

ON COMPACTNESS CONFORMALLY COMPACT EINSTEIN MANIFOLDS AND UNIQUENESS OF GRAHAM-LEE METRICS, III

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ABSTRACT. In this paper, we establish a compactness result for a class of conformally compact Einstein metrics defined on manifolds of dimension $d \geq 4$. As an application, we derive the global uniqueness of a class of conformally compact Einstein metric defined on the d -dimensional ball constructed in the earlier work of Graham-Lee [21] with $d \geq 4$. As a second application, we establish some gap phenomenon for a class of conformal invariants.

1. INTRODUCTION

1.1. **Statement of results.** Let X^d be a smooth manifold of dimension d with $d \geq 3$ with boundary ∂X . A smooth conformally compact metric g^+ on X is a Riemannian metric such that $g = r^2 g^+$ extends smoothly to the closure \bar{X} for some defining function r of the boundary ∂X in X . A defining function r is a smooth nonnegative function on the closure \bar{X} such that $\partial X = \{r = 0\}$ and the differential $Dr \neq 0$ on ∂X . A conformally compact metric g^+ on X is said to be conformally compact Einstein (CCE) if, in addition,

$$\text{Ric}[g^+] = -(d-1)g^+.$$

The most significant feature of CCE manifolds (X, g^+) is that the metric g^+ is “canonically” associated with the conformal structure $[\hat{g}]$ on the boundary at infinity ∂X , where $\hat{g} = g|_{T\partial X}$. $(\partial X, [\hat{g}])$ is called the conformal infinity of a conformally compact manifold (X, g^+) . It is of great interest in both the mathematics and theoretic physics communities to understand the correspondences between conformally compact Einstein manifolds (X, g^+) and their conformal infinities $(\partial X, [\hat{g}])$, especially due to the AdS/CFT correspondence in theoretic physics (cf. Maldacena [28, 29, 30] and Witten [34]).

The project we work on in this paper is to address the compactness issue that given a sequence of CCE manifolds $(X^d, M^{d-1}, \{g_i^+\})$ with $M = \partial X$ and $\{g_i\} = \{r_i^2 g_i^+\}$ a sequence of compactified metrics, denote $h_i = g_i|_M$, assuming $\{h_i\}$ forms a compact family of metrics in M , when is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M = h_i\}$ also forms a compact family of metrics in X ? We remark that, for a CCE manifold, given any conformal infinity, a special defining function which we call geodesic defining function r exists so that $|\nabla_{\bar{g}} r| \equiv 1$ in an asymptotic neighborhood $M \times [0, \epsilon)$ of M . We also remark that the eventual goal to study the compactness problem is to establish the existence of conformal filling in for some classes of Riemannian manifolds as the conformal infinity.

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One of the difficulties to address the compactness problem is due to the existence of some “non-local” term in the asymptotic expansion of the metric near the conformal infinity. To see this, when d is even, we look at the asymptotic behavior of the compactified metric g of CCE manifold (X^d, M^{d-1}, g^+) with conformal infinity $(M^{d-1}, [h])$ ([19], [17]) which takes the form

$$(1.1) \quad g := r^2 g^+ = h + g^{(2)} r^2 + \dots (\text{even powers}) + g^{(d-1)} r^{d-1} + g^{(d)} r^d + \dots$$

on an asymptotic neighborhood of $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g . The $g^{(j)}$ are tensors on M , and $g^{(d-1)}$ is trace-free with respect to a metric in the conformal class on M . For j even and $0 \leq j \leq d-2$, the tensor $g^{(j)}$ is locally formally determined by the conformal representative, but $g^{(d-1)}$ is a non-local term which is not determined by the boundary metric h , subject to the trace free condition.

When d is odd, the analogous expansion is

$$(1.2) \quad g := r^2 g^+ = h + g^{(2)} r^2 + \dots (\text{even powers}) + g^{(d-1)} r^{d-1} + k r^{d-1} \log r + \dots$$

where now the $g^{(j)}$ are locally determined for j even and $0 \leq j \leq d-2$, k is locally determined and trace-free, the trace of $g^{(d-1)}$ is locally determined, but the trace-free part of $g^{(d-1)}$ is formally undetermined. We remark that h together with $g^{(d-1)}$ determine the asymptotic behavior of g ([17], [5]).

A model case of a CCE manifold is the hyperbolic ball \mathbb{B}^d with the Poincaré metric g_H with the conformal infinity the standard $d-1$ sphere \mathbb{S}^{d-1} . In this case, it was proved by [33] (see also [15] and later a different proof by [27]) that (\mathbb{B}^d, g_H) is the unique CCE manifold with the standard canonical metric on \mathbb{S}^{d-1} as its conformal infinity.

Another class of examples of CCE manifolds was constructed by Graham-Lee [21] in 1991, where they have proved that for metrics on \mathbb{S}^{d-1} close enough in $C^{2,\alpha}$ norm to the standard metric on \mathbb{S}^{d-1} , is the conformal infinity of some CCE metric on the ball \mathbb{B}^d for all $d \geq 4$. In an earlier paper [11], in the special case when dimension $d = 4$, we have established a compactness result for a class of CCE manifolds and derived as a consequence the uniqueness of the CCE extension of Graham and Lee for the class of metrics on \mathbb{S}^3 $C^{3,\alpha}$ close to the standard canonical metric on \mathbb{S}^3 .

The goal of this paper to extend the above result in [11] to all dimensions $d \geq 4$.

In [10] and [11], we have considered a special choice of compactification $g^* = e^{2w} g^+$ on a CCE manifold (X^4, M^3, g^+) of dimension four, which we named as the Fefferman-Graham’s (FG) compactification, defined by solving the PDE:

$$(1.3) \quad -\Delta_{g^+} w = 3 \quad \text{on } X^4.$$

On a general d -dimensional CCE manifold (X^d, M^{d-1}, g^+) . When $d > 4$, we will consider a choice of compactification g^* which was considered earlier in a paper by Case-Chang, [9] and was named as the “adopted metric”. The metric was defined by solving the PDE:

$$(1.4) \quad -\Delta_{g^+} v - \frac{(d-1)^2 - 9}{4} v = 0 \quad \text{on } X^d,$$

then we define $g^* := v^{\frac{4}{d-4}}g^+$ with $g^*|_M = h$, the fixed metric on the conformal infinity of (X^d, g^+) . It is known that g^* has free Q-curvature (see [9], [12], section 5.2, note that in the notation in [12] $d = n + 1$, see also the statement of Lemma 2.1 below.)

We will begin with some results about compactness of some classes of CCE manifolds. In dimension 4, such results are obtained in [10, 11] (see also the related results in [2, 3]).

We first consider the case when d is even. More precisely, the first result we have is:

Theorem 1.1. *Suppose that X is a smooth oriented d -dimensional manifold with $d \geq 4$ even and with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X . Assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set \mathcal{C} of metrics that is of positive Yamabe type and compact in $C^{k,\gamma}$ Cheeger-Gromov topology with $k \geq d - 2$ when $d \geq 6$ and $k \geq 3$ when $d = 4$. Moreover, there exists some positive constant $C > 0$ such that the Yamabe constant of the conformal infinities is uniformly bounded from below by C . Assume there is $\delta_0 > 0$ such that if either*

$$(1') \quad \int_{X^d} (|W|^{d/2} d\text{vol})[g_i^+] < \delta_0, \quad \text{or}$$

$$(1'') \quad Y(\partial X, [h_i]) \geq Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}}]) - \delta_0,$$

then the set $\{g_i^\}$ of the adopted metrics (after diffeomorphisms that fix the boundary) is compact in $C^{k,\gamma'}$ Cheeger-Gromov topology for some $\gamma' < \gamma$.*

When the dimension d of the manifold X is odd, in general, we would not expect the strong estimate C^{d-1} as in the cases when d is even due to the term of $kr^{d-1} \log r$ term in the expansion of the metric g as in (1.2). This term k happens to be the obstruction tensor ([17, 20]) on the boundary of X and which may not vanish.

Our second result deals with general dimension d .

Theorem 1.2. *Suppose that X is a smooth oriented d -dimensional manifold with $d \geq 4$ and with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X . Assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set \mathcal{C} of metrics that is of positive Yamabe type and compact in C^6 Cheeger-Gromov topology. Moreover, there exists some positive constant $C > 0$ such that the Yamabe constant of the conformal infinities is uniformly bounded from below by C . Then under the above assumptions (1') or (1''), the set $\{g_i^*\}$ of the adopted metrics (after diffeomorphisms that fix the boundary) is compact in $C^{3,\gamma'}$ Cheeger-Gromov topology for all $0 < \gamma' < 1$.*

Remark 1. (1) *The results in Theorems 1.1 and 1.2 have been proved in [11] when $d = 4$.*

(2) *We can assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set \mathcal{C} in $C^{5,\gamma}$ Cheeger-Gromov topology. In such case, we have the compactness result in $C^{3,\gamma'}$ Cheeger-Gromov topology for all $0 < \gamma' < \gamma < 1$.*

(3) *We can expect the high order compactness result up to $C^{d-2,\gamma}$ in all dimensions $d \geq 4$, provided the Yamabe representative h of the conformal infinity has higher regularity. For example, the compactness results in $C^{k,\gamma'}$ with $2 \leq k \leq d - 2$ when the conformal infinity in $C^{k+1,\gamma}$ with $1 \geq \gamma > \gamma'$ (even in $C^{k,\gamma}$).*

As an application of Theorem 1.2, we are able to establish the global uniqueness for the CCE metrics on X^d with prescribed conformal infinities that are very close to the conformal round $(d - 1)$ -sphere as in the work of [21] (cf also [26, 27]). Namely,

Theorem 1.3. *For a given conformal $(d - 1)$ -sphere $(\mathbb{S}^{d-1}, [h])$ with $d \geq 4$ that is sufficiently close to the round one in C^6 topology, there is exactly one conformally compact Einstein metric g^+ on X^d whose conformal infinity is the prescribed conformal $(d - 1)$ -sphere $(\mathbb{S}^{d-1}, [h])$. Moreover, the topology of X should be a ball \mathbb{B}^n .*

Remark 2. *We remark*

- *In Theorems 1.1 and 1.2, we do not need the boundary condition $\partial X = \mathbb{S}^{d-1}$ for the compactness result.*
- *In Theorem 1.3, we could only assume in $C^{5,\gamma}$ topology for some $\gamma > 0$. Moreover, when $d = 6$ (resp. $d = 4$), we could consider the conformal infinity in $C^{4,\gamma}$ (resp. $C^{3,\gamma}$). Such result in dimension 4 has been obtained in [11]. In view of Lemma 3.1, when $d = 4$, we could get the compactness result in Theorem 1.1 under the assumption that the conformal infinity in $C^{2,\gamma}$. This will give the uniqueness result under such assumption on the conformal infinity (see [11, Remark1.10]). Thus, the global uniqueness result in all dimensions could be expected to be held when the conformal infinity is in $C^{2,\gamma}$.*

In this paper, we will present two different proofs of the regularity of the compactified metrics near the conformal infinity for the cases when d is even and when d is arbitrary. When d is even, we will take advantage that for the CCE manifold (X^d, g^+) , the d -th order obstruction tensor ([17, 20], see the definition in section 3) vanishes. Thus the 4-th Bach tensor for the compactified metric g^* satisfies an elliptic PDE (see (3.1)); this would lead to a gain of the regularity of the metric of g^* near the conformal infinity. We will describe this process in section 3.

