ON CONFORMALLY COMPACT EINSTEIN MANIFOLDS

SUN-YUNG A. CHANG* AND YUXIN GE

ABSTRACT. In this article, we survey some of the recent developement in the study of the compactness and uniqueness problems for a class of conformally compact Einstein manifolds.

1. INTRODUCTION

Let X^d be a smooth manifold of dimension d with $d \geq 3$ and with boundary $\partial X = M$. A smooth conformally compact metric g^+ on X is a Riemannian metric such that $g = r^2 g^+$ extends smoothly to the closure $\overline{X}$ for some defining function r of the boundary ∂X in X. A defining function r is a smooth nonnegative function on the closure $\overline{X}$ such that $\partial X = \{r = 0\}$ and the differential $Dr \neq 0$ on ∂X. A conformally compact metric g^+ on X is said to be conformally compact Einstein (CCE) if, in addition,

$$\mathrm{Ric}[g^+] = -(d-1)g^+.$$

where Ric denotes the Ricci curvature. The most significant feature of CCE manifolds (X, g^+) is that the metric g^+ is "canonically" associated with the conformal structure $[\hat{g}]$ on the boundary at infinity ∂X, where $\hat{g} = g|_{T\partial X}$. $(\partial X,\ [\hat{g}])$ is called the conformal infinity of a conformally compact manifold $(X,\ g^+)$. It is of great interest in both the mathematics and theoretic physics communities to understand the correspondences between conformally compact Einstein manifolds $(X,\ g^+)$ and their conformal infinities $(\partial X,\ [\hat{g}])$, especially due to the AdS/CFT correspondence in theoretic physics (cf. Maldacena [32, 33, 34] and Witten [37]).

For a CCE manifold, given any conformal infinity h and for any defining function r, we have always $|\nabla_g r| \equiv 1$ on M. In fact, it is known that the full Riemann curvature tensor $Rm[g_+]$ of the metric g^+ has the asymptotic expansion near the infinity, $\forall 1 \leq i, j, k, l \leq d$

$$Rm_{ijkl}[g_+] = -|\nabla r(x)|_g^2((g^+)_{ik}(g^+)_{jl} - (g^+)_{il}(g^+)_{jk}) + O(r^{-3})$$

which yields the above claim. A conformally compact metric g^+ on X is called asymptotically hyperbolic (AH) if in addition $|\nabla_g r| \equiv 1$ on M. Moreover, given any conformal infinity h, there exists a special defining function which we call geodesic defining function r such that $|\nabla_g r| \equiv 1$ in an asymptotic neighborhood $M \times [0, \epsilon)$ of M and $r^2 g^+|_{TM} = h$.

Under geodesic defining function r, we have a nice expansion for CCE metric g^+. It turns out the asymptotic behavior of the compactified metric is slightly different when the dimension d is even or odd.

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When d is even, the asymptotic behavior of the compactified metric g of CCE manifold (X^d, M^{d-1}, g^+) with conformal infinity $(M^{d-1}, [h])$ ([19, 18]) takes the form

$$g := r^2 g^+ = dr^2 + g_r = dr^2 + h + g^{(2)} r^2 + \cdots \text{(even powers)} + g^{(d-1)} r^{d-1} + g^{(d)} r^d + \cdots \tag{1.1}$$

on an asymptotic neighborhood of $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g. The $g^{(j)}$ are tensors on M, and $g^{(d-1)}$ is trace-free with respect to a metric in the conformal class on M. For j even and $0 \le j \le d-2$, the tensor $g^{(j)}$ is locally formally determined by the conformal representative, but $g^{(d-1)}$ is a non-local term which is not determined by the boundary metric h, subject to the trace free condition.

When d is odd, the analogous expansion is

$$g := r^2 g^+ = dr^2 + g_r = dr^2 + h + g^{(2)} r^2 + \cdots \text{(even powers)} + g^{(d-1)} r^{d-1} + k r^{d-1} \log r + \cdots \tag{1.2}$$

where now the $g^{(j)}$ are locally determined for j even and $0 \le j \le d-2$, k is locally determined and trace-free, the trace of $g^{(d-1)}$ is locally determined, but the trace-free part of $g^{(d-1)}$ is formally undetermined.

We remark that h together with $g^{(d-1)}$ determine the asymptotic behavior of g ([18, 2]).

In this paper, we will first briefly survey some of the recent development in this research area. We will then describe a series of joint works of the authors [7], with Jie Qing [8], and with Xiaoshang Jin and Jie Qing [9] in which we address the issues of compactness of sequences of CCE manifolds and uniqueness problem for a class of such manifolds constructed earlier by [22].

2. Basics and some short survey

Some basic examples

Example 1:
A model case of a CCE manifold is the hyperbolic ball $\mathbb{B}^d$ with the Poincaré metric $g_{\mathbb{H}} := \frac{4}{(1-|x|^2)^2} \sum_{i=1}^d dx_i^2$ where $|x| := \sqrt{\sum_{i=1}^d x_i^2}$ is the usual euclidean norm of $x = (x_1, \cdots, x_d) \in \mathbb{B}^d = \{y \in \mathbb{R}^d, |y| < 1\}$. For such metric, $r(x) = \frac{1-|x|}{1+|x|}$ is a geodesic defining function with the conformal infinity $h = \frac{1}{4} g_{\mathbb{S}^{d-1}}$ the standard metric on $d-1$ sphere $\mathbb{S}^{d-1}$ up to a constant and $g_r = (1 - r(x)^2)^2 h$.

