CONFORMAL INVARIANTS ASSOCIATED TO A MEASURE, I: POINTWISE INVARIANTS

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ABSTRACT. In this paper we study Riemannanian manifolds (M^n, g) equipped with a smooth measure m. In particular, we show that Riemannian invariants of (M^n, g) give rise to conformal densities of the *Riemannian measure space* (M^n, g, m). This leads to a natural definition of the Ricci and scalar curvatures of RM-spaces, both of which are conformally invariant. We also study some natural variational integrals.

1. INTRODUCTION

By a *Riemannian measure space* (or *RM-space*) we will mean a triple (M^n, g, m) consisting of a smooth oriented manifold M^n, a Riemannian metric g, and a smooth measure m defined on M^n. In this paper we show that every Riemannian invariant of (M^n, g) gives rise to a conformal density of the RM-space (M^n, g, m). In particular, we define conformally invariant notions of the Riemannian, Ricci, and scalar curvature associated to (M^n, g, m). When m is the Riemannian measure of g the conformally invariant curvatures agree with their classical counterparts (see Section 3).

In the Riemannian setting, the construction of conformal invariants is an important problem with connections to many fields in mathematics and physics; see the articles of Fefferman-Graham ([FG85], [FG02]) and Graham-Hirachi ([GH05]), for example. As we shall see, the additional structure of a measure gives rise to many conformal invariants which are not, in general, Riemannian invariants.

The dependence of our invariants on the measure m is mediated by the density function f, defined by

$$dm = e^{-f} dVol(g). \tag{1.1}$$

In particular the conformally invariant curvatures are local expressions in g and f. For example, the conformally invariant scalar curvature is defined by

$$\begin{aligned} R_n^m(g) &= e^{-\frac{2}{n}f} R(e^{-\frac{2}{n}f} g) \\ &= R(g) + \frac{2(n-1)}{n} \Delta_g f - \frac{(n-1)(n-2)}{n^2} |\nabla f|^2, \end{aligned}$$

where $R(g)$ is the scalar curvature of g. It is invariant in the sense that $R_n^m(e^{2w} g) = e^{-2w} R_n^m(g)$.

Geometric quantities associated a metric and measure are of course not new, and go back at least to the work of Bakry-Émery [BE85], who introduced a notion of the

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Ricci curvature in this setting. In Perelman's recent work on the Ricci flow ([Per02]) he defined the scalar curvature associated to the BE-Ricci tensor. As we explain below, the Ricci curvature of Bakry-Émery and the associated scalar curvature defined by Perelman can be viewed as the infinite-dimensional versions of the conformally invariant Ricci and scalar curvature.

The diffeomorphism group $\mathcal{D}$ of the manifold acts on an RM-space via pull-back: $\varphi \in \mathcal{D} \Rightarrow \varphi^*(M^n, g, m) = (M^n, \varphi^* g, \varphi^* m)$. This action gives rise to a notion of *RM-invariance*: $T = T(g,m)$ is an *RM-invariant* if $T(\varphi^* g, \varphi^* m) = \varphi^* T(g,m)$. The conformally invariant curvatures are RM-invariants, as are the Ricci curvature of Bakry-Émery and the scalar curvature of Perelman. However, if we allow $\mathcal{D}$ to act on the metric alone (keeping the measure m fixed), then in general $T(\varphi^* g, m) \neq \varphi^* T(g,m)$. This dependence on the diffeo-class of a metric leads to a fundamental result (Theorem 5.2): the behavior of a Riemannian invariant under conformal deformations of the metric is equivalent to the action of $\mathcal{D}$ (via conjugation) on the associated density.

Using this equivalence we study a natural class of variational integrals defined by scalar conformal densities. An example of particular interest is the (weighted) integral of the conformally invariant scalar curvature:

$$\mathcal{S}^m[g] = \int R_n^m(g) e^{-\frac{2}{n}f} \ dm.$$

Viewed as a functional on the space of metrics (with m fixed), critical points of $\mathcal{S}^m$ are conformal to an Einstein metric. We can also consider the constrained problem of restricting $\mathcal{S}^m$ to the orbit of g under the action of $\mathcal{D}$. By Theorem 5.2, this is equivalent to the Yamabe problem.

Many of the results of this paper were announced in [CGY06]. In the sequel to this paper [CGY07] we will construct conformally covariant operators on RM-spaces, based on the methods of Fefferman-Graham ([FG85]). These are analogues of the $GJMS$-operators ([GJMS92]) defined on Riemannian manifolds. They also give rise to a family of "scalar curvatures" associated to an RM-space, and the conformally invariant curvatures of this paper play an important role in the construction.

2. RM-invariants

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$. We denote the Riemannian curvature tensor by $Riem$, the Ricci tensor by Ric (or $Ric(g)$), and the scalar curvature by R (or $R(g)$). Finally, we denote the volume form of g by $dVol(g)$.

Let $\mathcal{G} \subset S^2 T^* M^n$ denote the ray bundle consisting of metrics in the conformal class of g; we also denote the conformal class by $[g] = \{\hat{g} = e^{2w} g \mid w \in C^\infty(M^n)\}$. If $\delta_s : \mathcal{G} \to \mathcal{G}$ denotes the dilations $\delta_s(g) = s^2 g$, then sections of $\mathcal{G}$ which are homogeneous of degree β with respect to δ_s are called *conformal densities of weight β*.

Suppose m is a smooth measure defined on M^n. We define the *density function associated to m* by

$$dm = e^{-f} dVol(g). \tag{2.1}$$

That is, f is the logarithm of the Radon-Nikodym derivative of $dVol(g)$ with respect to dm. Given a conformal metric $\hat{g} = e^{2w}g$, let $\hat{f}$ denote the density function associated to $\hat{g}$; i.e.,

$$dm = e^{-\hat{f}} dVol(\hat{g}). \tag{2.2}$$

Since $dVol(\hat{g}) = e^{nw}\, dVol(g)$, it follows that

$$\hat{f} = f + nw. \tag{2.3}$$

The measure m determines a canonical choice of conformal metric whose density function is zero; we call this the *canonical base metric*. The precise formulation of this fact is given by the following lemma:

Lemma 2.1. *Let (M^n, g, m) be an RM-space. Then there is a unique metric $g_m \in [g]$ with the property*

$$dm = dVol(g_m). \tag{2.4}$$

We call g_m the canonical base metric in $[g]$.