For all dimensions d , under the assumptions (namely C^6) of the boundary metrics, we will present a different strategy to reach the compactness result in the statement of Theorem 1.2. Namely we will apply the gauge fixing technique for Einstein metric. By choosing a fixed gauge, we will see how to gain the regularity of the compactified metric g^* in a neighborhood of any point on the conformal infinity. This is carried out in section 4 of the paper. As this is the most technical part of the paper, we will first outline the different steps of the proof at the beginning of section 4 before presenting the details.

Besides the separate argument in section 3 (for even dimensional d) and 4 (for all d), in the rest of the paper i.e. in sections 1, 2, 5 and 6, the argument for even and odd dimensional manifold are the same.

The paper is organized as follows: In section 2, we recall some basic ingredients in the proofs and list some of their key properties, in particular the estimate of injectivity radius there is the major technical step where we use the technique of blow-up analysis in Riemannian geometry. In section 3, we prove the boundary regularity for X^d when d is even. In section 4, we present a different proof for the boundary regularity for all d dimensional manifold X^d which works for all d . In section 5, we establish various compactness results for the adopted metrics and prove Theorems 1.1 and 1.2. In section 6, we prove Theorem 1.3 to obtain the global uniqueness for the conformally compact Einstein

metrics on X^d (or \mathbb{B}^d) constructed earlier in [21, 26] and in Corollary 6.1 we establish some gap phenomenon for classes of conformal invariants.

2. PRELIMINARIES

2.1. Adopted metrics g^* . We now consider a class of adopted metrics g^* by solving (1.4) to study the compactness problem of CCE manifolds when dimension is greater than 4.

Lemma 2.1. (*Chang-R. Yang[12]*) *Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with conformal infinity $(\partial X, [h])$, fix $h_1 \in [h]$ and r its corresponding geodesic defining function. Consider the solution v to (1.4), then v has the asymptotic behavior*

$$v = r^{\frac{d-4}{2}}(A + Br^3)$$

near ∂X , where A, B are functions even in r , such that $A|_{\partial X} \equiv 1$.

The proof of this lemma is a special case of the general scattering theory on CCE manifold as in Graham-Zworski [22], for self-contained purpose, we will sketch the proof of this special case here.

Proof of Lemma 2.1. Denote $s = \frac{d-1}{2} + \frac{3}{2}$, then it follows from a theorem of Mazzeo-Melrose [31], that solution v of 1.4 has the asymptotic behavior

$$v = r^{d-1-s}A + r^sB \quad \text{on } X^d$$

for some functions A, B even function in r defined on an asymptotic neighborhood of ∂X , with $A|_{\partial X} = 1$. \square

Lemma 2.2. (*Chang-R. Yang [12, Lemma 5.3]*) *With the same notation as in Lemma 2.1, Consider the metric $g^* = v^{\frac{4}{d-4}}g^+$, then g^* is totally geodesic on boundary with the free Q -curvature, that is, $Q_{g^*} \equiv 0$.*

Proof. Recall the fourth order Paneitz operator is given by

$$P_4 = (-\Delta)^2 + \delta(4A - \frac{d-2}{2(d-1)}R)\nabla + \frac{d-4}{2}Q_4$$

where $A = \frac{1}{d-2}(Ric - \frac{R}{2(d-1)}g)$ is the schouten tensor, δ is the dual operator of the differential ∇ and Q_4 is a fourth order Q -curvature. More precisely, let $\sigma_k(A)$ denote the k -th symmetric function of the eigenvalues of A and $Q_4 := -\Delta\sigma_1(A) + 4\sigma_2(A) + \frac{d-4}{2}\sigma_1(A)^2$. For a Einstein metric with $Ric_{g^+} = -(d-1)g^+$, thus we have $Q_4[g^+] = 0$ and

$$P_4[g^+] = (-\Delta_{g^+} - \frac{(d-1)^2 - 1}{4}) \circ (-\Delta_{g^+} - \frac{(d-1)^2 - 9}{4}).$$

Therefore

$$Q_4[g^*] = \frac{2}{d-4}P_4[g^*]1 = \frac{2}{d-4}v^{\frac{d+4}{d-4}}P_4[g^+]v = 0$$

It follows from the asymptotic behavior of v (Lemma 2.1) that g^* is totally geodesic on boundary since $\frac{\partial v}{\partial \nu} = 0$ on M where ν is the normal vector on the boundary. \square

Suppose that X is a smooth d -dimensional manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X . Let $g^* = \rho^2 g^+$ be the adopted metrics, that is, $v := \rho^{\frac{d-4}{2}}$ satisfies the equation (1.4). We recall some basic calculations for curvatures under conformal changes. Write $g^+ = r^{-2}g$ for some defining function r and calculate

$$\text{Ric}[g^+] = \text{Ric}[g] + (d-2)r^{-1}\nabla^2 r + (r^{-1}\Delta r - (d-1)r^{-2}|\nabla r|^2)g.$$

Then one has

$$R[g^+] = r^2(R[g] + \frac{2d-2}{r}\Delta r - \frac{d(d-1)}{r^2}|\nabla r|^2).$$

Here the covariant derivatives is calculated with respect to the metric g (or adopted metrics g^* in the following). Therefore, for adopted metrics g^* of a conformally compact Einstein metric g^+ , one has

$$(2.1) \quad R[g^*] = 2(d-1)\rho^{-2}(1 - |\nabla\rho|^2),$$

which in turn gives

$$(2.2) \quad \text{Ric}[g^*] = -(d-2)\rho^{-1}\nabla^2\rho + \frac{4-d}{4(d-1)}R[g^*]g^*$$

and

$$(2.3) \quad R[g^*] = -\frac{4(d-1)}{d+2}\rho^{-1}\Delta\rho.$$

Now we recall

Lemma 2.3. ([9]) *Suppose that X is a smooth d -dimensional manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X with the conformal infinity $(\partial X, [h])$ of nonnegative Yamabe type. Let $g^* = \rho^2 g^+$ be adopted metrics associated with the metric h with the positive scalar curvature in the conformal infinity. Then the scalar curvature $R[g^*]$ is positive in X . In particular,*

$$(2.4) \quad \|\nabla\rho\|[g^*] \leq 1.$$

2.2. Elliptic estimates for conformal Einstein metric and Q-flat metrics. Let R_{ijkl} , W_{ijkl} , R_{ij} and R be Riemann, Weyl, Ricci, Scalar curvature tensors respectively. We recall the definition of 4-th order Bach tensor B on manifolds of dimension d (X^d, g) as

$$(2.5) \quad B_{ij} := \frac{1}{d-3}\nabla^k\nabla^l W_{kijl} + \frac{1}{d-2}W_{kijl}R^{kl}.$$

Recall also the Cotten tensor \mathcal{C} is defined as

$$(2.6) \quad \mathcal{C}_{ijk} = A_{ij,k} - A_{ik,j}$$

where A is the schouten tensor

$$(2.7) \quad A_{ij} = \frac{1}{d-2}\left(\text{Ric}_{ij} - \frac{Rg_{ij}}{2(d-1)}\right)$$

It turns out there is a relation between the divergence of Weyl tensor to the Cotton tensor, namely

$$(2.8) \quad \nabla^l W_{ijkl} = (d-3)\mathcal{C}_{kij}$$

Applying this relation (2.8), we can write the Bach tensor into the following equations

$$(2.9) \quad (d-2)B_{ij} = \Delta R_{ij} - \frac{d-2}{2(d-1)}\nabla_i\nabla_j R - \frac{1}{2(d-1)}\Delta R g_{ij} + Q_1(Rm),$$

where $Q_1(Rm)$ is some quadratic term on Riemann curvature tensor

$$Q_1(Rm) := 2W_{ikjl}R^{kl} - \frac{d}{d-2}R_i{}^k R_{jk} + \frac{d}{(d-1)(d-2)}RR_{ij} + \left(\frac{1}{d-2}R_{kl}R^{kl} - \frac{R^2}{(d-1)(d-2)}\right)g_{ij}$$

We now recall that the adapted metric g^* which we haven chosen in Section 2.1 is Q -flat, i.e., $Q[g^*] = 0$, which can be rewritten into the following form

$$(2.10) \quad -\Delta R = -\frac{d^3 - 4d^2 + 16d - 16}{4(d-2)^2(d-1)}R^2 + \frac{4(d-1)}{(d-2)^2}|Ric|^2.$$

In this section, we will incorporate the Q -flat property of g^* to the Bach equation of g^* to derive estimates of the curvature of g^* under the assumptions of Theorem 1.1. To do so we first list properties of Bach tensor and Cotton tensor under conformal change of metrics. To see Bach equations coupled with Q -flat equation also provide estimates of Weyl curvature, one may rewrite Bach tensor equation as follows:

$$(2.11) \quad \Delta W_{ijkl} + (d-3)\nabla_l C_{kji} + (d-3)\nabla_k C_{lij} + \nabla_i C_{jkl} + \nabla_j C_{ilk} := K_{ijkl} + L_{ijkl},$$

where K is a quadratic of curvatures and $L_{ijkl} := B_{ki}g_{jl} + B_{lj}g_{ik} - B_{li}g_{jk} - B_{kj}g_{il}$ is some linear term on the Bach tensors. We recall some basic facts on the boundary $M = \partial X$. Let ∂_1 denote the boundary normal direction and $\alpha, \beta \in \{2, \dots, n\}$.