Example 2:
Another class of examples of CCE manifolds was constructed by Graham-Lee [22] in 1991, where they have proved that for metrics on $\mathbb{S}^{d-1}$ close enough in $C^{2,\alpha}$ norm to the standard metric on $\mathbb{S}^{d-1}$, is the conformal infinity of some CCE metric on the ball $\mathbb{B}^d$ for all $d \ge 4$.

Example 3: AdS-Schwarzchild space

$$On \ (R^2 \times S^2, g_m^+),$$

where

$$g_m^+ = V dt^2 + V^{-1} dr^2 + r^2 g_c,$$
$$V = 1 + r^2 - \frac{2m}{r},$$

m is any positive number, $r \in [r_h, +\infty)$, $t \in S^1(\lambda)$ and g_c the surface measure on S^2 and r_h is the positive root for $1 + r^2 - \frac{2m}{r} = 0$. We remark, it turns out that in this case, there are two different values of m so that both g_m^+ are conformal compact Einstein filling for the same boundary metric $S^1(\lambda) \times S^2$. This is the famous non-unique "filling in" example of Hawking-Page [27].

Existence and non-existence results

The most important existence result is the "Ambient Metric" construction by Fefferman-Graham ([16],[18]). As a consequence of their construction, for any given compact manifold (M^{d-1}, h) with an analytic metric h, some CCE metric exists on some tubular neighborhood $M^n \times (0, \epsilon)$ of M. This later result was recently extended to manifolds M with smooth metrics by Gursky-Székelyhidi [26].
As we have mentioned before, a perturbation result of Graham-Lee [22] asserts that in a neighborhood of the standard metric g_c on S^{d-1}, there exist a conformal compact Einstein metric on B^d with any given conformal infinity h.

Recent results of Gursky-Han and Gursky- Han-Stolz ([24], [25]) showed that when X is spin and of dimension $4k \geq 8$, and when the Yamabe invariant $Y(M, [h]) > 0$, then there are topological obstructions to the existence of a CCE metric g^+ defined in the interior of X with conformal infinity given by [h]. The basic idea is to adapt the classical Lichnerowicz result on the vanishing of the $\widehat{A}$-genus for spin manifolds of positive scalar curvature. Indeed, suppose g^+ is a CCE filling in of [h]; then one can use the compactification of Lee to obtain a metric $g = r^2 g^+$ with positive scalar curvature which is smooth up to the boundary, and such that M is totally geodesic with respect to g. It follows that the index of the Dirac operator (with respect to APS boundary conditions) is zero. However, using well known properties of the index, it is possible to construct examples of spin manifolds with boundary M and conformal classes [h] of positive Yamabe invariant on M such that the index of the Dirac operator (with respect to any extension of any metric in [h]) has non-vanishing index. For example, on the round sphere S^{4k-1} with $k \geq 2$, there are infinitely many such conformal classes.

The result of Gursky-Han and Gursky-Han-Stolz was based on a key fact pointed out earlier by J. Qing [36], which in turn relies on some earlier work of J. Lee [29].

Lemma 2.1. *On a CCE manifold (X^d, M^{d-1}, g^+), assuming $Y(M, [h]) > 0$, then there exists a compactification of g^+ with positive scalar curvature; hence $Y(X, \partial X, [r^2 g^+]) > 0$.*

Uniqueness and non-uniqueness results

Under the assumption of positive mass theorem, J. Qing [36] has established (B^d, g_H) as the unique CCE manifold with $(S^{d-1}, [g_c])$ as its conformal infinity. The proof of this result was later refined and established without using positive mass theorem by Li-Qing-Shi [31] (see also Dutta and Javaheri [15]). Later in sections 4 and 5 of this lecture notes, we will also prove the uniqueness of the CCE extension of the metrics constructed by Graham-Lee [22] for all $d \geq 4$.

As we have mentioned in the example 3 above, when the conformal infinity is $S^1(\lambda) \times S^2$ with product metric, Hawking-Page [27] have constructed non-unique CCE fill-ins.

In a recent series of joint works of ([7], [8] and [9]), we work to address the compactness issue of sequences of metrics on CCE manifolds. The question is as follows: given a sequence of CCE manifolds $(X^d, M^{d-1}, \{g_i^+\})$ with $M = \partial X$ and $\{g_i\} = \{r_i^2 g_i^+\}$ a sequence of compactified metrics, denote $h_i = g_i|_{TM}$, assuming $\{h_i\}$ forms a compact family of metrics in M, when is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M = h_i\}$ also forms a compact family of metrics in $\bar{X}$? One main difficulty to address the compactness problem is due to the existence of some "non-local" term in the asymptotic expansion of the metric near the conformal infinity. For example in the case $d = 4$, the $g^{(3)}$ term in the asymptotic expansion of $g = r^2 g^+$ in (1.1) is an "non-local" as it depends on both $h = g|_M$ and g^+.

