d-n}$ with the product metric $h = g + ds^2$. To define the conformally invariant Ricci and scalar curvatures on N^d we need to define a measure, so we take the product measure $d\mu = dm \times dVol(ds^2)$. Note that if $dm = e^{-f} dVol(g)$

and $d\mu = e^{-\tilde{f}} dVol(h)$, it follows that $\tilde{f} = f$. Therefore, given tangent vectors $X, Y \in TM^n$, if we denote their lift to TN^d also by X, Y, then by the definition of Ric_n^m

$$\begin{aligned}
\lim_{d\to\infty} Ric_d^{\mu}(h)(X,Y) &= \lim_{d\to\infty} \Big\{ Ric(h)(X,Y) + (\frac{d-2}{d})\nabla_h^2 \tilde{f}(X,Y) + \frac{1}{d}(\Delta_h \tilde{f})h(X,Y) \\
&\qquad + (\frac{d-2}{d^2})(d\tilde{f}\otimes d\tilde{f})(X,Y) - \frac{(d-2)}{d^2}|\nabla \tilde{f}|^2 h(X,Y)\Big\} \\
&= \lim_{d\to\infty} \Big\{ Ric(g)(X,Y) + (\frac{d-2}{d})\nabla_g^2 f(X,Y) + \frac{1}{d}(\Delta_g f)g(X,Y) \\
&\qquad + (\frac{d-2}{d^2})df(X)df(Y) - \frac{(d-2)}{d^2}|\nabla f|^2 g(X,Y)\Big\} \\
&= Ric(g)(X,Y) + \nabla^2 f(X,Y) \\
&= Ric^m(g)(X,Y).
\end{aligned}$$

Similarly,

$$\lim_{d\to\infty} R_d^m(h) = R^m(g).$$

Remark. This interpretation of the BE-Ricci tensor and the scalar curvature as infinite-dimensional limits of conformal invariants can be used to define the notion of the full curvature tensor associated to m. The details will be provided in a forthcoming paper.

5. $\mathcal{D}$-actions on RM-spaces

As we noted in the introduction, the diffeomorphism group $\mathcal{D}$ of the manifold acts on an RM-space (M^n, g, m) by pull-back:

$$\varphi^*(M^n, g, m) = (M^n, \varphi^* g, \varphi^* m). \tag{5.1}$$

This leads to a natural notion of invariance:

Definition 5.1. *A covariant tensor field $T = T(g,m)$ on M^n is a local* RM-invariant *if for every diffeomorphism $\varphi : M^n \to M^n$,*

$$T(\varphi^* g, \varphi^* m) = \varphi^* T(g,m). \tag{5.2}$$

What are examples of local RM-invariants? The first and most important is the density function f, along with its covariant derivatives. Consequently, the conformally invariant curvatures Ric_n^m, R_n^m, and τ^m are all RM-invariants, as well as the BE Ricci tensor Ric^m and the scalar curvature R^m of Perelman.

In certain problems we want to restrict the action of the diffeomorphism to the metric alone, leaving the measure fixed. Using (5.2), it is easy to see the effect of such an action: If $T = T(g,m)$ is a local RM-invariant, then for any $\varphi \in \mathcal{D}$

$$T(\varphi^* g, m) = \varphi^* T(g, (\varphi^{-1})^* m). \tag{5.3}$$

There are two ways of interpreting this formula. One is to view pull-backs of the metric as determining a change in the measure, $m \mapsto (\varphi^{-1})^* m$. By a Theorem of Moser, this is actually an equivalence; that is, all measures can be realized in this manner:

Proposition 5.1. *Let (M^n, g, m) be an RM-space and $T = T(g,m)$ a local RM-invariant. Given a measure μ with the same total mass, there is a diffeomorphism $\varphi \in \mathcal{D}$ with*

$$T(\varphi^* g, m) = \varphi^* T(g, \mu). \tag{5.4}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a measure μ (with the same total mass) satisfying (5.4).

Proof. Since μ and m have the same total mass, By Moser's theorem [12] there is a diffeomorphism $\varphi \in \mathcal{D}$ such that $\mu = (\varphi^{-1})^* m$. Therefore, (5.4) follows from (5.3).

The converse is obvious; just take $\mu = (\varphi^{-1})^* m$. □

Another interpretation of (5.3) comes from comparing the conformally invariant curvatures with their classical counterparts. The Ricci curvature of a Riemannian metric is not conformally invariant, but is invariant under pull-back of the metric by a diffeomorphism. The reverse is true for Ric^m_n; moreover, this "exchange of invariance" actually has a precise description:

Proposition 5.2. *Let (M^n, g, m) be an RM-space.*

(i) Given $\hat{g} \in [g]$, there is a diffeomorphism $\varphi \in \mathcal{D}$ such that

$$Ric(\varphi^* \hat{g}) = Ric^m_n(\varphi^* g). \tag{5.5}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]$ such that (5.5) holds.

(ii) A similar statement holds for τ^m : Given $\hat{g} \in [g]$, there is a diffeomorphism $\varphi \in \mathcal{D}$ such that

$$R(\varphi^* \hat{g}) = \tau^m(\varphi^* g). \tag{5.6}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]$ such that (5.6) holds.

An analogous result holds for the conformally invariant scalar curvature R^m_n, but one needs to take into account the scaling.