Proof. Let f be the density function associated to g:

$$dm = e^{-f} dVol(g).$$

If we take

$$g_m = e^{-\frac{2}{n}f} g, \tag{2.5}$$

then from formula (2.3) it follows that $dm = dVol(g_m)$. The uniqueness of g_m is clear. ☐

As we noted in the introduction, given an RM-space the diffeomorphism group $\mathcal{D}$ of the manifold acts on (M^n, g, m) by pull-back:

$$\varphi^*(M^n, g, m) = (M^n, \varphi^* g, \varphi^* m). \tag{2.6}$$

This leads to a natural notion of invariance:

Definition 2.1. *A covariant tensor field $T = T(g, m)$ on M^n is a local* RM-invariant *if for every diffeomorphism $\varphi : M^n \to M^n$,*

$$T(\varphi^* g, \varphi^* m) = \varphi^* T(g, m). \tag{2.7}$$

We will usually omit the adjective "local", unless the context requires it. We can also define RM-invariance for contravariant tensors:

$$T(\varphi^* g, \varphi^* m) = \varphi_* T(g, m),$$

and thus extend the notion to tensor fields of all types.

Example 1. The density function f associated to a measure, along with its covariant derivatives, is an example of a local RM-invariant:

Lemma 2.2. *Let (M^n, g, m) be a Riemannian measure space, and let f be given by*

$$dm = e^{-f} dVol(g). \tag{2.8}$$

Then f is an RM-invariant. In addition, $\nabla^k f$, all tensor products of the form

$$\sum_{I=(i_1,\ldots,i_p)} c_I \nabla^{i_1} f \otimes \cdots \otimes \nabla^{i_p} f, \tag{2.9}$$

and all contractions of (2.9) are RM-invariants.

Proof. Given a diffeomorphism $\varphi : M^n \to M^n$, write

$$\varphi^* dm = e^{-f_\varphi} dVol(\varphi^* g). \tag{2.10}$$

Comparing (2.8) and (2.10) and using the invariance of the Riemannian volume form under pull-back we conclude

$$f_\varphi = \varphi^* f = f \circ \varphi. \tag{2.11}$$

Thus, f is an RM-invariant. From this fact it follows that all covariant derivatives of f, and all tensor powers of its covariant derivatives along with their contractions, are also RM-invariants.

□

Example 2. The Bakry-Émery Ricci tensor and the scalar curvature introduced by Perelman are both examples of RM-invariants. This follows from Lemma 2.2, since the definitions are

$$\begin{aligned} Ric^m(g) &= Ric(g) + \nabla^2 f, \qquad \text{(Bakry-Émery)} \\ R^m(g) &= R(g) + 2\Delta f - |\nabla f|^2. \qquad \text{(Perelman)} \end{aligned} \tag{2.12}$$

Example 3. Given a Riemannian invariant, one can construct various RM-invariants through multiplication by a weight. More precisely, the (multiplicative) group of smooth positive functions $\mathcal{C}_+ = \{\gamma \in C^\infty(M^n) \mid \gamma > 0\}$ acts on a metric by conformal deformation:

$$\gamma : g \mapsto \gamma \cdot g. \tag{2.13}$$

Of course, $\mathcal{C}_+$ also acts on sections of the tensor bundles $\otimes^k T^*M^n \otimes^\ell TM^n$ by multiplication. Suppose $I = I(g)$ is a Riemannian invariant. To simplify matters we assume I is a scalar function or more generally a covariant tensor field; thus

$$I(\varphi^* g) = \varphi^* I(g). \tag{2.14}$$

Definition 2.2. *Given a pair of real numbers (α, β), let*

$$\mathcal{I}^m_{\alpha,\beta}(g) = e^{\beta f} I(e^{\alpha f} g) \tag{2.15}$$

That is, $\mathcal{I}^m_{\alpha,\beta}$ is obtained from I through 'conjugation' by the action of $\mathcal{C}_+$. It follows from Lemma 2.2 that $\mathcal{I}^m_{\alpha,\beta}$ is an RM-invariant. As we shall see, an example of particular importance is when $\alpha = -2/n$ and $\beta = 0$.

2.1. **Conformal densities.** An RM-conformal density is a conformal density (in the usual sense) which is also an RM-invariant. Using the construction of RM-invariants in Example 3 above, we now show that every Riemannian invariant induces such a conformal density, of any specified weight:

Proposition 2.1. *Let (M^n, g, m) be a Riemannian measure space with density function f:*

$$dm = e^{-f} dVol(g). \tag{2.16}$$

Suppose $I = I(g)$ is Riemannian invariant. Then for any real number β, the RM-invariant

$$\mathcal{I}^m_{-2/n,\beta/n}(g) = e^{(\beta/n)f} I(e^{-\frac{2}{n}f} g) \tag{2.17}$$

is a conformal density of weight β.

Proof. Let $\hat{g} = e^{2w} g$; by (2.3)

$$\begin{aligned} \mathcal{I}^m_{-2/n,\beta/n}(\hat{g}) &= e^{(\beta/n)\hat{f}} I(e^{-\frac{2}{n}\hat{f}} \hat{g}) \\ &= e^{(\beta/n)(f+nw)} I(e^{-\frac{2}{n}(f+nw)} e^{2w} g) \\ &= e^{\beta w} e^{(\beta/n)f} I(e^{-\frac{2}{n}f} g) \\ &= e^{\beta w} \mathcal{I}^m_{-2/n,\beta/n}(g). \end{aligned}$$

It follows that (2.17) defines a conformal density of weight β. □

It is clear from the definitions that

$$\mathcal{I}^{dVol(g)}_{\alpha,\beta}(g) = I(g). \tag{2.18}$$

In particular, for the canonical base metric we have

$$\mathcal{I}^m_{\alpha,\beta}(g_m) = I(g). \tag{2.19}$$

3. Conformally invariant curvatures

If we take I to be the curvature of a Riemannian metric (scalar, Ricci, etc.), then by Proposition 2.1 we can construct conformally invariant 'curvatures' associated to an RM-space:

Definition 3.1. *Let (M^n, g, m) be an RM-space. The* conformally invariant scalar curvature *of (M^n, g, m) is given by*

$$\begin{aligned} R^m_n(g) &= e^{-\frac{2}{n}f} R(g_m) \\ &= R(g) + \frac{2(n-1)}{n}\Delta_g f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2, \end{aligned} \tag{3.1}$$

where f is given by (2.16) and g_m is the canonical base metric. R^m_n is a conformal density of weight -2: If $\hat{g} = e^{2w} g$, then

$$R^m_n(\hat{g}) = e^{-2w} R^m_n(g). \tag{3.2}$$

In particular, R^m_n shares the same scaling properties as the scalar curvature.