One application of compactness is the uniqueness result of the CCE extension of Graham and Lee for the metrics on $\mathbb{S}^{d-1}$ close to the standard canonical metric on $\mathbb{S}^{d-1}$. As we have mentioned before, in the model case–the hyperbolic space form, it was proved by [36] (see also [15] and later a different proof by [31]) that $(\mathbb{B}^d, g_{\mathbb{H}})$ is the unique CCE manifold with the standard canonical metric on $\mathbb{S}^{d-1}$ as its conformal infinity. The compactness result permits us to generalize the global uniqueness in the above setting. Such result could be considered also as a stability result for the hyperbolic space.

In this work, if there is no confusion, we drop the argument g for the various curvature tensors Ric, Rm, etc...

3. Compactness result in high dimensions $d \geq 5$

On a general d-dimensional CCE manifold (X^d, M^{d-1}, g^+) with $d \geq 5$. A general consideration is what is a "good" choice of the compactification of g^+ one should use. A most natural consideration is the compactication of the Yamabe metric (i.e. the metric which minimize the L^1 norm of the scalar curvature in the compactified conformal class of metrics $[g^+]$ with fixed volume which we know exists). The problem with that choice is that, we do not see how to control the corresponding boundary metric of the Yamabe metric. Instead, in [9] and the earlier works of [7] and [8], we will consider a special choice of compactification with some given boundary metric to start with. In the case when $d \geq 5$, the metric we chose and denoted by g^* is the metric which was considered earlier in a paper by Case-Chang, [6] and was named as the "adopted metric".
Given a boundary metric h on the conformal infinity M, the metric was defined by solving the PDE:

$$-\Delta_{g^+} v - \frac{(d-1)^2 - 9}{4} v = 0 \quad on \;\; X^d, \tag{3.1}$$

then we define $g^* := v^{\frac{4}{d-4}} g^+ = \rho^2 g^+$ with $g^*|_M = h$, the fixed metric on the conformal infinity of (X^d, g^+). We now describe some special properties of the metric g^*.
Recall the fourth order Paneitz operator is given by (see [35, 5, 21])

$$P_4 = (-\triangle)^2 + \delta(4A - \frac{d-2}{2(d-1)}R)\nabla + \frac{d-4}{2}Q_4 \tag{3.2}$$

where $A = \frac{1}{d-2}(Ric - \frac{R}{2(d-1)}g)$ is the Schouten tensor, δ is the dual operator of the differential ∇, R denotes the scalar curvature and Q_4 is a fourth order Q-curvature. More precisely, let $\sigma_k(A)$ denote the k-th symmetric function of the eigenvalues of A and $Q_4 := -\triangle\sigma_1(A) + 4\sigma_2(A) + \frac{d-4}{2}\sigma_1(A)^2$. For a Einstein metric with $Ric_{g^+} = -(d-1)g^+$, thus we have $Q_4[g^+] = 0$ and

$$P_4[g^+] = (-\Delta_{g^+} - \frac{(d-1)^2-1}{4}) \circ (-\Delta_{g^+} - \frac{(d-1)^2-9}{4}).$$

Therefore

$$Q_4[g^*] = \frac{2}{d-4}P_4[g^*]1 = \frac{2}{d-4}v^{\frac{d+4}{d-4}}P_4[g^+]v = 0$$

Moreover, g^* is totally geodesic on boundary (see [9, Lemma2.6]).

We now recall some basic calculations for curvatures under conformal changes. Write $g^+ = r^{-2}g$ for some defining function r and calculate

$$Ric[g^+] = Ric[g] + (d-2)r^{-1}\nabla^2 r + (r^{-1}\triangle r - (d-1)r^{-2}|\nabla r|^2)g$$

so that

$$R[g^+] = r^2(R[g] + \frac{2d-2}{r}\triangle r - \frac{d(d-1)}{r^2}|\nabla r|^2).$$

Here the covariant derivatives is calculated with respect to the metric g (or adopted metrics g^* in the following). Therefore, for adopted metrics g^* of a conformally compact Einstein metric g^+, one has

$$R[g^*] = 2(d-1)\rho^{-2}(1 - |\nabla\rho|^2), \tag{3.3}$$

which in turn gives

$$Ric[g^*] = -(d-2)\rho^{-1}\nabla^2\rho + \frac{4-d}{4(d-1)}R[g^*]g^* \tag{3.4}$$

and

$$R[g^*] = -\frac{4(d-1)}{d+2}\rho^{-1}\triangle\rho. \tag{3.5}$$

When X is a smooth d-dimensional manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X with the conformal infinity $(\partial X, [h])$ of nonnegative Yamabe type, an important property of the g^* metric was proved in the earlier work of Case-Chang ([6, Lemma 4.2]) is that $g^* = \rho^2 g^+$ the adopted metrics associated with the metric h with the positive scalar curvature in the conformal infinity have the positive scalar curvature $R[g^*] > 0$ on X, which implies in particular,