6. Variational formulas

Let $\mathcal{M} = \mathcal{M}(M^n)$ denote the space of Riemannian metrics on M^n, and $\mathcal{P}$ the space of all probability measures. An *RM-functional* is a (differentiable) mapping $\mathcal{F} : \mathcal{M} \times \mathcal{P} \to \mathbb{R}$ which is invariant under the action of $\mathcal{D}$ in (5.2):

$$\mathcal{F}(\varphi^* g, \varphi^* m) = \mathcal{F}(g, m). \tag{6.1}$$

The basic example of an RM-functional is the integral of a scalar local RM-invariant $q = q(g, m)$:

$$\mathcal{F}(g,m) = \int q(g,m)\; dVol(g). \tag{6.2}$$

Of particular interest is the case where q is an expression involving the conformally invariant curvature.

As in the previous section, we will also be interested in the problem of fixing a measure m while varying the metric:

$$\mathcal{F}^m(\cdot) = \mathcal{F}(\cdot, m). \tag{6.3}$$

A constrained version of this problem arises from restricting $\mathcal{F}^m$ to the orbit of the metric g under the action of $\mathcal{D}$.

6.1. **The total τ^m-curvature.** Many classical variational problems from differential geometry and mathematical physics have counterparts in the setting of RM-spaces. For example, one can consider the RM-functional generalizing the total scalar curvature:

$$\begin{aligned} \mathcal{S}^m(g) &= \int R_n^m(g)\; d\nu_n \\ &= \int \tau^m(g)\; dm, \end{aligned} \tag{6.4}$$

where $d\nu_n = e^{-\frac{(n-2)}{n}f} dVol(g)$. When $n = 2$ then (6.4) reduces to the Gauss-Bonnet integral.

To study the variational properties of $\mathcal{S}^m$, we fix a measure $m \in \mathcal{P}$ and view $\mathcal{S}^m$ as a functional on the space of Riemannian metrics. As we shall see, the parallels with the total scalar curvature are considerable. We begin with a first variation calculation:

Theorem 6.1. *(i) $\mathcal{S}^m$ is conformally invariant:*

$$\mathcal{S}^m(e^{2w}g) = \mathcal{S}^m(g).$$

(ii) The $L^2(d\nu_n)$-gradient of $\mathcal{S}^m$ is (minus) the conformally invariant trace-free Ricci tensor: i.e.,

$$\begin{aligned} \mathcal{S}^m(g)'h &= \frac{d}{dt}\Big|_{t=0} \mathcal{S}^m(g+th) \\ &= \int -\langle h, E_n^m(g)\rangle\; d\nu_n. \end{aligned} \tag{6.5}$$

In particular, a metric g is critical for $\mathcal{S}^m$ if and only if it is conformal to an Einstein metric.

There is a well known mini-max scheme associated to the total scalar curvature functional, in which one minimizes in each conformal class (the Yamabe problem) then maximizes over all conformal classes (see [15]). There is an analogous mini-max scheme for $\mathcal{S}^m$, but due to its conformal invariance it is clear that the Yamabe problem (i.e., the "mini-" part) needs to be modified. Since Proposition 5.2 showed that conformal deformations of the scalar curvature are equivalent to deformations of τ^m under pull-back, apparently the appropriate replacement for "conformal class" is "diffeomorphism class"; that is, one should consider

$$\sigma(M^n, m) \equiv \sup_{g\in\mathcal{M}} \inf_{\varphi\in\mathcal{D}} \mathcal{S}^m(\varphi^* g). \tag{6.6}$$

As the next result shows, this value turns out to be the *same* as the σ-constant defined by the scalar curvature:

$$\sigma(M^n) = \sup_{g\in\mathcal{M}} Y(M^n, [g]), \tag{6.7}$$

where $Y(M^n, [g])$ denotes the Yamabe invariant of the conformal class of g.

Theorem 6.2. *Let (M^n, g, m) be a Riemannian measure space, where for simplicity we assume m is a probability measure. Then*

$$\inf_{\varphi\in\mathcal{D}} \mathcal{S}^m(\varphi^* g) = Y(M^n, [g]). \tag{6.8}$$

In particular, we have

$$\sigma(M^n, m) = \sigma(M^n), \tag{6.9}$$

independent of the measure m.

Moreover, the infimum on the left-hand side of (6.8) is attained by a diffeomorphism $\varphi_0 \in \mathcal{D}$ if and only if there is a conformal metric which attains the Yamabe invariant.

Like the Yamabe problem, critical points of the constrained problem satisfy a scalar equation:

Proposition 6.1. *The metric g is critical for $\mathcal{S}^m\big|_{\mathcal{D}(g)}$ if and only if $\tau^m(g) = const.$*

The proof of Proposition 6.1 relies on the fact that the formal tangent space to $\mathcal{D}(g)$ is given by Lie derivatives of the metric with respect to smooth vector fields. Consequently, it is natural to introduce a linear functional $\mathcal{G}^m(g) : \mathfrak{X}(M^n) \to \mathbb{R}$, where $\mathfrak{X}(M^n)$ denotes the Lie algebra of vector fields on M^n:

$$\mathcal{G}^m(g)(X) = \int X\tau^m(g)\, dm. \tag{6.10}$$

This is the obvious extension of the Futaki invariant from Kähler geometry [4], or the Kazadan-Warner integral from conformal geometry [7].

Proposition 6.2. (i) *The functional $\mathcal{G}^m(g)$ is conformally invariant: if $\hat{g} = e^{2w}g$, then*

$$\mathcal{G}^m(\hat{g}) = \mathcal{G}^m(g).$$

(ii) If X is a conformal Killing vector field, then $\mathcal{G}^m(g)(X) = 0$.

(iii) If $E_n^m(g) = 0$, then $\mathcal{G}^m(g) = 0$. In particular, unless $\mathcal{G}^m(g) \equiv 0$ for some measure m, the conformal class of g does not contain an Einstein metric.

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