Notice when $n = 2$, the conformally invariant scalar curvature is given by

$$R^m_2(g) = R(g) + \Delta f. \tag{3.3}$$

It will be convenient to define a scale-invariant version of R^m_n.

Definition 3.2. *Let (M^n, g, m) be an RM-space. The* τ^m-curvature *of (M^n, g, m) is the quantity*

$$\begin{aligned} \tau^m(g) &= e^{\frac{2}{n}f} R^m_n(g) \\ &= R(g_m). \end{aligned} \tag{3.4}$$

$\tau^m(g)$ is a conformal density of weight 0: If $\hat{g} = e^{2w} g$, then

$$\tau^m(\hat{g}) = \tau^m(g). \tag{3.5}$$

Definition 3.3. *Let (M^n, g, m) be an RM-space. The* conformally invariant Ricci curvature *of (M^n, g, m) is given by*

$$\begin{aligned} Ric^m_n(g) &= Ric(g_m) \\ &= Ric(g) + (\frac{n-2}{n})\nabla^2_g f + \frac{1}{n}(\Delta_g f)g + (\frac{n-2}{n^2})df \otimes df - \frac{(n-2)}{n^2}|\nabla f|^2 g. \end{aligned} \tag{3.6}$$

Ric^m_n is a conformal density of weight 0: If $\hat{g} = e^{2w} g$, then

$$Ric^m_n(\hat{g}) = Ric^m_n(g). \tag{3.7}$$

In particular, Ric^m_n shares the same scaling properties as the Ricci curvature.

Note that $R^m_n(g)$ is the contraction of $Ric^m_n(g)$. Moreover, by (2.18)

$$\begin{aligned} Ric^{dVol(g)}_n(g) &= Ric(g), \\ R^{dVol(g)}_n(g) &= R(g), \\ \tau^{dVol(g)}(g) &= R(g). \end{aligned} \tag{3.8}$$

for any metric g. In particular, by (2.19), for the canonical base metric g_m we have

$$\begin{aligned} Ric^m_n(g_m) &= Ric(g_m), \\ R^m_n(g_m) &= R(g_m), \\ \tau^m(g_m) &= R(g_m). \end{aligned} \tag{3.9}$$

From these identities we see that certain properties of the usual Ricci and scalar curvatures are retained by their conformally invariant counterparts:

Proposition 3.1. *Let (M^n, g, m) be an RM-space.*

(i) If $Ric^m_n(g) > 0$ (respectively, ≥ 0, $= 0$, ≤ 0, < 0), then $[g]$ contains a metric whose Ricci curvature is positive (resp., non-negative, zero, non-positive, negative).

(ii) If $R^m_n(g) > 0$ (resp., ≥ 0, $= 0$, ≤ 0, < 0), then $[g]$ contains a metric whose scalar curvature is positive (resp., non-negative, zero, non-positive, negative).

Proof. Both statements are immediate consequences of (3.9). □

The converse of Proposition 3.1 is false. For example, suppose we are given a metric g of positive Ricci curvature; then it is easy to construct a measure m so that $Ric_n^m(g)$ has negative eigenvalues on an open set.

3.1. **The Bianchi identity.** The contracted second Bianchi identity implies that

$$(\delta Ric)_j = \nabla^i R_{ij} = \frac{1}{2}\nabla_j R, \tag{3.10}$$

where δ is the divergence operator. If we introduce the *Bianchi operator* $\mathcal{B} = \mathcal{B}_g$ on symmetric $(0,2)$-tensors by

$$\mathcal{B}h = \delta h - \frac{1}{2}d(tr_g\ h), \tag{3.11}$$

then (3.10) can be written

$$\mathcal{B}Ric = 0. \tag{3.12}$$

Similar identities hold for the conformally invariant Ricci and scalar curvatures. First, a simple calculation gives

$$\nabla^i Ric_n^m(g)_{ij} = \frac{1}{2}\nabla_j R_n^m(g) + (\frac{n-2}{n})Ric_n^m(g)_{ij}\nabla_i f. \tag{3.13}$$

To make this more compact, we define the operator

$$\delta_n^m = \delta - (\frac{n-2}{n})\nabla f \lrcorner \tag{3.14}$$

where $\lrcorner$ denotes the interior product. Note that δ_n^m is the adjoint of ∇ relative to a weight. Let

$$d\nu_n = e^{-(\frac{n-2}{n})f} dVol(g), \tag{3.15}$$

and for functions $u, v \in L^2(M^n)$ define

$$\langle u, v\rangle_{L^2(g,d\nu_n)} = \int uv\ d\nu_n. \tag{3.16}$$

If X, Y are smooth vector fields, we define

$$\langle X, Y\rangle_{L^2(g,d\nu_n)} = \int \langle X, Y\rangle_g\ d\nu_n,$$

and use this definition to extend the L^2-inner product to tensors of all types. In particular, for a symmetric $(0,2)$-tensor h and 1-form η we have

$$\int \langle h, \nabla\eta\rangle d\nu_n = \int -\langle \delta_n^m h, \eta\rangle d\nu_n. \tag{3.17}$$

δ_n^m can be extended to act on sections of other bundles; for example, if $\omega \in \Omega^p(TM^n)$ and $\theta \in \Omega^{p-1}(TM^n)$ are forms, then

$$\int \langle \omega, d\theta\rangle d\nu_n = -\int \langle \delta_n^m \omega, \theta\rangle d\nu_n \tag{3.18}$$

defines δ_n^m on the exterior algebra.

Using δ_n^m, (3.13) becomes

$$\delta_n^m Ric_n^m(g) = \frac{1}{2} dR_n^m(g). \tag{3.19}$$

We will refer to the operator δ_n^m as the *m-adjoint*. We can also define the Bianchi operator associated to a measure:

$$\mathcal{B}_n^m h = \delta_n^m h - \frac{1}{2} d(tr_g\ h). \tag{3.20}$$

Then (3.19) is equivalent to

$$\mathcal{B}_n^m Ric_n^m(g) = 0. \tag{3.21}$$

It is interesting to note that $\mathcal{B}^m$ is conformally invariant:

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad \mathcal{B}_{n,\hat{g}}^m = e^{-2w} \mathcal{B}_{n,g}^m. \tag{3.22}$$

Acting on forms, $\delta_n^m : \Omega^p(TM^n) \to \Omega^{p-1}(TM^n)$ is conformally "covariant":

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad \delta_{n,\hat{g}}^m (e^{2(1-p)w} \eta) = e^{2(p-2)w} \delta_{n,g}^m \eta. \tag{3.23}$$

In particular, δ_n^m acting on 1-forms is invariant: $\delta_{n,\hat{g}}^m = e^{-2w} \delta_{n,g}^m$.