$$\|\nabla\rho\|[g^*] \leq 1. \tag{3.6}$$

This property is one of main ingredients in our blow-up analysis.
Another important property in blow-up analysis is the non-collapsing result for adopted metrics g^* when the conformal infinity $(\partial X, [h])$ is of positive Yamabe type (see [8, Lemma 3.3] and [9, Lemma 2.11]). That is, the volume of any geodesic ball with radius equals to 1 is uniformly bounded below by some positive constant when the curvature tensor is bounded.
We recall the Yamabe invariant of the conformal infinity $(\partial X, [h])$ is defined as follows

$$Y(\partial X, [h]) = \inf_{\widetilde{h}\in[h]} \frac{\int_{\partial X} R[\widetilde{h}] dvol[\widetilde{h}]}{Vol(\partial X, \widetilde{h})^{(d-3)/(d-1)}}$$

We now split the discussion into two cases.

Case I, when d is even

We first consider the case when d is even. In this case due to the vanishing obstruction tensor ([18, 20]) for CCE manifolds, the curvature tensor satisfies an elliptic system. More precisely, let R_{ikjl}, R_{ij} and W_{ikjl} be Riemann, Ricci, Weyl curvature tensors respectively. We recall the definition of 4-th order Bach tensor B on d-dimensional manifolds (X^d, g) as

$$B_{ij} := \frac{1}{d-3}\nabla^k\nabla^l W_{kijl} + \frac{1}{d-2} W_{kijl} R^{kl}. \tag{3.7}$$

Recall also the Cotten tensor $\mathcal{C}$ is defined as

$$\mathcal{C}_{ijk} = A_{ij,k} - A_{ik,j} \tag{3.8}$$

where A is the schouten tensor. It turns out there is a relation between the divergence of Weyl tensor to the Cotton tensor, namely

$$\nabla^l W_{ijkl} = (d-3)\mathcal{C}_{kij} \tag{3.9}$$

Applying this relation (3.9), we can write the Bach tensor into the following equations

$$(d-2)B_{ij} = \Delta R_{ij} - \frac{d-2}{2(d-1)}\nabla_i\nabla_j R - \frac{1}{2(d-1)}\triangle R g_{ij} + Q_1(Rm), \tag{3.10}$$

where $Q_1(Rm)$ is some quadratic term on Riemann curvature tensor

$$Q_1(Rm) := 2W_{ikjl}R^{kl} - \frac{d}{d-2} R_i{}^k R_{jk} + \frac{d}{(d-1)(d-2)} R R_{ij} + (\frac{1}{d-2} R_{kl}R^{kl} - \frac{R^2}{(d-1)(d-2)}) g_{ij}$$

We now recall that the adapted metric g^* have flat Q_4-curvature, i.e., $Q_4[g^*] = 0$, which can be rewritten into the following form

$$-\triangle R = -\frac{d^3 - 4d^2 + 16d - 16}{4(d-2)^2(d-1)} R^2 + \frac{4(d-1)}{(d-2)^2}|Ric|^2. \tag{3.11}$$

We will now incorporate the Q_4-flat property of g^* to the Bach equation of g^* to derive estimates of the curvature of g^*.

One first recall that if follows from [18, 20] when d is even, the metrics conformal to Einstein ones have the vanishing the obstruction tensors $\mathcal{O}_{ij}$ (see also [28]), that is

$$\mathcal{O}_{ij} = (\triangle)^{d-4/2}\frac{1}{d-3}\nabla^j\nabla^l W_{ijkl} + lots = (\triangle)^{d-4/2}B_{ij} + lots = 0 \tag{3.12}$$

For example, when $d = 6$, we have

$$\begin{aligned} B_{ij,k}{}^{k} = \; & 2W_{kijl}B^{kl} + 4A_k{}^k B_{ij} - 8A^{kl}\mathcal{C}_{(ij)k,l} \\ & +4\mathcal{C}_{ki}{}^{l}\mathcal{C}_{lj}{}^{k} - 2\mathcal{C}_i{}^{kl}\mathcal{C}_{jkl} - 4A^k{}_{k,l}\mathcal{C}_{ij}{}^{l} + 4W_{kijl}A^k{}_m A^{ml} \end{aligned} \tag{3.13}$$

where $2\mathcal{C}_{(ij)k} = \mathcal{C}_{ijk} + \mathcal{C}_{jik}$. Gathering (3.10),(3.11) and (3.12), the Ricci tensor satisfies a $d-2$ order elliptic system. This allows us to apply some standard elliptic PDE techniques, to obtain a ε-regularity result for the Ricci tensor, and then for the metrics g^*. This is the key step which permits us to do various blow-up analysis and derives the following compactness result (see [9, Theorem 1.1]).