Lemma 2.4. *On (X^d, g) , suppose $\tilde{g} = e^{2w}g$, we have*

$$(2.12) \quad \widetilde{C_{ijk}} := C_{ijk}[\tilde{g}] = C_{ijk}[g] - g^{ml}W_{kjim}[g]w_l$$

$$(2.13) \quad \widetilde{B_{ij}} := B_{ij}[\tilde{g}] = e^{-2w}B_{ij}[g] + e^{-2w}(d-4)\langle \nabla[g]w, C_{i\cdot} + C_{\cdot i} \rangle_g + e^{-2w}(d-4)\nabla^k[g]w\nabla^l[g]wW_{kijl}[g]$$

(2.12) and (2.13) are derived by a routine but tedious computations. If we apply Lemma 2.4 to the adopted metrics $g^* = \rho^2 g^+$ and notice since g^+ is Einstein. Both Bach tensor and Cotton tensor of g^+ vanish, we obtain from the fact $W_{jkil}[g^*] = \rho^2 W_{jkil}[g^+]$

Corollary 2.5. *Suppose $(X^d, \partial X, g^+)$ is a conformally compact Einstein with adopted metrics $g^* = \rho^2 g^+$. Then, we have*

$$(2.14) \quad B_{ij}[g^*] = \rho^{-2}(d-4)\nabla^k[g^*]\rho\nabla^l[g^*]\rho W_{ikjl}[g^*] = -(d-4)\rho^{-1}\nabla^k[g^*]\rho C_{ikj}[g^*]$$

$$(2.15) \quad C_{ijk}[g^*] = \rho^{-1}\nabla^l[g^*]\rho W_{jkil}[g^*]$$

Lemma 2.6. *Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with conformal infinity $(\partial X, [h])$ with C^3 compactification. Then, for the adopted metrics g^* , we have on the boundary $M = \partial X$*

(1) ∂X is totally geodesic;

(2) $R = \frac{2(d-1)}{d-2}\hat{R}$;

(3) $R_{11} = \frac{d}{2(d-2)}\hat{R}$, $R_{1\alpha} = 0$, $R_{\alpha\beta} = \frac{d-2}{d-3}\hat{R}_{\alpha\beta} - \frac{1}{2(d-2)(d-3)}\hat{R}g_{\alpha\beta}$

- (4) $W_{\alpha\beta\gamma\delta} = \hat{W}_{\alpha\beta\gamma\delta}$ and Weyl tensor vanishes for all other indices;
(5) $\mathcal{C}_{\alpha\beta\gamma} = \hat{\mathcal{C}}_{\alpha\beta\gamma}$ and cotten tensor vanishes for all other indices;
(6) $\nabla_1 A_{11} = \frac{\nabla_1 R}{2(d-1)}$, $\nabla_\alpha A_{\beta\gamma} = \hat{\nabla}_\alpha \hat{A}_{\beta\gamma}$ and the first covariant derivatives of Schouten tensor A vanishes for all other indices;
(7) $\nabla_\sigma W_{\alpha\beta\gamma\delta} = \hat{\nabla}_\sigma \hat{W}_{\alpha\beta\gamma\delta}$, $\nabla_1 W_{\alpha\beta\gamma 1} = -\nabla_1 W_{\alpha\beta 1\gamma} = \nabla_1 W_{\gamma 1\alpha\beta} = -\nabla_1 W_{1\gamma\alpha\beta} = \hat{\mathcal{C}}_{\gamma\alpha\beta}$ and the first covariant derivatives of Wyel tensor W vanishes for all other indices;

Proof. The part (1) is well known.

It follows from [9, section 6] that $\frac{R}{2(d-1)} = 2\frac{\hat{R}}{2(d-2)}$ on the boundary. Therefore (2) is proved.

For the second part, we remark first from Gauss-Codazzi aquations that

$$R_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} \text{ and } R_{1\beta\gamma\delta} = 0$$

so that

$$R_{1\alpha} = 0.$$

To prove the rest, we write $g_1 = r^2 g^+$ he compactified metric under some geodesic defining function r and $g^* = \rho^2 g^+$ the adopted metric. We assume both g_1 and g^* have the same boundary metric h and totally geodesic boundary. We write $g^* = w^{-2} g_1 := (\frac{r}{\rho})^{-2} g_1$. Thus, on the boundary ∂X , we have $w \equiv 1$, and $\nabla w \equiv 0$. As a consequence, we infer on the boundary

$$A_{\alpha\beta}[g^*] = A_{\alpha\beta}[g_1], \quad W[g^*] = W[g_1]$$

since $\nabla_\alpha \nabla_\beta w = 0$ on M . Now we study the schouten tensor and Weyl tensor for the compactified metric g_1 . We note the full indices $i, j, k \in \{1, \dots, d\}$. As before, we have $r^2 g^+ =: g = ds^2 + g_r$, $g_r = h + g^{(2)} r^2 + O(r^4)$, $g_{\alpha\beta}^{(2)} = -\hat{A}_{\alpha\beta}$ where \hat{A} is the schouten tensor of the metrc h (see [19]). For the convenience of readers, we scketch proof. Let (x_1, x_2, \dots, x_d) be coordinates on the boundary $M = \partial X$. We have $g_{11} = 1$, $g_{1\alpha} = 0$ and $g_{\alpha\beta} = h_{\alpha\beta} + O(r^2)$. A direct calculation leads to the Christoph symbols $\Gamma_{j1}^i = 0$ on the boundary M , that is $(\nabla g)_{\frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_1}} = 0$ on the boundary M , which implies the desired claim. we fix a point P on the boundary M and let $(x_1, x_2, , \dots, x_d)$ be normal coordinates at P . At P , we have the Christoph symbols $\Gamma_{jk}^i = 0$. Hence, we can write at P

$$R_{ijk}{}^l = \frac{1}{2} g^{lm} (g_{im,kj} + g_{jk,mi} - g_{ik,mj} - g_{jm,ki})$$

so that

$$(2.16) \quad R_{1\alpha 1}{}^\gamma = -\frac{1}{2} g_{\alpha\gamma,11} = -g_{\alpha\gamma}^{(2)} = \hat{A}_{\alpha\gamma}$$

On the other hand, on the boundary M , we have the following Gauss-Codazzi equations

$$R_{\alpha\beta\gamma}{}^\delta = \hat{R}_{\alpha\beta\gamma}{}^\delta \text{ and } R_{1\beta\gamma}{}^\delta = 0$$

since the boundary is totally geodesic. Therefore, at P , we have $R_{\alpha 1} = 0$, $R_{\alpha\beta} = \hat{R}_{\alpha\beta} + R_{\alpha 1\beta}{}^1$, and $R = \hat{R} + 2R_{11}$. On the other hand, it follows from (2.16) that

$$R_{11} = \frac{\hat{R}}{2(d-2)} \text{ and } R = \frac{d-1}{d-2} \hat{R}$$

Gathering the above relations from (2.16), we infer

$$\begin{aligned} A_{11} &= 0 \\ A_{1\alpha} &= \frac{1}{d-2} R_{1\alpha} = 0 \\ A_{\alpha\beta} &= \frac{1}{d-2} (\hat{R}_{\alpha\beta} + R_{1\alpha 1}{}^\beta - \frac{R}{2(d-1)} g_{\alpha\beta}) = \frac{1}{d-2} (\hat{R}_{\alpha\beta} + \hat{A}_{\alpha\beta} - \frac{\hat{R}}{2(d-2)} g_{\alpha\beta}) = \hat{A}_{\alpha\beta} \end{aligned}$$

By the decomposition of Riemann curvature, we have

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - (A \otimes g)_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} - (\hat{A} \otimes \hat{g})_{\alpha\beta\gamma\delta} = \hat{W}_{\alpha\beta\gamma\delta}$$

On the other hand, we know

$$R_{1\beta\gamma\delta} = 0 = (A \otimes g)_{1\beta\gamma\delta}$$

so that

$$W_{1\beta\gamma\delta} = 0$$

Moreover, by (2.16) and the decomposition of Riemann curvature, we infer

$$W_{1\beta 1\delta} = R_{1\beta 1\delta} - (A \otimes g)_{1\beta 1\delta} = R_{1\beta 1\delta} - A_{\beta\delta} = R_{1\beta 1\delta} - \hat{A}_{\beta\delta} = 0.$$

It is clear that $W_{111\delta} = W_{1111} = 0$.

Using (2.15), we infer

$$(2.17) \quad \rho \mathcal{C}_{ijk} = \nabla^l \rho W_{jkil}$$

so that, by taking the covariant derivative, we get

$$\nabla^m \rho \mathcal{C}_{ijk} + \rho \nabla^m \mathcal{C}_{ijk} = \nabla^m \nabla^l \rho W_{jkil} + \nabla^l \rho \nabla^m W_{jkil}$$

Hence, together with (2.2) and by choosing $m = 1$, we deduce on the boundary M

$$\mathcal{C}_{ijk} = \nabla^1 W_{jki1} = \nabla^p W_{jkip} - \nabla^\alpha W_{jkia} = (d-3)\mathcal{C}_{ijk} - \hat{\nabla}^\alpha W_{jkia}$$

That is,

$$(d-4)\mathcal{C}_{ijk} = \hat{\nabla}^\alpha W_{jkia}$$

Therefore

$$(d-4)\mathcal{C}_{\beta\gamma\delta} = \hat{\nabla}^\alpha \hat{W}_{\gamma\delta\beta\alpha} = (d-4)\hat{\mathcal{C}}_{\beta\gamma\delta}.$$

When the indices ijk contain 1, it follows from (4) that

$$(d-4)\mathcal{C}_{\beta\gamma\delta} = 0$$

Hence, we prove (5).

By the expression of the schouten tensor on the boundary, we have

$$\nabla_\alpha A_{\beta\gamma} = \hat{\nabla}_\alpha \hat{A}_{\beta\gamma}, \nabla_\alpha A_{11} = \nabla_\alpha A_{1\beta} = 0$$

Together with the expression of the Cotten tensor on the boundary, we infer

$$\nabla_1 A_{1\alpha} = 0, \nabla_1 A_{\alpha\beta} = 0.$$

Thanks of the second Bianchi identity, we obtain

$$\nabla_\alpha A_{1\alpha} + \nabla_1 A_{11} = \frac{\nabla_1 R}{2(d-1)}$$

Therefore, the results in (6) are proved.

The first equality in (7) is a direct result of ones in (4). On the other hand, because of (4), we have also when the indices $ijkl$ contain 1

$$\nabla_\alpha W_{ijkl} = 0$$

Hence

$$\nabla_1 W_{\alpha\beta\gamma 1} = \nabla_k W_{\alpha\beta\gamma k} - \nabla_\delta W_{\alpha\beta\gamma\delta} = \nabla_k W_{\alpha\beta\gamma k} - \hat{\nabla}_\delta \hat{W}_{\alpha\beta\gamma\delta} = (d-3)\mathcal{C}_{\gamma\alpha\beta} - (d-4)\hat{\mathcal{C}}_{\gamma\alpha\beta} = \hat{\mathcal{C}}_{\gamma\alpha\beta}$$

and

$$\nabla_1 W_{\alpha 1\gamma 1} = \nabla_k W_{\alpha 1\gamma k} - \nabla_\delta W_{\alpha 1\gamma\delta} = \nabla_k W_{\alpha 1\gamma k} = (d-3)\mathcal{C}_{\gamma\alpha 1} = 0.$$

It is clear that $\nabla_1 W_{\alpha 111} = 0$. Finally, we prove the result. \square

Our first step is to handle the regularity of the Bach tensors. For this purpose, we want to exploit the obstruction tensor found by Fefferman and Graham [16] and express the curvature tensors on the boundary relating to the boundary metric.

Lemma 2.7. *Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with conformal infinity $(\partial X, [h])$ with $d \geq 6$. Then, under the C^{d-1} adopted metrics g^* , we have on the boundary $M = \partial X$ for the all multi-index $i = (i_1, \dots, i_l)$ of the length $|i| := l \leq d-3$ with $1 \leq i_1, \dots, i_l \leq d$*

$$\nabla_i A = P(\hat{\nabla}_\gamma \hat{A}, \hat{\nabla}_\delta \hat{W}, \hat{\nabla}_\kappa(\nabla_1 R)|_M), \quad \nabla_i W = P_1(\hat{\nabla}_\gamma \hat{A}, \hat{\nabla}_\delta \hat{W}, \hat{\nabla}_\kappa(\nabla_1 R)|_M)$$

where P and P_1 are some homogenous polynomials on $(\hat{\nabla}_\gamma \hat{A}, \hat{\nabla}_\delta \hat{W}, \hat{\nabla}_\kappa(\nabla_1 R)|_M)$ with the multi-indices γ, δ, κ satisfying $|\gamma| + |\delta| + |\kappa| \leq l$ for each term in the polynomials, each component of γ, δ, κ taking values from 2 to d , where $|\cdot|$ designates the length of the multi-indice.