3.2. **The Einstein condition.** There is a natural notion of an Einstein metric in the RM-setting.

Definition 3.4. *Let (M^n, g, m) be an RM-space. The* conformally invariant trace-free Ricci tensor *of (M^n, g, m) is given by*

$$\begin{aligned} E_n^m(g) &= Ric_n^m(g) - \frac{1}{n} R_n^m(g) g \\ &= Ric(g) - \frac{1}{n} R(g) g + (\frac{n-2}{n}) \nabla_g^2 f - (\frac{n-2}{n^2})(\Delta_g f) g \\ &\quad + (\frac{n-2}{n^2}) df \otimes df - (\frac{n-2}{n^3}) |\nabla f|^2 g, \end{aligned} \tag{3.24}$$

Obviously E_n^m enjoys the same invariance properties as Ric_n^m. In the same way that the sign of the conformally invariant Ricci curvature detects the existence of a conformal metric with the same sign, the tensor E_n^m detects the existence of a conformal Einstein metric:

Proposition 3.2. *Let (M^n, g) be a smooth Riemannian manifold of dimension $n \geq 3$.*

(i) If m is a measure for which $E_n^m(g) = 0$, then $[g]$ contains an Einstein metric, namely, the canonical base metric.

(ii) If g is an Einstein metric and m is a measure for which $E_n^m(g) = 0$, then either $dm = \lambda dVol(g)$ for some constant $\lambda > 0$, or (M^n, g) is homothetically isometric to (S^n, g_c), where g_c denotes the round metric, and $dm = \lambda \varphi^* dVol(g_c)$ for some conformal map φ and constant $\lambda > 0$.

Proof. The proof of (i) follows from the conformal invariance of E_n^m and the definition of the canonical base metric.

To prove (ii), suppose g is Einstein and $E_n^m(g) = 0$. By part (i), the canonical base metric $g_m = e^{-\frac{2}{n}f}g$ is also Einstein. This leads to two possibilities: either g_m is homothetic to g, in which case $f = const.$ and $dm = \lambda dVol(g)$ as claimed, or the conformal class of g admits two distinct Einstein metrics. In this latter case it follows from a theorem of Obata ([Oba71], Proposition 6.2) that (M^n, g) is homothetically isometric to (S^n, g_c), and (also up to homothety) g_m is obtained from g by a conformal transformation. In particular,

$$\begin{aligned} m = dVol(g_m) &= \lambda dVol(\varphi^* g_c) \\ &= \lambda \varphi^* dVol(g_c). \end{aligned}$$

□

One consequence of the classical second Bianchi identity is that the scalar curvature of an Einstein metric must be constant. An analogous result holds in the setting of RM-spaces. By the Bianchi identity (3.19),

$$\delta_n^m E_n^m(g) = \frac{(n-2)}{2n}\nabla R_n^m(g) + \frac{(n-2)}{n^2}R_n^m(g)\nabla f. \tag{3.25}$$

Also,

$$\begin{aligned} dR_n^m(g) &= d\big(e^{-\frac{2}{n}f}\tau^m(g)\big) \\ &= -\frac{2}{n}R_n^m(g)df + e^{-\frac{2}{n}f}d\tau^m(g). \end{aligned}$$

Substituting this into (3.25) gives

$$\delta_n^m E_n^m(g) = \frac{(n-2)}{2n}e^{-\frac{2}{n}f}d\tau^m(g). \tag{3.26}$$

Corollary 3.1. *If (M^n, g, m) is an RM-space with $n \geq 3$ and $E_n^m(g) = 0$, then $\tau^m(g)$ is constant.*

Remark 3.1. By (3.5) the condition $\tau^m(g) = const.$ is conformally invariant, while by (3.2) the condition $R_n^m(g) = const.$ is not.

4. Relation to the curvatures of Bakry-Émery and Perelman

In this section we explore the relationship of the conformally invariant Ricci and scalar curvatures to the curvatures defined by Bakry-Émery [BE85] and Perelman [Per02]. To recall the definition of these latter invariants, once again we let (M^n, g, m) denote an RM-space, and let f be defined by (2.1):

$$dm = e^{-f}dVol(g).$$

The Bakry-Émery Ricci curvature is defined by

$$Ric^m(g) = Ric(g) + \nabla_g^2 f. \tag{4.1}$$

This definition arose in the analysis of infinite-dimensional diffusion processes, though this interpretation will not be relevant in the present paper. Instead, we will emphasize the role it plays as the curvature term in a "weighted" Wietzenböck formula, as we now explain.

In [Per02] (see also Lott, [Lot03]), Perelman defined the adjoint d^{*m} of the exterior derivative d relative to the measure m[1]. More precisely, given a p-form η and smooth function φ, d^{*m} is defined by

$$\int \langle d\varphi, \eta \rangle \; dm = - \int \varphi \; d^{*m}\eta \; dm. \tag{4.2}$$

By (2.1),

$$d^{*m} = d^* - \nabla f \lrcorner \, . \tag{4.3}$$

In case $p = 1$, in local coordinates we have

$$d^{*m}\eta = g^{kl}\nabla_k \eta_l - g^{kl}\nabla_k f \eta_l. \tag{4.4}$$

Similarly, one can define the adjoint of the covariant derivative, denoted ∇^{*m}. Using these operators we can form the associated 'Hodge laplacian' and 'rough laplacian':

$$\Box^m = dd^{*m} + d^{*m}d, \tag{4.5}$$

$$\Delta^m = \nabla^{*m}\nabla. \tag{4.6}$$

The following Weitzenböck formula for one-forms follows from the definitions of $\Box^m$ and Δ^m (see [Per02], §1.3; [Lot03], §2):

$$\Delta^m = \Box^m + Ric^m. \tag{4.7}$$

Perelman introduced a notion of the scalar curvature associated to the BE-Ricci tensor:

$$R^m = R + 2\Delta f - |\nabla f|^2. \tag{4.8}$$

In particular, R^m is *not* the trace of Ric^m. Perelman justified this definition in two ways: first, by noting that the classical contracted second Bianchi identity holds for Ric^m and R^m:

$$d^{*m} Ric^m = \frac{1}{2} dR^m. \tag{4.9}$$

Second, Perelman showed that the Dirac operator defined relative to dm in an analogous fashion leads to a Wetizenböck formula for the spin laplacian, where the usual scalar curvature term replaced by R^m.