Theorem 3.1. *Suppose that X is a smooth oriented d-dimensional manifold with $d \geq 6$ even and with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in $C^{k,\alpha}$ Cheeger-Gromov topology with $k \geq d-2$. Moreover, assume there exits some positive constant $C > 0$ such that the Yamabe invariant of the conformal infinities is uniformly bounded below by C. Assume there is $\delta_0 > 0$ such that if either*

(1′) $\int_{X^d}(|W|^{d/2}dvol)[g_i^+] < \delta_0,$ *or*

(1″) $Y(\partial X, [h_i]) \geq Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}}]) - \delta_0,$

then the set $\{g_i^*\}$ of the adopted metrics (after diffeomorphisms that fix the boundary) is compact in $C^{k,\alpha'}$ Cheeger-Gromov topology for all $0 < \alpha' < \alpha$.

Case II, when d is odd

When the dimension d of the manifold X is odd, in general, we would not expect the strong estimate C^{d-1} as in the cases when d is even due to the term of $kr^{d-1}\log r$ term in the expansion of the metric g as in (1.2). This term k happens to be the obstruction tensor ([18, 20]) on the boundary of X and which may not vanish. For all dimensions d, under the assumptions (namely C^6) of the boundary metrics, we will present a different strategy to reach the compactness result. The main difference from even dimension case consists to the boundary ε regularity. In odd dimensions, we have no the obstruction tensor to exploit. Hence, we go back to the Einstein equation for the metric g^+. As same as many geometric problems, Einstein equation is invariant under the diffeomorphism group. We will apply the gauge fixing technique for Einstein metric to prove ε regularity (see [9, Lemma 4.6 and Lemma 4.7]). We need to choose a fixed gauge to get the regularity in the neighborhood of the boundary. Another difficulty comes from the degeneration of such elliptic equation at the infinity. To overcome it, we choose the suitable weighted functional spaces for which the linearized operator of above "Gauged Einstein equation" is a isomorphism.

Let us introduce some notations. We firstly choose smooth local coordinates $\theta = (\theta^2, \theta^2, \cdots, \theta^d)$ on an open set $U \subset \partial X$. It can extend to $(\theta^1, \theta) = (\rho, \theta^2, \theta^2, \cdots, \theta^d)$ on the open subset $\Omega = [0, \epsilon) \times U \subset \overline{X}$, where ρ is the above defining function and $\epsilon > 0$ is some small positive number.

For any fixed point $p \in \partial X$, let Ω be a neighbourhood and (ρ, θ) be the background coordinates such that $\theta(p) = 0$. For each $R > 0$ sufficiently small, we define $Z_R(p) \subset \Omega \subset \overline{X}$:

$$Z_R(p) = \{(\rho, \theta) \in \Omega : |\theta| < R, 0 < \rho < R\}$$

In [14], Chrúsciel-Delay-Lee-Skinner use gauged Einstein equation to study the regularity problem and later on Biquard-Herzlich [4] prove a local version. Let us consider the nonlinear functional on d-dimensional open set $Z_R(p)$ with $p \in \partial X$ introduced by Biquard [3]: for two asymptotically hyperbolic metrics g^+ and k^+

$$F(g^+, k^+) := Ric[g^+] + (d-1)g^+ - \delta^*_{g^+}(B_{k^+}(g^+)), \tag{3.14}$$

where $B_{k^+}(g^+)$ is a linear condition, essentially the infinitesimal version of the harmonicity condition

$$B_{k^+}(g^+) := \delta_{k^+} g^+ + \frac{1}{2} d\mathrm{tr}_{k^+}(g^+).$$

We have for any asymptotically hyperbolic metrics k^+

$$D_1 F(k^+, k^+) = \frac{1}{2}(\triangle_L + 2(d-1)),$$

where D_1 denotes the partial differentiation of F with respective to its first variable, and the Lichnerowicz Laplacian $\triangle_L$ on symmetric 2-tensors is given by

$$\triangle_L := \nabla^* \nabla[k^+] + 2\overset{\circ}{Ric}[k^+] - 2\overset{\circ}{Rm}[k^+];$$

where

$$\overset{\circ}{Ric}[k^+](u)_{ij} = \frac{1}{2}(R_{im}[g^+]u_j{}^m + R_{jm}[k^+]u_i{}^m),$$

and

$$\overset{\circ}{Rm}[k^+](u)_{ij} = R_{imjl}[k^+]u^{ml}.$$

It is clear for any CCE metrics g^+

$$F(g^+, g^+) = 0.$$

Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with positive conformal infinity $(\partial X, [h])$ and with dimension $d \geq 5$. Assume that, under the adopted metrics g^*, we assume

(1) $\|Rm_{g^*}\|_{C^0} \leq 1$;
(2) $\|h\|_{C^6} \leq N$ for some positive constants $N > 0$;

We will prove the ε regularity. Namely, Rm_{g^*} is in Hölder space $C^{1,\alpha}$ for all $\alpha \in (0,1)$ (or equivalently, the adopted metric g^* is in Hölder space $C^{3,\alpha}$) near the boundary ∂X.