Proof. We prove the result by induction.

For $l = 0, 1$, it follows from Lemma 2.6.

Assume the result is true for $l = r$. When i_1, \dots, i_{r+1} are not all equal to 1, we could change the order of the covariant derivative such that

$$\nabla_i A = \nabla_{i_j} \nabla_{i'} A + P_r(\nabla_m Rm)$$

where $i_j \neq 1$, i' designates the multi-index removed i_j , $|m| \leq r$, and P_r involves only the derivatives of Riemann curvature of the order less than r . In such case, the results follow from the induction. It is similar for the Weyl tensor W . Now we treat the $r+1$ order the normal derivatives $\nabla_1^{(r+1)} A$ and $\nabla_1^{(r+1)} W$. For this purpose, we study first $\nabla_1^{(r)} \mathcal{C}_{ijk}$. Recall (2.17) and take the r order normal derivatives so that

$$\begin{aligned} r \nabla_1^{(r)} \mathcal{C}_{ijk} &= \nabla_1^{(r)} W_{jki1} + Q_r(\nabla_m Rm) \\ &= \nabla_1^{(r-1)} \delta W_{jki} - \nabla_1^{(r-1)} \nabla^\beta W_{jki\beta} + Q_r(\nabla_m Rm) \\ &= \nabla_1^{(r-1)} \delta W_{jki} - \nabla^\beta \nabla_1^{(r-1)} W_{jki\beta} + \bar{Q}_r(\nabla_m Rm) \\ &= (d-3) \nabla_1^{(r)} \mathcal{C}_{ijk} - \nabla^\beta \nabla_1^{(r-1)} W_{jki\beta} + \bar{Q}_r(\nabla_m Rm) \end{aligned}$$

Here Q_r, \bar{Q}_r involves only the derivatives of Riemann curvature of the order less than r and we use the relations (2.1) to (2.3) and the assumption in the induction. Therefore, we deduce

$$(d-3-r) \nabla_1^{(r)} \mathcal{C}_{ijk} = \nabla^\beta \nabla_1^{(r-1)} W_{jki\beta} - \bar{Q}_r(\nabla_m Rm)$$

which yields the desired result for the cotten tensor C . Applying the equations (2.9) to (2.11), we obtain

$$\begin{aligned}\nabla_1^{(r+1)} A &= \nabla_1^{(r-1)} \Delta A - \nabla_1^{(r-1)} \nabla_\beta \nabla^\beta A, \\ \nabla_1^{(r+1)} W &= \nabla_1^{(r-1)} \Delta W - \nabla_1^{(r-1)} \nabla_\beta \nabla^\beta W\end{aligned}$$

Hence, the claims follows. Finally, we prove the result. \square

2.3. Cheeger-Gromov convergences for manifolds with boundary. Our approach to establish the compactness of conformally compact Einstein d -dimensional manifolds is to prove by contradiction. We will analyze and eliminate the causes of possible non-compactness by the method of blow-up. This method has been essential and powerful in many compactness problems in geometric analysis, particularly in Riemannian geometry. The fundamental tool in the context of Riemannian geometry is the so-called Cheeger-Gromov convergences of Riemannian manifolds developed from Gromov-Hausdorff convergences (see, for example, [13, 1], for Cheeger-Gromov convergences of Riemannian manifolds without boundary). In this subsection, for later uses in our paper, we will present the Cheeger-Gromov convergences for manifolds with boundary. Good references in the subject are for examples in [32, 24, 25, 35, 4].

Let us first recall the definition of harmonic radius for a Riemannian manifold with boundary (cf. [32]). Assume (X, g) is a complete Riemannian d -dimensional manifold with the boundary ∂X . A local coordinates

$$(x_1, \dots, x_d) : B(p, r) \rightarrow \Omega \subset \mathbb{R}^d$$

is said to be harmonic if,

- $\Delta x_i = 0$ for all $1 \leq i \leq d$ in $B(p, r) \subset X$, when $p \in X$ is in the interior;
- $\Delta x_i = 0$ for all $1 \leq i \leq d$ in $B(p, r) \cap X$ and, on the boundary $B(p, r) \cap \partial X$, (x_2, \dots, x_d) is a harmonic coordinate in ∂X at p while $x_1 = 0$, when $p \in \partial X$ is on the boundary.

For $\alpha \in (0, 1)$ and $M \in (1, 2)$, we define the harmonic radius $r^{1,\alpha}(M)$ to be the biggest number r satisfying the following properties:

- If $\text{dist}(p, \partial X) > r$, there is a harmonic coordinate chart on $B(p, r)$ such that

$$(2.18) \quad M^{-2} \delta_{jk} \leq g_{jk}(x) \leq M^2 \delta_{jk}$$

and

$$(2.19) \quad r^{1+\alpha} \sup |x - y|^{-\alpha} |\partial g_{jk}(x) - \partial g_{jk}(y)| \leq M - 1$$

in $\overline{B(p, \frac{r}{2})}$.

- If $p \in \partial X$, there is a boundary harmonic coordinate chart on $B(p, 4r)$ such that (2.18) and (2.19) hold in $\overline{B(p, 2r)}$.

The following is the extension of the $C^{1,\alpha}$ convergence theorem of Anderson [13, 1] to manifolds with boundary (cf. [24, 4]).

Lemma 2.8. ([4, Theorem 3.1],[11, Remark 2.7]) *Suppose that $\mathcal{M}(R_0, i_0, h_0, d_0)$ is the set of all compact Riemannian manifolds (X, g) with boundary such that*

$$\begin{aligned} |Ric_X| &\leq R_0, & |Ric_{\partial X}| &\leq R_0 \\ i_{int}(X) &\geq i_0, & i_{\partial}(X) &\geq 2i_0, & i(\partial X) &\geq i_0, \\ Diam(X) &\leq d_0, & \|H\|_{Lip(\partial X)} &\leq h_0, \end{aligned}$$

where $Ric_{\partial X}$ is the Ricci curvature of the boundary, $i(\partial X)$ is the injectivity radius of the boundary, $i_{int}(X)$ is the interior injectivity radius, $i_{\partial}(X, g)$ is the boundary injectivity radius and H is the mean curvature of the boundary. Then $\mathcal{M}(R_0, i_0, h_0, d_0)$ is pre-compact in the $C^{1,\alpha}$ Cheeger-Gromov topology for any $\alpha \in (0, 1)$. Moreover, if the Ricci curvatures are bounded in $C^{k,\alpha}$ norm and the boundaries are all totally geodesic with $k \geq 0$, then one has the pre-compactness in $C^{k+2,\alpha'}$ Cheeger-Gromov topology with $\alpha' < \alpha$. Furthermore, one has the pre-compactness in the Cheeger-Gromov topology with base points if dropping the assumption on the diameter $Diam(X)$.

2.4. Injectivity radii: blow-up before blow-up. Our main results in this subsection concern the injectivity radius estimates for manifolds with boundary. For our purpose we may always assume that the geometry of the boundary is compact in Cheeger-Gromov sense. The following Lemma will be established as a consequence of [4, Theorem 3.1], which was mentioned as Lemma 2.8.

Lemma 2.9. *Suppose that (X^d, g^+) is a conformally compact Einstein d -dimensional manifold with the conformal infinity of Yamabe constant $Y(\partial X, [h]) \geq Y_0 > 0$. And suppose that (X^d, g^*) is the adopted metric associated with a representative metric h in the conformal infinity on the boundary with the non-negative scalar curvature such that the intrinsic injectivity radius $i(\partial X, h) \geq i_0 > 0$, and that $i_{\partial}(X, g^*) \leq i_{int}(X, g^*)$. Then there is a constant $C_{\partial} = C(d) > 0$, depending of i_0 , such that*

$$(2.20) \quad \max_X |Rm|(i_{\partial}(X, g^*))^2 + i_{\partial}(X, g^*) \geq C_{\partial}$$

where Rm is Riemann curvature of g^* .

The proof of the above property is as same as the one of [11, Lemma 3.1]. One of key facts is the result in [10, Lemma 4.4]: $\rho(x) \geq Cd(x, \partial X)$ provided $d(x, \partial X) \leq i_{\partial}(X, g^*)$ and when $i_{\partial}(X, g^*) \geq 1$, then $\rho(x) \geq C$ provided $d(x, \partial X) \geq \frac{1}{2}$. We leave the details to the interested readers.

Next we would like to get the lower bound estimates for the interior injectivity radius i_{int} of a compact Riemannian manifold with boundary. The real reason for having no interior collapsing follows from the following recent work in [27], which plays also an important role in Theorems 1.3 and Corollary 6.1.

Lemma 2.10. (Li-Qing-Shi [27, Theorem 1.3]) *Suppose that (X^d, g^+) is a conformally compact Einstein manifold with the conformal infinity of Yamabe constant $Y(\partial X, [h]) > 0$.*

Then, for any $p \in X^d$,

$$(2.21) \quad 1 \geq \frac{\text{vol}_{g^+}(B(p, r))}{\text{vol}_{g_{\mathbb{H}^d}}(B(r))} \geq \left(\frac{Y(\partial X, [h])}{Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}^{d-1}}])} \right)^{\frac{d-1}{2}}$$

We now will derive similar estimate like Lemma 2.9 for the lower bound of the injectivity radius as a consequence of Lemma 2.10.

Lemma 2.11. *Suppose that (X^d, g^+) is a conformally compact Einstein d -dimensional manifold with the conformal infinity of Yamabe constant $Y(\partial X, [h]) \geq Y_0 > 0$. And suppose that (X^d, g^*) is the adopted metric associated with the Yamabe metric h on the boundary such that the intrinsic injectivity radius $i(\partial X, h) \geq i_o > 0$, and that $i_\partial(X, g^*) \geq i_{\text{int}}(X, g^*)$. Then there is a constant $C_{\text{int}} > 0$, depending of Y_0 and i_o , such that*

$$(2.22) \quad \max_X |Rm|(i_{\text{int}}(X, g^*))^2 + i_{\text{int}}(X, g^*) \geq C_{\text{int}}$$

where Rm is the Riemann curvature of g^* .

The proof of the above Lemma is similar to the one of [11, Lemma 3.3]. Here we omit the details.

2.5. Möbius coordinates and weighted function spaces. Let (X, g^+) be a conformally compact Einstein d -manifold with a continuous conformal compactification $g = \rho^2 g^+$. ρ is a C^1 defining function for (\bar{X}, g) . For any $\epsilon > 0$, let X_ϵ denote the open subset of \bar{M} where $0 < \rho < \epsilon$ and \bar{X}_ϵ denote the open subset where $0 \leq \rho < \epsilon$.