As we now explain, the BE-Ricci curvature, and the scalar curvature defined by Perelman, can both be viewed as the infinite-dimensional limit of their conformally invariant counterparts. To see this we make use of a common construction, that of taking the product of M^n with a manifold of large dimension (see, e.g., [Lot03]).

[1]So that our laplacians agree with the Euclidean laplace operator, we add the minus sign.

First, however, let us argue informally: If we let the dimension $n \to \infty$ in (3.6), then in the limit we obtain the BE-Ricci curvature:

$$\begin{aligned}\lim_{n\to\infty} Ric_n^m(g) \text{ " = " } & Ric + \nabla_g^2 f \\ &= Ric^m(g).\end{aligned}$$

Similarly, letting $n \to \infty$ in the definition of R_n^m gives the scalar curvature defined by Perelman:

$$\begin{aligned}\lim_{n\to\infty} R_n^m(g) \text{ " = " } & R + 2\Delta_g f - |\nabla f|^2 \\ &= R^m(g).\end{aligned}$$

In these formulas we used quotation marks to emphasize the fact that the process of letting the dimension go to infinity is not meant to be taken literally. However, we can make these formal observations more concrete by the aforementioned construction.

Let $(\mathbb{T}^{d-n}, ds^2)$ denote the flat $(d-n)$-dimensional torus, and consider the product manifold $N^d = M^n \times \mathbb{T}^{d-n}$ with the product metric $h = g + ds^2$. To define the conformally invariant Ricci and scalar curvatures on N^d we need to define a measure, so we take the product measure $d\mu = dm \times dVol(ds^2)$. Note that if $dm = e^{-f} dVol(g)$ and $d\mu = e^{-\tilde{f}} dVol(h)$, it follows that $\tilde{f} = f$. Therefore, given tangent vectors $X, Y \in TM^n$, if we denote their lift to TN^d also by X, Y, then by the definition of Ric_n^m

$$\begin{aligned}\lim_{d\to\infty} Ric_d^\mu(h)(X,Y) &= \lim_{d\to\infty} \Big\{ Ric(h)(X,Y) + (\frac{d-2}{d})\nabla_h^2 \tilde{f}(X,Y) + \frac{1}{d}(\Delta_h \tilde{f})h(X,Y) \\ &\qquad + (\frac{d-2}{d^2})(d\tilde{f} \otimes d\tilde{f})(X,Y) - \frac{(d-2)}{d^2}|\nabla \tilde{f}|^2 h(X,Y)\Big\} \\ &= \lim_{d\to\infty} \Big\{ Ric(g)(X,Y) + (\frac{d-2}{d})\nabla_g^2 f(X,Y) + \frac{1}{d}(\Delta_g f)g(X,Y) \\ &\qquad + (\frac{d-2}{d^2})df(X)df(Y) - \frac{(d-2)}{d^2}|\nabla f|^2 g(X,Y)\Big\} \\ &= Ric(g)(X,Y) + \nabla^2 f(X,Y) \\ &= Ric_g^m(X,Y).\end{aligned}$$

Similarly,

$$\lim_{d\to\infty} R_d^m(h) = R^m(g).$$

A similar argument can be used to relate the m-divergence defined in Section 3.1 with the operator d^{*m}. Comparing (3.14) and (4.3) one sees that formally

$$\text{" } \lim_{n\to\infty} \delta_n^m = d^{*m}. \text{ "}$$

This can also be made precise using the construction above.

5. $\mathcal{D}$ and $\mathcal{C}_+$-actions on RM-spaces

As we observed in Section 2, there are two natural group actions on RM-spaces: the natural $\mathcal{D}$-action defined by (2.6),

$$\varphi : (M^n, g, m) \mapsto (M^n, \varphi^* g, \varphi^* m),$$

and the action of $\mathcal{C}_+$ via conformal rescaling:

$$\gamma : (M^n, g, m) \mapsto (M^n, \gamma g, m). \tag{5.1}$$

Note that $\mathcal{C}_+$ acts on the metric alone, while $\mathcal{D}$ acts by pull-back on *both* the metric and the measure. On the other hand, if $T = T(g, m)$ is an RM-invariant and we pull back the metric g by a diffeomorphism $\varphi \in \mathcal{D}$ (leaving m fixed), then by (2.6),

$$T(\varphi^* g, m) = \varphi^* T(g, (\varphi^{-1})^* m). \tag{5.2}$$

One way of interpreting (5.2) is to view pull-backs of the metric as determining a change in the measure, $m \mapsto (\varphi^{-1})^* m$. By a Theorem of Moser, this is actually an equivalence; that is, *all* measures can be realized in this manner:

Theorem 5.1. *Let (M^n, g, m) be an RM-space and $T = T(g, m)$ a local RM-invariant. Given a measure μ with*

$$\int d\mu = \int dm, \tag{5.3}$$

there is a diffeomorphism $\varphi \in \mathcal{D}$ with

$$T(\varphi^* g, m) = \varphi^* T(g, \mu). \tag{5.4}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a measure μ satisfying (5.3) and (5.4).

Proof. By Moser's theorem [Mos65], equation (5.3) implies the existence of a diffeomorphism $\varphi \in \mathcal{D}$ such that $\mu = (\varphi^{-1})^* m$. Therefore, (5.4) follows from (5.2).

The converse is obvious; just take $\mu = (\varphi^{-1})^* m$. $\square$

Our next result shows that the behavior of a Riemannian invariant $I = I(g)$ under conformal deformations of the metric is *equivalent* to the behavior of the associated conformal density $\mathcal{I}^m_{-2/n,0}(g)$ under conjugation by a diffeomorphism. To simplify the statement we normalize the measure and the metric:

Theorem 5.2. *Let (M^n, g, m) be an RM-space, where for convenience we assume g and m have the same total mass:*

$$\int dVol(g) = \int dm. \tag{5.5}$$

Suppose $I = I(g)$ is a Riemannian invariant, and $\mathcal{I}^m = \mathcal{I}^m_{-2/n,0}(g)$ is the conformal density of weight 0 defined in (2.17). Given $\hat{g} \in [g]$ of the same volume, there is a diffeomorphism $\varphi \in \mathcal{D}$ such that

$$I(\hat{g}) = (\varphi^{-1})^* \mathcal{I}^m (\varphi^* g). \tag{5.6}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]$ (of the same volume) such that (5.6) holds.