We can identify $\{p \in \bar{X}, \rho(p) \leq r_1\} = [0, r_1] \times \partial X$ for some $r_1 > 0$ as a submanifold with the boundary. We consider a C^4 compacitified AH manifold on $[0, r_1/2] \times \partial X$

$$t = d\rho^2 + h + \rho^2 h^{(2)},\ t^+ = \rho^{-2} t$$

where $h^{(2)} = g^{(2)}$ is the Fefferman-Graham expansion term and intrinsically determined by the boundary metric h ($g^{(2)}$ is the schouten tensor of h for the adopted metric). Given $2R < r_1/2$, we look for a local diffeomorphism $\Phi : Z_R(p) \to \Phi(Z_R(p)) \subset Z_{2R}(p)$ such that $\Phi^* g^+$ solves the gauged Einstein equation in $Z_{R/2}(p)$

$$F(\Phi^* g^+, t^+) = 0 \tag{3.15}$$

We divide the boundary $\partial Z_R(p) := \partial^\infty Z_R(p) \cup \partial^{int} Z_R(p) = (\{\rho = 0\} \cap \partial Z_R(p)) \cup (\{\rho > 0\} \cap \partial Z_R(p))$. Recall CCE g^+ and regular AH t^+ have the same conformal infinity h on ∂X. We try to find a $C^{2,\alpha}$ (with $\alpha \in (0,1)$) local diffeomorphism $\Phi : Z_R(p) \to Z_{2R}(p)$ fixing the boundary $\partial^\infty Z_R(p)$ such that the gauged condition is satisfied in $Z_{R/2}(p)$ up to the diffeomorphism Φ, that is

$$B_{t^+}(\Phi^* g^+) = 0 \text{ in } Z_{R/2}(p)$$

Thus, the gauged Einstein equation (3.15) is satisfied in $Z_{R/2}(p)$. Such equation permits us to prove $\rho^2(\Phi^* g^+ - t^+)$ in Hölder space $C^{3,\alpha}$ for all $\alpha \in (0,1)$, which gives $\rho^2 \Phi^* g^+$ in $C^{3,\alpha}$. Using the fact g^+ is CCE, we derive the regularity result for the Cotton tensor in Hölder space $C^{0,\alpha}$. Hence, it follows from (3.7) and (3.10) that the Ricci tensor Ric is in Hölder space $C^{1,\alpha}$ in $Z_{R/2}(p)$ which yields the desired the ε-regularity in $Z_{R/2}(p)$. Once the ε-regularity is established, the rest proof is as same as even dimensions case. Finally, we prove the following compactness result. For the more details, see [9, Theorem 1.2].

Theorem 3.2. *Suppose that X is a smooth oriented d-dimensional manifold with $d \geq 4$ and with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in C^6 Cheeger-Gromov topology. Moreover, assume there exits some positive constant $C > 0$ such that the Yamabe invariant of the conformal infinities is uniformly bounded below by C. Then under the above assumptions $(1')$ or $(1'')$, the set $\{g_i^*\}$ of the adopted metrics (after diffeomorphisms that fix the boundary) is compact in $C^{3,\alpha}$ Cheeger-Gromov topology for all $0 < \alpha < 1$.*

4. Uniqueness of Graham-Lee metrics in high dimension $d \geq 5$

As an application of Theorem 3.2, we are able to establish the global uniqueness for the CCE metrics on X^d with prescribed conformal infinities that are very close to the conformal round $(d-1)$-sphere as in the work of [22] (cf also [30]). Namely, (cf [9, Theorem 1.3])

Theorem 4.1. *For a given conformal $(d-1)$-sphere $(\mathbb{S}^{d-1}, [h])$ with $d \geq 5$ that is sufficiently close to the round one in C^6 topology, there is exactly one conformally compact Einstein metric g^+ on X^d whose conformal infinity is the prescribed conformal $(d-1)$-sphere $(\mathbb{S}^{d-1}, [h])$. Moreover, the topology of X should be a ball $\mathbb{B}^d$.*

We remark that there exists a unique CCE filling in metric when the conformal infinity is the standard sphere [36] (see also [15, 31]). The above uniqueness result is stability one for the model case–hyperbolic space.

The above theorem could be proved by contradiction. Assume otherwise there is a sequence of conformal $(d-1)$-dimensional sphere $(\mathbb{S}^{d-1}, [h_i])$ that converges to the round sphere such that, for each i, there exist two non-isometric conformally compact Einstein metrics g_i^+ and $\widetilde{g}_i^+$. And g_i^* and $\widetilde{g}_i^*$ are the corresponding adopted metrics.

Up to a subsequence, both g_i^* and $\widetilde{g}_i^*$ converge to the adopted metric $g_{\mathbb{H}}^*$ of hyperbolic space in $C^{3,\alpha}$ Cheeger-Gromov sense due to Theorem 3.2. On the other hand, there exists a diffeomorphism φ_i of class $C^{2,\alpha}$ for all $\alpha \in (0,1)$ (equal to the identity on the boundary), such that

$$F(\varphi_i^*\widetilde{g}_i^+, g_i^+) = 0$$

Moreover $\|\varphi_i(x) - x\|_{C^{2,\alpha}} \to 0$ and $\|\varphi_i^*\widetilde{g}_i^* - g_i^*\|_{C^{1,\alpha}} \to 0$ when $i \to \infty$. By the implicit function theorem, we have local uniqueness around each g_i^+, which implies, for large i, we have

$$g_i^+ = \varphi_i^*\widetilde{g}_i^+.$$

5. Compactness and uniqueness in dimension $d = 4$

In this section, we report results in dimension 4 established in [7, 8]. On a 4-dimensional CCE manifold (X^4, M^3, g^+), we will consider a special choice of compactification $g^* = g_{FG} = \rho^2 g^+$, called "Fefferman-Graham's compactification" (denoted also by FG metric or FG compactification). We call it the FG metric as the PDE (5.1) which was used later by Chang-Qing-Yang [12] to define the metric was first introduced in [17] in connection with their study of integral conformal invariants for CCE manifolds.