We firstly choose smooth local coordinates $\theta = (\theta^1, \theta^2, \dots, \theta^d)$ on an open set $U \subset \partial X$. It can extend to $(\theta^1, \theta) = (\rho, \theta^1, \theta^2, \dots, \theta^d)$ on the open subset $\Omega = [0, \epsilon) \times U \subset \bar{X}$. Choose finitely many U_i to cover ∂X . The resulting coordinates on $\Omega_i = [0, \epsilon_i) \times U_i$ will be called background coordinates for \bar{X} . Let R be the smallest number of such ϵ_i , then any point in \bar{X}_R is contained in some background coordinate chart.

Now we consider upper half-space model of hyperbolic space, i.e. $\mathbb{H}^d = \{(y, x) = (y, x^1, x^2, \dots, x^d) \in \mathbb{R}^d : y > 0\}$, with $x^1 = y$ and with the hyperbolic metric \check{g} given in coordinates by

$$\check{g} = \frac{1}{y^2}(dy^2 + dx^2).$$

We let B_1 and B_2 denote the hyperbolic geodesic ball of radius 1 and 2 centred at point $(y, x) = (1, 0)$. For any point $p \in M_R$, let (ρ_0, θ_0) be the coordinate representation of p in some fixed background chart. We can define a diffeomorphism $\Phi_p : B_2 \rightarrow X$ by

$$(\rho, \theta) = \Phi_p(y, x) = (\rho_0 y, \theta_0 + \rho_0 x).$$

As is shown in [26], Φ_{p_0} maps B_2 diffeomorphically onto a neighborhood of p_0 in M_R if $p_0 \in M_{R/8}$. And there exists a countable set of points $\{p_i\} \subset M_{R/8}$ such that the sets $\Phi_{p_i}(B_2)$ form a uniformly locally finite covering of $M_{R/8}$, and the sets $\{\Phi_{p_i}(B_1)\}$ still cover $M_{R/8}$. We set

$$\Phi_i = \Phi_{p_i}, \quad V_1(p_i) = \Phi_i(B_1), \quad V_2(p_i) = \Phi_i(B_2).$$

We call $(V_2(p_i), \Phi_i^{-1})$ a Möbius coordinate chart of $M_{R/8}$.

In order to study the geometric elliptic operator near the boundary, we need to use the boundary Möbius coordinates, which was first introduced in [26].

For any fix $p \in \partial X$, let Ω be a neighbourhood and (ρ, θ) be the background coordinates such that $\theta(p) = 0$. We can assume that the compacted metric $g_p = \delta_{ij}$ in the coordinates. For each $a > 0$ and R sufficiently small, we define $Y_a \subset \mathbb{H}$ and $Z_R(p) \subset \Omega \subset \bar{X}$:

$$\begin{aligned} Y_a &= \{(y, x) \in \mathbb{H} : |x| < a, 0 < y < a\} \\ Z_R(p) &= \{(\rho, \theta) \in \Omega : |\theta| < R, 0 < \rho < R\} \end{aligned}$$

Define a chart $\Psi_{p,R} : Y_1 \rightarrow Z_R(p)$ by

$$(x, y) \mapsto (Ry, Rx) = (\rho, \theta)$$

We will call $\Psi_{p,R}$ a boundary Möbius chart of radius R centered at p .

We consider the tensor bundle E of co-variant rank p and contra-variant rank q over \bar{X} . We will also use the same symbol E to denote the restriction of this bundle to X . Define the weight of such a bundle $E \subset T_q^p$ to be $r = p - q$. If $g = \rho^2 g^+$, then

$$|T|_{g^+} = \rho^r |T|_g \quad \text{for all } T \in T_q^p X.$$

Next we will define weighted Hölder spaces of tensor fields. For any $0 < \alpha < 1$ and m a nonnegative integer, let $C_{(0)}^{m,\alpha}(\bar{X})$ denote the usual Banach space of functions on \bar{X} with m derivatives that are Hölder continuous of degree α up to the boundary in each background coordinate chart, with the obvious norm. Let s be a real number satisfying $0 \leq s \leq m + \alpha$, we can define weighted space $C_{(s)}^{m,\alpha}(\bar{X}) \subset C_{(0)}^{m,\alpha}(\bar{X})$ by

$$C_{(s)}^{m,\alpha}(\bar{X}) = \{u \in C_{(0)}^{m,\alpha}(\bar{X}) : u = O(\rho^s)\}.$$

It is showed in [26] that we can consider $C_{(s)}^{m,\alpha}(\bar{X})$ as a Banach space with the norm inherited from $C_{(0)}^{m,\alpha}(\bar{X})$.

Let E be a geometric tensor bundle over \bar{X} , we define $C_{(0)}^{m,\alpha}(\bar{X}; E)$ to be the space of tensor fields whose components in each background coordinate chart are in $C_{(0)}^{m,\alpha}(\bar{X})$.

There are the following relationships between the Hölder spaces on X and those on \bar{X} :

Lemma 2.12. [26] *Let E be a geometric tensor bundle of weight r over \bar{X} , and suppose $0 < \alpha < 1, 0 < m + \alpha \leq l + \beta$, and $0 \leq s \leq k + \alpha$. The following inclusions are continuous.*

- (a) $C_{(s)}^{m,\alpha}(\bar{X}; E) \hookrightarrow C_{s+r}^{m,\alpha}(X; E)$
- (b) $C_{m+\alpha+r}^{m,\alpha}(X; E) \hookrightarrow C_{(0)}^{m,\alpha}(\bar{X}; E)$.

Remark 2.13. *Let (X, g_i^+) be a sequence of AH manifolds with the compactified metrics (\bar{X}, g_i) . When (\bar{X}, g_i) converges to some Riemannian metric in $C^{l,\beta}$ topology in the sense of Cheeger-Gromov and the defining functions ρ_i converge to some defining functions of the corresponding AH manifold (X, g_∞^+) in $C^{l+1,\beta}$ topology, then the norm of the above linear embeddings is uniformly bounded above. Such result could be proved directly or follows from Banach-Steinhaus Theorem because of the convergence of the manifolds and the defining functions.*

2.6. Interior regularity in all dimensions. Although the interior regularity for CCE manifolds is well known, we derive the interior regularity for convenience.

Theorem 2.14. *Suppose $(X^d, \partial X, g^+)$ is a conformally compact Einstein with the $C^{k-2, \gamma}$ adopted metrics $g^* = \rho^2 g^+$ for $k \geq 2$ and $\gamma \in (0, 1)$. Assume that*

- (1) *Given $M > 1$ and $\gamma \in (0, 1)$ there exists some $r_0 > 0$ such that the harmonic radius $r^{1, \gamma}(M) \geq r_0$;*
- (2) *there exist positive constants $C, C_1 > 0$ such that $\rho(x) \geq C_1$ provided $d_{g^*}(x, \partial X) \geq C$;*

Then for all $x \in \bar{X}$ with $d_{g^}(x, \partial X) \geq C$ and for all $r \leq r_1 := \min(r_0, C/2)$, we have*

$$(2.23) \quad \|Ric_{g^*}\|_{C^{k, \gamma}(B(x, r/2))} \leq C(M, \gamma, r_0, C_1, k, \|Rm_{g^*}\|_{C^{k-2, \gamma}(B(x, r_1))})$$

which yields also in harmonic coordinates

$$(2.24) \quad \|g^*\|_{C^{k+2, \gamma}(B(x, r/2))} \leq C(M, \gamma, r_0, C_1, k, \|g^*\|_{C^{k, \gamma}(B(x, r_1))})$$

Proof. In view of equation (2.10), it follows from [18, Theorem 6.2], that the estimate (2.23) holds for the scalar curvature since $Rm_{g^*} \in C^{k-2, \gamma}$, that is, $R \in C^{k, \gamma}$. Using Lemma 2.3 and the formula (2.14), (2.2) and (2.3), the Bach tensor $B \in C^{k-2, \gamma}(B(x, r))$. Recall the elliptic system (2.9). By the classical regularity theory [18, Theorem 6.2], we derive $Ric_{g^*} \in C^{k, \gamma}(B(x, 3r/4))$ and the estimate (2.23) holds. Finally, the estimate (2.24) comes from Lemma 2.8. \square

Remark 3. *We notice the metric g^* is smooth in the interior.*

3. BOUNDARY REGULARITY IN EVEN DIMENSION

To obtain the regularity result, we use some elliptic PDEs for the AHE manifolds in the even dimensions. More precisely, it follows from [17, 20] when d is even, the metrics conformal to Einstein metric have the vanishing the obstruction tensors \mathcal{O}_{ij} (see also [23]), that is

$$(3.1) \quad \mathcal{O}_{ij} = (\Delta)^{d-4/2} \frac{1}{d-3} \nabla^j \nabla^l W_{ijkl} + \text{lots} = (\Delta)^{d-4/2} B_{ij} + \text{lots} = 0$$

For example, when $d = 6$, we have

$$(3.2) \quad \begin{aligned} B_{ij,k}{}^k &= 2W_{kijl} B^{kl} + 4A_k{}^k B_{ij} - 8A^{kl} \mathcal{C}_{(ij)k,l} \\ &\quad + 4\mathcal{C}_{ki}{}^l \mathcal{C}_{lj}{}^k - 2\mathcal{C}_i{}^{kl} \mathcal{C}_{jkl} - 4A_{k,l}^k \mathcal{C}_{ij}{}^l + 4W_{kijl} A^k{}_m A^{ml} \end{aligned}$$

where $2\mathcal{C}_{(ij)k} = \mathcal{C}_{ijk} + \mathcal{C}_{jik}$. Our main results in this part can be stated as follows.

Lemma 3.1. *Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with positive conformal infinity $(\partial X, [h])$ with even dimension $d \geq 6$. Assume that, under the C^{d-2} adopted metrics g^* , we have*

- (1) $\|Rm_{g^*}\|_{C^{d-4}} \leq 1$;
- (2) *Given $M > 1$ and $\gamma \in (0, 1)$ there exists some $r_0 > 0$ such that the harmonic radius $r^{1, \gamma}(M) \geq r_0$;*
- (3) $\|h\|_{C^{d-1, \gamma}} \leq N$ for some positive constants $N > 0$ and $\gamma \in (0, 1)$.