Proof. Given $\hat{g} \in [g]$ with the same volume of g (and consequently, the same mass as m), by Moser's Theorem there is a diffeomorphism $\varphi \in \mathcal{D}$ such that

$$(\varphi^{-1})^* dm = dVol(\hat{g}). \tag{5.7}$$

Therefore, by (5.2),

$$\begin{aligned} \mathcal{I}^m(\varphi^* g) &= \varphi^*\big(\mathcal{I}^{(\varphi^{-1})^* m}(g)\big) \\ &= \varphi^*\big(\mathcal{I}^{dVol(\hat{g})}(g)\big) \\ &= \varphi^*\big(\mathcal{I}^{dVol(\hat{g})}(\hat{g})\big) \quad \text{(by conformal invariance of } \mathcal{I}^m) \\ &= \varphi^*\big(I(\hat{g})\big). \qquad \text{(by (2.18))} \end{aligned}$$

Since I is a Riemannian invariant, pulling back both sides by φ^{-1} gives (5.6).

Conversely, suppose $\varphi \in \mathcal{D}$ is given. Then

$$(\varphi^{-1})^* m = e^{nw} dVol(g) \tag{5.8}$$

for some function w. Let $\hat{g} = e^{2w} g$; then (5.8) can be rewritten

$$(\varphi^{-1})^* m = dVol(\hat{g}). \tag{5.9}$$

Therefore,

$$\begin{aligned} \mathcal{I}^m(\varphi^* g) &= \varphi^*\big(\mathcal{I}^{(\varphi^{-1})^* m}(g)\big) \\ &= \varphi^*\big(\mathcal{I}^{dVol(\hat{g})}(g)\big) \\ &= \varphi^*\big(\mathcal{I}^{dVol(\hat{g})}(\hat{g})\big) \quad \text{(by conformal invariance of } \mathcal{I}^m) \\ &= \varphi^*\big(I(\hat{g})\big), \qquad \text{(by (2.18))} \end{aligned}$$

and pulling back by φ we get (5.6). □

Remark 5.3. Theorem 5.2 can be regarded as the fundamental theorem of the geometry of RM-spaces, in that it gives the relationship between RM-conformal densities and Riemannian invariants.

Corollary 5.1. *Let (M^n, g, m) be an RM-space, where for convenience we assume g and m have the same total mass. Given $\hat{g} \in [g]$ with the same volume as g, there is a diffeomorphism $\varphi \in \mathcal{D}$ such that*

$$R(\hat{g}) = (\varphi^{-1})^* \tau^m(\varphi^* g). \tag{5.10}$$

Conversely, given a diffeomorphism $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]$ such that (5.10) holds.

6. Variational theory

In this section we consider variational problems associated to RM-spaces. We begin with some general features of variational integrals, and describe a natural constrained problem. In the succeeding section we explore in detail an interesting example associated to the scalar curvature.

Let $\mathcal{M} = \mathcal{M}(M^n)$ denote the space of Riemannian metrics on M^n, and $\mathcal{P}$ the space of all probability measures. Sometimes we may need to normalize the volume, so we denote by $\mathcal{M}_1$ the subspace of unit-volume metrics.

Definition 6.1. *An* RM-functional *is a (differentiable) mapping* $\mathcal{F} : \mathcal{M} \times \mathcal{P} \to \mathbb{R}$ *which is invariant under the action of* $\mathcal{D}$ *in (2.7):*

$$\mathcal{F}(\varphi^* g, \varphi^* m) = \mathcal{F}(g, m). \tag{6.1}$$

The basic example of an RM-functional is the integral of a scalar local RM-invariant $q = q(g, m)$:

$$\mathcal{F}(g, m) = \int q(g, m)\ dVol(g). \tag{6.2}$$

Of particular interest will be the case where q is the conformal density $\mathcal{I}^m(g) = \mathcal{I}^m_{-2/n,0}(g)$ associated to a Riemannian invariant $I(g)$, as defined in Proposition 2.1. Let

$$\mathcal{J}^m[g] = \int \mathcal{I}^m(g)\ dm, \tag{6.3}$$

and denote the corresponding Riemannian functional by

$$J[g] = \int I(g)\ dVol(g). \tag{6.4}$$

Example. Let $I(g) = R(g)$, the scalar curvature. Then

$$\mathcal{S}^m[g] = \int \tau^m(g)\ dm, \tag{6.5}$$

while the Riemannian functional is the total scalar curvature:

$$S[g] = \int R(g)\ dVol(g). \tag{6.6}$$

Let ∇J denote the L^2-gradient of the functional $J[\cdot]$ (assuming it exists):

$$\frac{d}{ds} J[g + sh]\Big|_{s=0} = \int \big\langle \nabla J(g), h \big\rangle_g\ dVol(g). \tag{6.7}$$

Proposition 6.1. *Suppose* (M^n, g, m) *is an* RM*-space, and* $I = I(g)$ *is a scalar Riemannian invariant. Let* $\mathcal{I}^m = \mathcal{I}^m_{-2/n,0}$ *denote the associated conformal density, and define the functionals* J *and* $\mathcal{J}^m$ *by (6.4) and (6.3), respectively. Then*

(*i*) *The functional* $\mathcal{J}^m$ *is conformally invariant:*

$$\mathcal{J}^m[e^{2w} g] = \mathcal{J}^m[g]. \tag{6.8}$$

(ii) If the functional J is differentiable with L^2-gradient ∇J, then $\mathcal{J}^m$ is also differentiable, and

$$\nabla \mathcal{J}^m(g) = e^{\frac{2}{n}f}\Big\{\nabla J(g_m) - \frac{1}{n}tr_{g_m}\big(\nabla J(g_m)\big)g_m\Big\}. \tag{6.9}$$