In [17], they studied the function w which is a solution of the following PDE

$$-\triangle_{g^+} w = d - 1 \tag{5.1}$$

We remark that in the special case when $d = 4$, the FG metric $g^* = e^{2w} g^+$ on a CCE 4-manifold (X^4, M^3, g^+) is a natural dimensional continuation of the adopted metrics on CCE d-manifold (X^d, M^{d-1}, g^+) when $d \geq 5$ in the following sense: Fixed a boundary metric h, if we name the solution v as v_s of the Poisson equation

$$-\Delta_{g^+} v - s(d - 1 - s)v = 0 \ on \ X^d, \tag{5.2}$$

when we choose $s = \frac{d}{2} + 1$, then for $d \geq 5$, the adopted metric on X^d which we have introduced earlier in section 4 is defined as $g^* = {v_s}^{\frac{d-4}{2}} g^+ = \rho_s^2 g^+$ with $g^*|_M = h$. While when $d = 4$, we have $s = \frac{d}{2} + 1 = 3 = d - 1$, then the solution w of (5.1) satisfies

$$w = -\frac{d}{ds}|_{s=d-1} v_s, \tag{5.3}$$

and the FG metric is defined as the compactified metric $g^* = e^{2w} g^+ = \rho^2 g^+$. Note that when $s = d - 1$, the natural solution of the Poisson equation (5.2) is $v_s \equiv 1$, thus ρ is the limiting function of ρ_s when s tends to $d - 1$. We refer the readers to the expository article [13] for further explanation of the relationship between the adopted metric and the FG metric, and the connection between FG metric and the notion of renormalized volume and other integral conformal invariants in the CCE settings.

Thus the FG metric on X^4 satisfies properties similar to the "adopted metrics" defined on X^d when $d \geq 5$. The most important among them are the properties that the FG metric g^* has free Q_4 curvature, and positive scalar curvature, and its restriction to the boundary M is totally geodesic. For simplicity, we choose the boundary metric h be the

Yamabe metric representative of the conformal infinity. With the same arguments as Theorem 3.2, we then obtain the same result in dimension 4 (see [8, Theorem1.3]).

Theorem 5.1. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. We assume*

(1) The set $\{h_i\}$ of Yamabe metrics that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in $C^{k,\alpha}$ Cheeger-Gromov topology with $k \geq 3$ and with some $\alpha \in (0,1)$;
there is $\delta_0 > 0$ such that if
either
(2) $\quad \int_{X^4} (|W|^2 dvol)[g_i^+] < \delta_0$,
or
(2′) $\quad Y(\partial X, [h_i]) \geq Y(\mathbb{S}^3, [g_{\mathbb{S}}]) - \delta_0$,

then the set $\{g_i^*\}$ of the FG compactifications (after diffeomorphisms that fix the boundary) is compact in $C^{k,\alpha'}$ Cheeger-Gromov topology for all $\alpha' \in (0, \alpha)$.

We now present some general compactness results in [7, 8] on X^4 without the assumptions that Weyl tensor is small in L^2 norm or the Yamabe invariant of the conformal infinity be close to that one of the standard sphere.

We first introduce some geometric quantities. In [7, Lemma 2.1], for a CCE manifold (X^4, M^3, g^+) with any compactification g, we introduce the notion of 2-tensor S which on a 3-manifold M^3

$$(S[g])_{\alpha,\beta} := \nabla^i (W[g])_{i\alpha\mathbf{n}\beta} + \nabla^i (W[g])_{i\beta\mathbf{n}\alpha} - \nabla^{\mathbf{n}} (W[g])_{\mathbf{n}\alpha\mathbf{n}\beta} - \frac{4}{3} H[g] (W[g])_{\alpha\mathbf{n}\beta}{}^{\mathbf{n}}$$

where $W[g]$ denotes the Weyl tensor, $H[g]$ the mean curvature on the boundary M, letter i is full indices, Greek indices α, β represent the tangential indices and $\mathbf{n}$ is the outward unit normal of the boundary under the metric g. When the compactified metric g has totally geodesic boundary, it takes the form:

$$(S[g])_{\alpha,\beta} = \frac{1}{2} \partial_{\mathbf{n}} \operatorname{Ric}[g]_{\alpha,\beta} - \frac{1}{12} \partial_{\mathbf{n}} R[g]\, h_{\alpha,\beta}.$$

The 2-tensor S is conformally invariant in the sense that

$$S[r^2 g] = r^{-1} S[g].$$

The connection of the S tensor to that of $g^{(3)}$ in (1.1) is that (see [7, Remark 2.2, (2.7)]): Under any compactification by a geodesic defining function r, $g = r^2 g^+$ has $\partial_{\mathbf{n}} R[g] = 0$ on M, thus

$$(S[g])_{\alpha,\beta} = -\frac{3}{2} g^{(3)}_{\alpha,\beta}. \tag{5.4}$$

This shows that $g^{(3)}$ is also a local conformal invariant, which has been stated by Graham [19].