Then for all $x \in \bar{X}$ and for all $r \leq r_0$, we have

(3.3)

$$\|Ric_{g^*}\|_{C^{d-3,\gamma}(B(x,r/2)\cap\bar{X})} \leq C(M, \gamma, r_0, d, \|Rm_{g^*}\|_{C^{d-4}(B(x,r_0)\cap\bar{X})}, \|h\|_{C^{d-1,\gamma}(B(x,r_0)\cap\partial X)})$$

which yields also

$$(3.4) \quad \|g^*\|_{C^{d-1,\gamma}(B(x,r/2)\cap\bar{X})} \leq C(M, \gamma, r_0, d, \|Rm_{g^*}\|_{C^{d-4}(B(x,r_0)\cap\bar{X})}, \|h\|_{C^{d-1,\gamma}(B(x,r_0)\cap\partial X)})$$

Moreover, if we assume only instead of (3) that

$$(3') \quad \|h\|_{C^{d-2,\gamma}} \leq N \text{ for some positive constants } N > 0,$$

then there $\alpha(M, r_0, \gamma)$ such that for all $x \in \bar{X}$ and for all $r \leq r_0$, we have

$$(3.5) \quad \|Ric_{g^*}\|_{C^{d-4,\alpha}(B(x,r/2)\cap\bar{X})} \leq C(M, r_0, d, \|Rm_{g^*}\|_{C^{d-4}(B(x,r_0)\cap\bar{X})}, \|h\|_{C^{d-2,\gamma}(B(x,r_0)\cap\partial X)})$$

which implies also in harmonic coordinates

$$(3.6) \quad \|g^*\|_{C^{d-2,\alpha}(B(x,r/2)\cap\bar{X})} \leq C(M, r_0, d, \|Rm_{g^*}\|_{C^{d-4}(B(x,r_0)\cap\bar{X})}, \|h\|_{C^{d-2,\gamma}(B(x,r_0)\cap\partial X)})$$

Proof. We use the harmonic coordinate and boundary conditions in Lemma 2.6. In view of equation (2.10), it follows from [18, Theorem 6.6], that the estimate (3.3) holds for the scalar curvature since $\|Rm_{g^*}\|_{C^{d-4}} \leq 1$, that is, $R \in C^{d-3,\gamma}$. Using Lemma 2.7, the restriction of the schouten tensor A and the Weyl tensor W on the boundary in the space of functions $C^{d-3,\gamma}$. Now we want to estimate the Bach tensor via the obstruction tensor. Recall the elliptic system (3.1) or (3.2). We try to use the classical regularity theory for the laplacian operator successively. It follows from [18, Theorem 8.32] that $B \in C^{1,\gamma}$ (when $d = 6$) or more generally $B \in C^{d-5,\gamma}$. Using equation (2.9), and again from [18, Theorem 6.6], the estimate (3.3) holds for the Ricci curvature since it is so for the scalar curvature. Therefore, the estimate (3.4) comes from Lemma 2.8. Similarly, we prove estimates (3.5) and (3.6) by [18, Theorem 8.29]. \square

Remark 4. In Lemma 3.1,

- we have high order estimates, that is, if $h \in C^{k,\gamma}$ with $k \geq d-1$, we have $g^* \in C^{k,\gamma}$.
- we could expect the weak regularity on the boundary for example $h \in C^{d/2+1}$ when $d = 6$.

4. BOUNDARY REGULARITY IN ALL DIMENSIONS

We now use a different strategy to gain boundary regularity for general dimension d , namely the "gauged Einstein equations" as in the work of Chruściel-Delay-Lee-Skinner [14]. The eventual goal is to gain the regularity of the compactified metric through the choice of a suitable local gauge, from there we gain the regularity of the Weyl and Cotton tensor near conformal infinity, which in turn implies the regularity of the 4-th Bach tensor. This section is organized as follows. In subsection 4.1, we present the concept of local gauge for Einstein metric introduced by Biquard [6], and derive some $C^{3,\alpha}$ regularity of the defining function ρ in the adapted harmonic coordinate in Lemma 4.1, and from which we derive the closeness of the metric g^+ related to the approximated metric t^+ in Lemma 4.2. In subsection 4.2, we prove the existence the suitable local gauge in the neighborhood of any point on the conformal infinity and derive the suitable estimates for such local gauge in Lemma 4.4, once we establish some uniform estimates for the linearized operator of gauge condition in Lemma 4.3. In subsection 4.3, we prove first

some uniform estimates for the linearized operator with respect to the first variable for the gauged Einstein functional in Lemma 4.5 and derive some ε -regularity result of the gauged metric in Lemma 4.6, which leads to the regularity in a neighborhood of any point on the conformal infinity in Lemma 4.7. In subsection 4.4, we apply the estimates in subsection 4.3 to derive estimates of the Weyl and Cotton tensor of the compactified metric g^* in Lemma 4.8, and finally passing such information to the $C^{1,\lambda}$ estimates of $Rm[g^*]$ in a local neighborhood of the conformal infinity in Lemma 4.9.

4.1. Gauged Einstein equation. In [14], the authors use gauged Einstein equation to study the regularity and later on Biquard-Herzlich [7] prove a local version. Let us consider the nonlinear functional on d -dimensional open set $Z_R(p)$ with $p \in \partial X$ introduced by Biquard [6] for two asymptotically hyperbolic metrics g^+ and k^+ .

$$(4.1) \quad F(g^+, k^+) := Ric[g^+] + (d-1)g^+ - \delta_{g^+}(B_{k^+}(g^+)),$$

where $B_{k^+}(g^+)$ is a linear condition, essentially the infinitesimal version of the harmonicity condition

$$B_{k^+}(g^+) := \delta_{k^+}g^+ + \frac{1}{2}d\text{tr}_{k^+}(g^+).$$

We have for any asymptotically hyperbolic metrics k^+

$$D_1F(k^+, k^+) = \frac{1}{2}(\Delta_L + 2(d-1)),$$

where D_1 denotes the differentiation of F with respect to its first variable, and where the Lichnerowicz Laplacian Δ_L on symmetric 2-tensors is given by

$$\Delta_L := \nabla^*\nabla[k^+] + 2\overset{\circ}{Ric}[k^+] - 2\overset{\circ}{Rm}[k^+];$$

where

$$\overset{\circ}{Ric}[k^+](u)_{ij} = \frac{1}{2}(R_{im}[g^+]u_j^m + R_{jm}[k^+]u_i^m),$$

and

$$\overset{\circ}{Rm}[k^+](u)_{ij} = R_{imjl}[k^+]u^{ml}.$$

It is clear for any asymptotically hyperbolic Einstein metrics g^+

$$F(g^+, g^+) = 0$$

Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with positive conformal infinity $(\partial X, [h])$ with dimension $d \geq 4$. Assume that, under the C^3 adopted metrics g^* , we assume

- (1) $\|Rm_{g^*}\|_{C^0} \leq 1$;
- (2) there exists some $r_0 > 0$ such that the injectivity radius $i_{\text{int}}(X) \geq r_0$, $i_{\partial}(X) \geq 2r_0$, $i(\partial X) \geq r_0$;
- (3) $\|h\|_{C^6} \leq N$ for some positive constants $N > 0$;

Hence, we can identify $\{p \in \bar{X}, \rho(p) \leq r_1\} = [0, r_1] \times \partial X \subset \{d_{g^*}(p, \partial X) \leq r_0\}$ for some $r_1 > 0$ (we could decrease r_1 if necessary) as a submanifolds with the boundary. We consider a C^4 compactified AH manifold on $[0, r_1/2] \times \partial X$

$$t = d\rho^2 + h + \rho^2h^{(2)}, \quad t^+ = \rho^{-2}t$$

where $h^{(2)}$ is the Fefferman-Graham expansion term and intrinsically determined by the boundary metric h . Given $2R < r_1/2$, we look for a local diffeomorphism $\Phi : Z_R(p) \rightarrow Z_{2R}(p)$ such that Φ^*g^+ solves the gauged Einstein equation in $Z_{R/2}(p)$

$$(4.2) \quad F(\Phi^*g^+, t^+) = 0$$

We divide the boundary $\partial Z_R(p) := \partial^\infty Z_R(p) \cup \partial^{int} Z_R(p) = (\{\rho = 0\} \cap \partial Z_R(p)) \cup (\{\rho > 0\} \cap \partial Z_R(p))$. Given a CCE g^+ and a regular AH t^+ with the same conformal infinity on the local boundary $\Psi_{p,R}(Y_1^\infty)$, we try to find a local diffeomorphism $\Phi : Z_R(p) \rightarrow Z_{2R}(p)$ such that the gauged condition is satisfied in $Z_{R/2}(p)$ up to the diffeomorphism Φ , that is

$$B_{t^+}(\Phi^*g^+) = 0 \text{ in } Z_{R/2}(p)$$

Thus, the gauged Einstein equation (4.2) is satisfied in $Z_{R/2}(p)$. We know $\rho \in C_{loc}^{3,\gamma}$ for all $\gamma \in (0, 1)$ under the adapted harmonic coordinates for the metric g^* . More precisely, we have

Lemma 4.1. *Under the above the assumptions, there exists some positive constant C depending on γ but independent of p (and the sequence of the metrics) such that for all $p \in \partial X$ under the adapted harmonic coordinates*

$$\|\rho\|_{C^{3,\gamma}(Z_{r_1/2}(p))} \leq C$$

Proof. By the classical elliptic regularity [18, Theorem 8.33], it follows from (2.10) that the scalar curvature $R \in C_{loc}^{1,\gamma}$ and we have

$$\|R\|_{C^{1,\gamma}(Z_{r_1}(p))} \leq C$$

for all $p \in \partial X$ and for all $\gamma \in (0, 1)$. Thanks of (2.3) and Lemma 2.3, we infer that ρ is $C^{3,\gamma}$ smooth in $Z_{r_1/2}(p)$ under the adapted harmonic coordinates for the metric g^* , and

$$\|\rho\|_{C^{3,\gamma}(Z_{r_1/2}(p))} \leq C$$

Therefore, we prove the desired results. \square

Remark 5. *Under the assumptions that the metric is in $C^{2,\gamma}$ and the scalar curvature in $C^{2,\gamma}$, there holds ρ in $C^{4,\gamma}$ under the adapted harmonic coordinates for the metric g^* .*

In the above lemma we consider the partial differential derivatives for $C^{3,\gamma}$ norm, do not consider the covariant derivatives. We could identify the neighborhood $\{p \in X | \rho(p) \leq r_1/2\}$ of ∂X in X as $[0, r_1/2] \times \partial X$. In fact, let $(\theta^2, \dots, \theta^d)$ be the harmonic chart of ∂X . We extend them as harmonic functions (x^2, \dots, x^d) in X so that a local chart of $\{p \in X | \rho(p) \leq r_1/2\}$ could be given by (ρ, x^2, \dots, x^d) . In view of Lemma 4.1, such chart is $C^{3,\gamma}$ compatible with the harmonic coordinates of X . Thus, recall the C^4 compactified AH manifold on $[0, r_1/2] \times \partial X$

$$t = d\rho^2 + h + \rho^2 h^{(2)}, \quad t^+ = \rho^{-2}t$$

We suppose for t , one has $i_\partial(X) \geq 2r_1$, $i(\partial X) \geq r_1$ (we could decrease r_1 if necessary). We consider t^+ as a reference AH metric with the given conformal infinity h . For simplicity, we drop the index i for the family of metrics t_i and t_i^+ if there is no confusion. Recall near the boundary (in $[0, r_1/2] \times \partial X$), t_i^+ is a family of class C^4 AH manifolds, and moreover

the family of metrics t_i is compact in $C^{3,\gamma}$ Cheeger-Gromov's topology in $Z_R(p)$ for all $R < r_1/2$, for any $p \in \partial X$ and for all $\gamma \in (0, 1)$. We define a map $H_v : Z_R(p) \rightarrow M$ by

$$H_v(q) = \exp_q(v(q)),$$

where \exp denotes the Riemannian exponential map of t^+ . It is showed that H_v is diffeomorphism if v is sufficiently small, and by [14, lemma 4.1] it extends to a homeomorphism of $Z_R(p)$ fixing the boundary at infinity pointwise if v is small in $C_\delta^{1,0}(Z_R(p); TX)$ for $\delta > 0$.