Proof. The conformal invariance of $\mathcal{J}^m$ is obvious from the definition. To prove (6.9), we first note that the gradient of J can be expressed in terms of the L^2-adjoint of the linearized operator:

$$\begin{aligned}\frac{d}{ds}J[g+sh]\Big|_{s=0} &= \int \frac{d}{ds}I(g+sh)\Big|_{s=0} dVol(g) + \int I(g)\ \frac{d}{ds}dVol(g+sh)\Big|_{s=0} \\ &= \int I'(g)[h]\ dVol(g) + \int \frac{1}{2}I(g)(tr_g h)\ dVol(g) \\ &= \Big\langle I'(g)[h], 1\Big\rangle_{L^2} + \Big\langle h, \frac{1}{2}I(g)g\Big\rangle_{L^2} \\ &= \int \big\langle h, I'(g)^*(1) + \frac{1}{2}I(g)g\big\rangle_g\ dVol(g),\end{aligned}$$

which implies

$$\nabla J(g) = I'(g)^*(1) + \frac{1}{2}I(g)g. \tag{6.10}$$

Similarly, for $\mathcal{J}^m$ we find

$$\begin{aligned}\frac{d}{ds}\mathcal{J}^m[g+sh]\Big|_{s=0} &= \int \frac{d}{ds}\mathcal{I}^m(g+sh)\Big|_{s=0}\ dm \\ &= \int (\mathcal{I}^m)'(g)[h]\ dm \\ &= \Big\langle (\mathcal{I}^m)'(g)[h], 1\Big\rangle_{L^2(g,dm)} \\ &= \Big\langle h, (\mathcal{I}^m)'(g)^*(1)\Big\rangle_{L^2(g,dm)},\end{aligned}$$

hence

$$\nabla \mathcal{J}^m(g) = (\mathcal{I}^m)'(g)^*(1). \tag{6.11}$$

Claim 6.1.

$$(\mathcal{I}^m)'(g)^*u = e^{\frac{2}{n}f}\Big\{I'(g_m)^*u - \frac{1}{n}tr_{g_m}\Big(I'(g_m)^*u\Big)g_m\Big\}, \tag{6.12}$$

where $g_m = e^{-\frac{2}{n}f}g$ is the canonical base metric.

Proof. The claim follows from the identity

$$(\mathcal{I}^m)'(g)[h] = I'(g_m)[e^{-\frac{2}{n}f}\mathring{h}], \tag{6.13}$$

where

$$\mathring{h} = h - \frac{1}{n}(tr_g\ h)g,$$

by simply integrating by parts. To verify (6.13), let f_s be defined by

$$dm = e^{-f_s} dVol(g + sh). \tag{6.14}$$

Then

$$\frac{d}{ds} f_s\big|_{s=0} = \frac{1}{2} tr_g\ h. \tag{6.15}$$

Since $\mathcal{I}^m(g) = I(e^{-\frac{2}{n}f} g)$, we find

$$\begin{aligned}
(\mathcal{I}^m)'(g)[h] &= \frac{d}{ds}\mathcal{I}^m(g+sh)\Big|_{s=0} \\
&= \frac{d}{ds} I\big(e^{-\frac{2}{n}f_s}(g+sh)\big)\Big|_{s=0} \\
&= I'(e^{-\frac{2}{n}f} g)[e^{-\frac{2}{n}f}(h - \frac{1}{n}(tr_g\ h)g)].
\end{aligned}$$

□

Using (6.12) and (6.10) we conclude

$$\begin{aligned}
\nabla \mathcal{J}^m(g) &= (\mathcal{I}^m)'(g)^*(1) \\
&= e^{\frac{2}{n}f}\Big\{ I'(g_m)^*(1) - \frac{1}{n} tr_{g_m}\Big(I'(g_m)^*(1)\Big) g_m \Big\} \\
&= e^{\frac{2}{n}f}\Big\{ \nabla J(g_m) - \frac{1}{n} tr_{g_m}\big(\nabla J(g_m)\big) g_m \Big\}.
\end{aligned}$$

□

Example. The gradient of the total scalar curvature is the 'gravitational tensor':

$$\nabla S(g) = -Ric(g) + \frac{1}{2} R(g) g,$$

whose trace is

$$tr_g \nabla S(g) = \frac{(n-2)}{2} R(g).$$

Therefore, by Proposition 6.1 the gradient of $\mathcal{S}^m$ is

$$\begin{aligned}
\nabla \mathcal{S}^m(g) &= e^{\frac{2}{n}f}\Big\{ - Ric(g_m) + \frac{1}{2} R(g_m) g_m - \frac{1}{n}\big[\frac{(n-2)}{2} R(g_m)\big] g_m \Big\} \\
&= e^{\frac{2}{n}f}\Big\{ - Ric(g_m) + \frac{1}{n} R(g_m) g_m \Big\} \\
&= -e^{\frac{2}{n}f} E_n^m(g),
\end{aligned} \tag{6.16}$$

where $E_n^m(g)$ is the conformally invariant trace-free Ricci tensor (see Section 3.2).

Proposition 6.2. *Let (M^n, gm) be an RM-space of dimension $n \geq 3$. Then g is critical for $\mathcal{S}^m$ if and only if it is conformal to an Einstein metric.*

6.1. **A constrained variational problem.** Since we are fixing the measure when studying $\mathcal{J}^m$, a natural constrained variational problem is to restrict the functional to the orbit of a metric g under the action of $\mathcal{D}$. By Theorem 5.2, this is equivalent to restricting the associated functional J to the conformal class of g:

Theorem 6.1. *Let (M^n, g, m) be an RM-space, where m is a probability measure and we normalize g to have unit volume. Let $I, J, \mathcal{I}^m$, and $\mathcal{J}^m$ be as above. Then*

$$\mathcal{J}^m\Big|_{\mathcal{D}(g)} = J\Big|_{[g]_1}, \tag{6.17}$$

where $\mathcal{D}(g) = \{\varphi^* g \mid \varphi \in \mathcal{D}\}$ is the orbit of g under the action of $\mathcal{D}$, and $[g]_1$ denotes metrics conformal to g of unit volume.

More precisely: Given $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]_1$ with

$$\mathcal{J}^m[\varphi^* g] = J[\hat{g}]. \tag{6.18}$$

Conversely, given $\hat{g} \in [g]_1$, there is a diffeomorphism $\varphi \in \mathcal{D}$ such that (6.18) holds.