The compactness result in general case can be stated as follows (see [8, Theorem 1.1] and also [7, Theorem 1.1]):

Theorem 5.2. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the same condition* (1) *in Theorem 5.1 holds. Assume the following conditions:*

($2''$) The FG compactifications $\{g_i^* = \rho_i^2 g_i^+\}$ associated with the Yamabe representatives $\{h_i\}$ on the boundary satisfies:

$$\lim_{r\to 0}\sup_i \sup_{x\in\partial X}\oint_{B(x,r)} |S_i|[g_i^*]dvol[h_i] = 0$$

(3) $H_2(X,\mathbb{Z}) = 0$.

Then, the set $\{g_i^*\}$ of FG compactifications (after diffeomorphisms that fix the boundary) forms a compact family in the $C^{k,\alpha'}$ Cheeger-Gromov topology for all $\alpha' \in (0,\alpha)$.

We remark that we were aware that in the paper [1] by M. Anderson, he had asserted similar compactness results in the CCE setting under no assumptions on the (analogue of the) nonlocal tensor S. We have difficulty understanding some key estimates in his arguments.

The key points for compactness result in general case on 4 dimensional CCE manifolds are the following: on one hand, the condition ($2''$) in Theorem 5.2 rules out the boundary blow up; on the other hand, the topological condition (3) in Theorem 5.2 rules out the interior blow up.

We now explain the connection of the S tensor to other scalar curvature invariants for the metric g^*, which plays a key role in the results in [7, Theorem 1.7] and [8, Theorem 1.2].

Recall that on a 4-manifold $(X^4,\ g)$, a 4-th order Q_4-curvature is given by

$$Q_4[g] := -\frac{1}{6}\triangle R - \frac{1}{2}|Ric|^2 + \frac{1}{6}R^2. \tag{5.5}$$

Q_4 curvature is naturally associated with a 4th-order Paneitz operator (3.2). The relation of the pair $\{Q_4, P_4\}$ in 4 dimensions is like that of the well known pair $\{K, -\Delta\}$ in 2 dimensions, where K denotes the Gaussian curvature:

$$-\Delta[g] + K[g] = K[e^{2w}g]e^{2w} \quad \text{on}\ \ X^2,$$
$$P_4[g]w + Q_4[g] = Q_4[e^{2w}g]e^{4w} \quad \text{on}\ \ X^4$$

for conformal changes of the metric. For a 4-manifold $(X^4,\ g)$ with boundary, in the earlier works of Chang-Qing [10, 11], in connection with the 4th order Q curvature, a 3rd order "non-local" boundary curvature T was introduced on ∂X to study the boundary behavior of g. The relation between the pair (Q_4, T) is a generalization of that of the Dirichlet-Neumann pair $(-\Delta, \partial_{\mathbf{n}})$. The expression of T curvature is in general complicated, but in the special case when g is totally geodesic, the expression T takes the simple form:

$$T[g] := \frac{1}{12}\partial_{\mathbf{n}}R. \tag{5.6}$$

We can state another compactness result (see [8, Theorem 1.2] and also [7, Theorem 1.7]).

Theorem 5.3. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the same conditions* (1) *as in Theorems 5.1, (3) as in Theorem 5.2 and*

> *(2''') For the associated Fefferman-Graham's compactifications $\{g_i^* = \rho_i^2 g_i^+\}$ with the Yamabe representatives $\{h_i\}$ on the boundary,*
> $$\lim_{r\to 0}\inf_i \inf_{x\in\partial X} \oint_{B(x,r)} T[g_i^*] dvol[h_i] \geq 0.$$

Then, the set $\{g_i^*\}$ is compact in $C^{k,\alpha'}$ Cheeger-Gromov topology for all $\alpha' \in (0,\alpha)$ up to diffeomorphisms that fix the boundary, provided $k \geq 7$.

We remark that with the same arguments as in high dimensions (see Theorem 4.1), we also reach a global uniqueness result in dimension 4 (see [8, Theorem 1.9]). Namely,

Theorem 5.4. *For a given conformal 3-sphere $(\mathbb{S}^3, [h])$ that is sufficiently close to the round one in $C^{3,\alpha}$ Cheeger-Gromov topology with some $\alpha \in (0,1)$, there is exactly one conformally compact Einstein metric g^+ on $\mathbb{B}^4$ whose conformal infinity is the prescribed conformal 3-sphere $(\mathbb{S}^3, [h])$.*

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Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail address: chang@math.princeton.edu

IMT, Université Toulouse 3, 118, route de Narbonne 31062 Toulouse, France
E-mail address: yge@math.univ-toulouse.fr