Let Σ^2 denote the bundle of symmetric covariant 2-tensors over X . Let φ_R be the cut-off function in \overline{X} such that

$$\text{supp}\varphi_R \in Z_R(p), \varphi_R \equiv 1 \text{ on } Z_{\frac{R}{2}}(p), \|\varphi_R\|_{C_{k+\lambda}^{k,\lambda}(Z_R(p))} \leq C_0 R^{-k-\lambda} \forall 0 \leq k \leq 2\forall \lambda \in (0, 1).$$

We set $g_\varphi^+ = t^+ + \varphi(g^+ - t^+)$. We try to find a local gauge H_v

$$B_{t^+}((H_v)^* g_\varphi^+) = 0 \text{ in } Z_R(p)$$

The linearized operator on v is $B_{t^+}(\delta_{t^+})^* = \frac{1}{2}((\nabla)^* \nabla[t^+] - Ric[t^+])$ which is an isomorphism from $C_\delta^{k+2,\lambda}$ into $C_\delta^{k,\lambda}$ provided $\delta \in (-1, d)$.

Now we consider the boundary harmonic chart (x^1, \dots, x^d) . Let ϕ be a chart such that $\phi^{-1}(q) = (\rho(q), x^2, \dots, x^d)$.

Lemma 4.2. *Under the above the assumptions, there exists some positive constant C and $R_0 < r_1/2$ independent of $p \in \partial X$ (and the sequence of the metrics) such that for all $p \in \partial X$*

$$\begin{aligned} i) \quad & g = t + O(\rho^\lambda) \quad \forall \lambda \in (0, 1) \\ ii) \quad & g^+ - t^+ \in C_{1+\lambda}^{1,\lambda} \quad \forall 0 < \lambda < \lambda' < 1. \text{ Furthermore,} \\ & \|g^+ - t^+\|_{C_{1+\lambda}^{1,\lambda}(Z_{R_0}(p))} \leq C R_0^{\lambda' - \lambda} \end{aligned}$$

Proof. We use the above chart ϕ . We want to prove on the boundary $g_{1\gamma} = 0$, $\partial_1 g_{ij} = 0$ for all $\gamma = 2, \dots, d$ and for all $i, j = 1, \dots, d$. For the first one, we note on the boundary

$$g(\partial_1, \partial_\gamma) = \partial_\gamma \rho = 0$$

since ρ vanishes on the boundary ∂X . Using (2.1), $g_{11} = g(\partial_1, \partial_1) = O(\rho^2)$ so that $\partial_1 g_{11} = 0$ on ∂X .

Again $g(\partial_1, \partial_1) \equiv 1$ on the boundary ∂X , which yield $g(\nabla_{\partial_\gamma} \partial_1, \partial_1) = 0$ on the boundary. Together with the fact the boundary is totally geodesic. Thus, $\nabla_{\partial_\gamma} \partial_1 = 0$ on the boundary. On the other hand, by (2.2), we deduce $\partial_1 g_{1\gamma} = \partial_1 \partial_\gamma \rho = D^2 \rho(\partial_1, \partial_\gamma) - (\nabla_{\partial_1} \partial_\gamma) \rho = D^2 \rho(\partial_1, \partial_\gamma) - (\nabla_{\partial_\gamma} \partial_1) \rho = D^2 \rho(\partial_1, \partial_\gamma) = 0$. Again it follows from (2.2) that $\rho_{,\alpha\beta} = 0$ so that the Christoffel symbols $\Gamma_{\alpha\beta}^1 = 0$ on the boundary, that is, $0 = \frac{1}{2}(\partial_\beta g_{\alpha 1} + \partial_\alpha g_{1\beta} - \partial_1 g_{\alpha\beta}) = -\frac{1}{2} \partial_1 g_{\alpha\beta}$ since $\partial_\beta g_{\alpha 1} = \partial_\alpha g_{1\beta} = 0$ on the boundary. Thus, we prove the claim. We know the metric g^* is in $C^{1,\lambda}$ so that (i) is an immediate result of the above claim. By the Taylor's expansion, the second property comes from the fact g^* is bounded in $C^{1,\lambda}$ topology for all $\lambda \in (0, 1)$. \square

4.2. Local gauge. We want to find a diffeomorphism H fixing the boundary in $Z_R(p)$ such that $B_{t^+}H^*g^+ = 0$ which is equivalent to $B_{(H^{-1})^*t^+}g^+ = 0$ in $H^{-1}(Z_R(p))$. Given small $R > 0$ and $p \in \partial X$, let $\Psi_{p,R} : Y_1 \subset \mathbb{H} \rightarrow Z_R(p)$ be a boundary Möbius chart. It follows from [26, Lemma 6.1] that for any $\lambda \in (0, 1)$

$$\|\Psi_{p,R}^*t^+ - g_{\mathbb{H}}\|_{3,\lambda;Y_1} \leq CR$$

where the positive constant $C > 0$ independent of the sequence and the point $p \in \partial X$. We denote φ some non-negative smooth cut-off function such that $\varphi \equiv 1$ on $Y_{1/2}$ and $\varphi \equiv 0$ on $\mathbb{H} \setminus Y_1$. We want to glue the metric $\Psi_{p,R}^*t^+$ with the standard hyperbolic metric $g_{\mathbb{H}}$ as follows

$$t_{p,R}^+ = \varphi\Psi_{p,R}^*t^+ + (1 - \varphi)g_{\mathbb{H}}$$

There exists some small $\bar{R}_0 > 0$ such that the sectional curvature of $t_{p,R}^+$ is negative and $t_{p,R}^+$ is a compact family of AH metrics in $C^{3,\lambda}$ Cheeger-Gromov topology for all $R \leq \bar{R}_0$, for all $p \in \partial X$ and for the sequence (for adopted metrics) since t^+ is compact family of AH metrics $C^{3,\lambda}$ Cheeger-Gromov topology (for compactified metrics). We denote $\tilde{Z}_R(p)$ such AH metric. We consider the following mapping Ψ .

$$\begin{aligned} \Psi : C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2) &\rightarrow C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2) \\ (v, w) &\mapsto (B_{(H_v^{-1})^*t^+}(t_w^+), w) \end{aligned}$$

where $t_w^+ = t^+ + w$. It is clear that

$$D_1\Psi_1(0, 0)(Y) = B_{t^+}((\delta_{t^+})^*Y) = \left(\frac{1}{2}((\nabla)^*\nabla[t^+] - Ric[t^+])Y\right)$$

Here $\Psi = (\Psi_1, \Psi_2)$ and δ^* is the symmetrized covariant derivative of the vector field. It is known [26, Theorem C] that $D_1\Psi_1(0, 0) : C_{\delta}^{k,\lambda} \rightarrow C_{\delta}^{k-2,\lambda}$ is an isomorphism provided $\delta \in (-1, d)$. In the following, if there is no confusion, the set $Z_R(p)$ is always related to the metric t^+ .

Lemma 4.3. *Under the above the assumptions, there exists some positive constant C and $\eta > 0$ small independent of $p \in \partial X$ (and the sequence of the metrics) such that Ψ is a C^1 map for $(v_i, w_i) \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2)$ with $\|v_i\| + \|w_i\| \leq \eta$ for $i = 1, 2$ and satisfies*

$$\begin{aligned} i) \quad &\|D_1\Psi_1(0, 0)\| + \|(D_1\Psi_1(0, 0))^{-1}\| \leq C \\ ii) \quad &\|D\Psi_1(v_1, w_1) - D\Psi_1(v_2, w_2)\| \leq C(\|v_1 - v_2\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} + \|w_1 - w_2\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}) \end{aligned}$$

where C is some positive constant independent of the sequence and $p \in \partial X$.

Proof. If there is no confusion, we denote the extension metric $t_{p,R}^+$ as t^+ in the proof. It follows from [26, Lemma 4.6] that $\|D_1\Psi_1(0, 0)\| \leq C$ for some positive constant C independent of $p \in \partial X$ (and the sequence of the metrics) since the family of metrics t_i (resp. t_i^+) is compact in $C^{3,\gamma}$ Cheeger-Gromov topology for all $\gamma \in (0, 1)$.

Now we prove $\|(D_1\Psi_1(0, 0))^{-1}\| \leq C$ by the contradiction. Recall the sectional curvature is negative on $\tilde{Z}_{\bar{R}_0}(p)$. Therefore, there is no L^2 kernel for the linear operator $\frac{1}{2}((\nabla)^*\nabla[t^+] - Ric[t^+])$. As a consequence, it follows from [26, Theorem C] that

$D_1\Psi_1(0,0) : C_{1+\lambda}^{2,\lambda} \rightarrow C_{1+\lambda}^{0,\lambda}$ is an isomorphism since $1 + \lambda \in (-1, d)$. We suppose

$$\|(D_1\Psi_1(0,0))^{-1}[t_i^+]\| \rightarrow \infty$$

Thus, we choose some vector field $v_i \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\tilde{R}_0}(p); T\tilde{Z}_{\tilde{R}_0}(p))$ with $\|v_i\|_{C_{1+\lambda}^{2,\lambda}} = 1$ and

$$\|(D_1\Psi_1(0,0)[t_i^+])v_i\|_{C_{1+\lambda}^{0,\lambda}} \rightarrow 0.$$

Up to a subsequence, t_i converges to t_∞ in $C^{3,\gamma}$ Cheeger-Gromov topology for all $\gamma \in (0, 1)$. Modulo a subsequence, t_i^+ converges also to a $C^{3,\gamma}$ AH $t_\infty^+ = \rho^{-2}t_\infty$ in $C^{3,\gamma}$ Cheeger-Gromov topology. On the other hand, by [26, Lemma 6.4],

$$\|v_i\|_{C_{1+\lambda}^{2,\lambda}} \leq C(\|(D_1\Psi_1(0,0)[t_i^+])v_i\|_{C_{1+\lambda}^{0,\lambda}} + \|v_i\|_{C_{1+\lambda'}^{0,0}})$$

where $\lambda' \in (0, \lambda)$ and C is some positive constant independent of p and the sequence since t_i is in some compact set in $C^{3,\gamma}$ Cheeger-Gromov topology. Thus, we have for large i

$$\|v_i\|_{C_{1+\lambda'}^{0,0}} \geq 1/2C$$

By the Rellich Lemma [26, Lemma 3.6], the mapping $C_{1+\lambda}^{2,\lambda} \hookrightarrow C_{1+\lambda'}^{0,0}$ is a compact embedding so that we infer $\|v_\infty\|_{C_{1+\lambda'}^{0,0}} \geq 1/2C$. On the other hand, we have

$$(D_1\Psi_1(0,0)[t_\infty^+])v_\infty = 0$$

As above, we have $D_1\Psi_1(0,0)[t_\infty^+] : C_{1+\lambda}^{2,\lambda} \rightarrow C_{1+\lambda}^{0,\lambda}$ is an isomorphism so that $v_\infty = 0$. This contradiction yields the desired result (i).