Proof. This equivalence is essentially a corollary of Theorem 5.2 and its proof. Given $\varphi \in \mathcal{D}$, there is a conformal metric $\hat{g} \in [g]_1$ with

$$(\varphi^{-1})^* \mathcal{I}^m(\varphi^* g) = I(\hat{g}).$$

Pulling back by φ gives

$$\mathcal{I}^m(\varphi^* g) = \varphi^* I(\hat{g}). \tag{6.19}$$

Also, by (5.9),

$$dm = \varphi^* dVol(\hat{g}). \tag{6.20}$$

Therefore,

$$\begin{aligned} \mathcal{J}^m[\varphi^* g] &= \int \mathcal{I}^m(\varphi^* g)\ dm \\ &= \int \varphi^* I(\hat{g})\ \varphi^* dVol(\hat{g}) \\ &= \int I(\hat{g})\ dVol(\hat{g}) \\ &= J[\hat{g}]. \end{aligned} \tag{6.21}$$

Conversely, by Theorem 5.2, if $\hat{g}$ is a conformal metric of unit volume then there is a diffeomorphism $\varphi \in \mathcal{D}$ satisfying (6.19) and (6.20). Then (6.21) implies

$$J[\hat{g}] = \mathcal{J}^m[\varphi^* g],$$

and (6.18) follows. □

7. The total τ^m-curvature

Now let $I = I(g)$ be the scalar curvature, and consider the variational integral associated to its RM-counterpart; i.e., the τ^m-curvature. As a corollary of Theorem 6.1, we can give a different characterization of the Yamabe invariant:

Theorem 7.1. *Let (M^n, g, m) be an RM-space of dimension $n \geq 3$, where m is a probability measure and we normalize g to have unit volume. Then*

$$\inf_{\varphi \in \mathcal{D}} \int \tau^m(\varphi^* g)\, dm = Y(M^n, [g]), \tag{7.1}$$

where $Y(M^n, [g])$ is the Yamabe invariant of g. Moreover, the infimum on the left-hand side of (7.1) is attained by a diffeomorphism if and only if there is a conformal metric which attains the Yamabe invariant.

Remark 7.2. Recall the *σ-constant* of a manifold M^n is defined to be

$$\sigma(M^n) = \sup_{g \in \mathcal{M}} Y(M^n, [g]). \tag{7.2}$$

If we let

$$\sigma(M^n, m) \equiv \sup_{g \in \mathcal{M}_1} \inf_{\varphi \in \mathcal{D}} \mathcal{S}^m(\varphi^* g), \tag{7.3}$$

then it follows from Theorem 7.1 that

$$\sigma(M^n, m) = \sigma(M^n), \tag{7.4}$$

independent of the measure m.

More generally, a metric is a critical point of the total scalar curvature constrained to a fixed conformal class if and only if it has constant scalar curvature. The corresponding result for the total τ^m-curvature is

Proposition 7.1. *When the dimension $n \geq 3$, a metric g is critical for $\mathcal{S}^m\big|_{\mathcal{D}(g)}$ if and only if $\tau^m(g) = const.$*

Proof. Let $\{\varphi_t\}$ be the 1-parameter family of diffeomorphisms generated by a vector field X. By (6.16),

$$\begin{aligned} 0 &= \frac{d}{dt} \mathcal{S}^m(\varphi_t^* g)\big|_{t=0} \\ &= \mathcal{S}^m(g)'[L_X g] \\ &= -\int \langle e^{\frac{2}{n} f} E_n^m(g), L_X g \rangle\, dm \\ &= -\int \langle E_n^m(g), L_X g \rangle\, d\nu_n. \end{aligned} \tag{7.5}$$

The Proposition will follow from the next Lemma:

Lemma 7.1. *For any vector field X,*

$$\int -\langle E_n^m(g), L_X g\rangle \; d\nu_n = \frac{(n-2)}{n} \int X\tau^m(g) \; dm. \tag{7.6}$$

Proof. Let $\alpha = X^\flat$ be the one-form dual to the vector field X, then

$$\langle E_n^m(g), L_X g\rangle = 2\langle E_n^m(g), \nabla\alpha\rangle.$$

Therefore,

$$\begin{aligned}
\int -\langle E_n^m(g), L_X g\rangle \; d\nu_n &= \int -2\langle E_n^m(g), \nabla\alpha\rangle \; d\nu_n \\
&= \int 2\langle \delta_n^m E_n^m(g), \alpha\rangle \; d\nu_n \quad \text{(by (3.17))} \\
&= \int \frac{(n-2)}{n} \langle e^{-\frac{2}{n}f} \nabla\tau^m(g), \alpha\rangle \; d\nu_n \quad \text{(by (3.26))} \\
&= \frac{(n-2)}{n} \int \langle \nabla\tau^m(g), \alpha\rangle \; dm \\
&= \frac{(n-2)}{n} \int X\tau^m(g) \; dm.
\end{aligned}$$

□

To complete the proof of the Proposition, for any vector field X (7.6) implies

$$\begin{aligned}
0 &= \frac{d}{dt} \mathcal{S}^m(\varphi_t^* g)\big|_{t=0} \\
&= -\int \langle E_n^m(g), L_X g\rangle \; dVol(g) \\
&= \frac{(n-2)}{n} \int X\tau^m(g) \; dm,
\end{aligned}$$

and it follows that $\tau^m(g)$ must be constant. □

In view of Proposition 7.1, it is natural to introduce the following linear functional $\mathcal{G}^m(g) : \mathfrak{X}(M^n) \to \mathbb{R}$:

$$\mathcal{G}^m(g)(X) = \int X\tau^m(g) \; dm. \tag{7.7}$$

This is the obvious extension of the Futaki invariant from Kähler geometry [Fut83], or the Kazadan-Warner integral from conformal geometry [KW74].

Proposition 7.2. (*i*) *The functional $\mathcal{G}^m(g)$ is conformally invariant: if $\hat{g} = e^{2w} g$, then*

$$\mathcal{G}^m(\hat{g}) = \mathcal{G}^m(g).$$

(*ii*) *If X is a conformal Killing vector field, then $\mathcal{G}^m(g)(X) = 0$.*

(iii) If $E_n^m(g) = 0$*, then* $\mathcal{G}^m(g) = 0$*. In particular, unless* $\mathcal{G}^m(g) \equiv 0$ *for some measure* m*, the conformal class of* g *does not contain an Einstein metric.*

Proof. (i) The conformal invariance of $\mathcal{G}^m(g)$ follows from the conformal invariance of τ^m.

(ii) If X is a conformal vector field, then $L_X g = \psi g$ for some function ψ. Therefore, $\langle E_n^m(g), L_X g\rangle = 0$, and from (7.6) it follows that $\mathcal{G}^m(g)(X) = 0$.

(iii) If $E_n^m(g) = 0$, then (7.6) obviously implies $\mathcal{G}^m(g) = 0$.

□

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