Now we prove Ψ is a C^1 map with the property (ii). The proof is similar as in [14, Lemmas 4.2 and 4.4]. Let Φ_j be any Möbius chart around some point $p_j \in \tilde{Z}_{\tilde{R}_0}(p)$. We sketch the proof here. When v is small, we have H_v maps $\overline{V_1(p_j)}$ into $V_2(p_j)$ and $H_v(V_2(p_j))$ contains $\overline{V_1(p_j)}$. We denote Möbius coordinates by x or (x^1, x^2, \dots, x^d) and the associated standard fiber coordinates on $T\tilde{Z}_R(p)$ by v or (v^1, v^2, \dots, v^d) . Let $E^j(x, v)$ denote the component functions of the t^+ -exponential map in Möbius coordinates. As the Christoffel symbols are of class C^3 , it follows from the standard ODE theory that $E^j(x, v)$ is of class C^3 . Thus, H_v has component functions given by $H^j(x) = E^j(x, v(x))$ with $E^j(x, 0) = x^j$. Set $A^j(x) = H^j(x) - x^j$. For sufficiently small $v \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_R(p); T\tilde{Z}_R(p))$, $H_v : \tilde{Z}_{\tilde{R}_0}(p) \rightarrow \tilde{Z}_{\tilde{R}_0}(p)$ is a diffeomorphism. Moreover, we have

$$\|A(x)\|_{2,\lambda;\tilde{B}_2} \leq C\|\Phi_i^*v\|_{2,\lambda;\tilde{B}_2} \leq C\rho(p_i)^{1+\lambda}\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\tilde{R}_0}(p))}$$

Here C is some positive constant independent of the sequence and $p \in \partial X$ since the Christoffel symbols are bounded in C^3 topology for t_i^+ . We denote $K(x) = H^{-1}(x) = x + B(x)$ so that the differential $dK(x) = (dH)^{-1}(K(x))$. Thus, in Möbius coordinates, we have $K(x) + A(K(x)) = x$, that is, $B(x) = -A(K(x))$, which implies

$$\|B\|_{2,\lambda;\tilde{B}_1} = \|-A(K(x))\|_{2,\lambda;\tilde{B}_1} \leq C\|A\|_{2,\lambda;\tilde{B}_2}(1 + \|B\|_{2,\lambda;\tilde{B}_1})$$

so that

$$\|B\|_{2,\lambda;\tilde{B}_1} \leq C\rho(p_i)^{1+\lambda}\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\tilde{R}_0}(p))}$$

provided $C\|A\|_{2,\lambda;\bar{B}_2} \leq 1/2$. Here C is still some positive constant independent of the sequence and $p \in \partial X$. Now, we write

$$\begin{aligned} \Phi_l^*((H_v^{-1})^*t^+ - t^+) &= ((\Phi_l^*t^+)_{jk}(K(x)) - (\Phi_l^*t^+)_{jk}(x))dx^j \otimes dx^k \\ &\quad + 2(\Phi_l^*t^+)_{jk}(K(x))\frac{\partial B^k}{\partial x^q}dx^j \otimes dx^q \\ &\quad + (\Phi_l^*t^+)_{jk}(K(x))\frac{\partial B^j}{\partial x^m}\frac{\partial B^k}{\partial x^q}dx^m \otimes dx^q \end{aligned}$$

Recall that t_i is bounded in C^4 topology (or simply C^3 topology) near the boundary so that by [14, Lemma2.1]

$$\|((\Phi_l^*t^+)_{jk}(K(x)) - (\Phi_l^*t^+)_{jk}(x))\|_{1,\lambda;\bar{B}_1} \leq C\|B(x)\|_{1,\lambda;\bar{B}_1}$$

On the other hand, we can estimate

$$\|2(\Phi_l^*t^+)_{jk}(K(x))\frac{\partial B^k}{\partial x^q}dx^j \otimes dx^q + (\Phi_l^*t^+)_{jk}(K(x))\frac{\partial B^j}{\partial x^m}\frac{\partial B^k}{\partial x^q}dx^m \otimes dx^q\|_{1,\lambda;\bar{B}_1} \leq C\|B(x)\|_{2,\lambda;\bar{B}_1}$$

As a consequence, we infer

$$\|\Phi_l^*((H_v^{-1})^*t^+ - t^+)\|_{1,\lambda;\bar{B}_1} \leq C\|\Phi_l^*v\|_{2,\lambda;\bar{B}_2} \leq C\rho(p_i)^{1+\lambda}\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}$$

which implies

$$\|(H_v^{-1})^*t^+ - t^+\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq C\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}$$

where C is some positive constant independent of the sequence and $p \in \partial X$. In Möbius chart, we know

$$\Phi_l^*t^+ = g_{jk}(x)dx^j \otimes dx^k, \quad \Phi_l^*(H_v^{-1})^*t^+ = (g_{-v})_{jk}(x)dx^j \otimes dx^k = (g_{jk}(x) + \alpha_{jk}(x))dx^j \otimes dx^k$$

with

$$\|\alpha_{jk}(x)dx^j \otimes dx^k\|_{1,\lambda;\bar{B}_1} \leq C\rho(p_i)^{1+\lambda}\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}$$

Hence, it follows directly

$$\begin{aligned} \|B_{\Phi_l^*(H_v^{-1})^*t^+}\Phi_l^*(t_w^+)\|_{0,\lambda;\bar{B}_1} &\leq C(\|\Phi_l^*v\|_{2,\lambda;\bar{B}_2} + \|\Phi_l^*w\|_{1,\lambda;\bar{B}_2}) \\ &\leq C\rho(p_i)^{1+\lambda}(\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} + \|w\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}) \end{aligned}$$

which implies

$$\|B_{(H_v^{-1})^*t^+}(t_w^+)\|_{C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq C(\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} + \|w\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))})$$

Recall $E(x, v)$ the component functions of the t^+ -exponential map is bounded in C^3 topology near the boundary, (that is, in the set $\{(x, v)|x \in \tilde{Z}_{\bar{R}_0}(p), |v| \leq 1\}$ with the uniform bound w.r.t. p and sequence), since t is bounded in C^4 topology near the boundary. Given small $(v_1, w_1), (v_2, w_2) \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2)$, let $H_{v_1+v_2}(x) = E(x, v_1 + v_2)$ be a diffeomorphism related to the vector field $v_1 + v_2$ and its inverse map can be written as $H_{v_1+v_2}^{-1} - H_{v_1}^{-1} = H_{v_1+v_2}^{-1} \circ (H_{v_1} - H_{v_1+v_2}) \circ H_{v_1}^{-1}$. Similarly, in Möbius Chart, we can estimate

$$\|H_{v_1+v_2}(x) - H_{v_1}(x)\|_{2,\lambda;\bar{B}_1} + \|H_{v_1+v_2}^{-1}(x) - H_{v_1}^{-1}(x)\|_{2,\lambda;\bar{B}_1} \leq C\|\Phi_l^*v_2\|_{2,\lambda;\bar{B}_2}$$

so that

$$\|(H_{v_1+v_2}^{-1})^*t^+ - (H_{v_1}^{-1})^*t^+\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq C\|v_2\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}$$

Moreover, there exists some small $\eta > 0$ independent of the sequence and $p \in \partial X$ such that for all small $(v_1, w_1), (v_2, w_2) \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2)$ with $\|v_i\| + \|w_i\| \leq \eta$ and for all $(v, w) \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p)) \times C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2)$, we have

$$\begin{aligned} & \|D\Psi_1(v_1, w_1)(v, w) - D\Psi_1(v_2, w_2)(v, w)\|_{C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \\ & \leq C(\|v_1 - v_2\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} + \|w_1 - w_2\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))})(\|v\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} + \|w\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))}) \end{aligned}$$

where C is some positive constant independent of the sequence and $p \in \partial X$. Here we use the fact that $D_2\Psi_1(v_1, w_1)w = B_{(H_{v_1}^{-1})^*t^+}w$ and t is compact in C^3 topology near the boundary. On the other hand, we remark $\Psi_2(\cdot, \cdot)$ is a linear continuous projection on the second variable. From the above estimates, Ψ is a C^1 map satisfying (ii). Therefore, we finish the proof. \square

Lemma 4.4. *Under the above the assumptions, there exists some positive constant C and $R_1 < \bar{R}_0/2$ independent of $p \in \partial X$ (and the sequence of the metrics) such that for all $p \in \partial X$ and $R \leq R_1$ there exists a local gauge vector field $\tilde{v} \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p))$ small satisfies*

- i) $H_v^*g^+$ solves local gauge for gauged Einstein equation in $Z_{R/2}(p)$
- ii) $\|H_v\|_{C^{2,\lambda}(Z_{\bar{R}_0/2}(p))} \leq C$
- iii) $\|\tilde{v}\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq CR^{\lambda'-\lambda}\bar{R}_0^{1+\lambda} \forall 0 \leq \lambda < \lambda' < 1$

where $v = (\Psi_{p, \bar{R}_0})_*\tilde{v}$

Proof. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-negative cut-off function satisfying $\varphi(s) \equiv 1$ for all $s < 1/2$ and $\varphi(s) \equiv 0$ for all $s > 1$. We consider

$$w_R(x) = \varphi(d_t(x, p)/R)(g^+ - t^+).$$

Set $\tilde{w}_R = (\Psi_{p, \bar{R}_0})^*w_R$. Thanks of Lemma 4.2, we have $\tilde{w}_R \in C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p); \Sigma^2)$ and

$$\|\tilde{w}_R\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq CR^{\lambda'-\lambda}\bar{R}_0^{1+\lambda}$$

so that $\|\tilde{w}_R\|_{C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \rightarrow 0$ as $R \rightarrow 0$. In view of Lemma 4.3, it follows from the inverse function theorem there exists some small $R_1 < \bar{R}_0/2$ such that for all $R \leq R_1$ one could $\tilde{v} \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p); T\tilde{Z}_{\bar{R}_0}(p))$ which solves $\Psi(\tilde{v}, \tilde{w}_R) = (0, \tilde{w}_R)$. Moreover, we estimate

$$\|\tilde{v}\|_{C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq CR^{\lambda'-\lambda}\bar{R}_0^{1+\lambda}.$$

To see that, we have $\Psi(0, 0) = 0$ and $\Psi_1(0, \tilde{w}_R) = B_{(\Psi_{p, \bar{R}_0})^*t^+}((\Psi_{p, \bar{R}_0})^*(t^+ + w_R)) = (\Psi_{p, \bar{R}_0})^*B_{t^+}(w_R)$ so that

$$\|\Psi_1(0, \tilde{w}_R)\|_{C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq CR^{\lambda'-\lambda}\bar{R}_0^{1+\lambda}.$$

Thus, we prove (iii). As a consequence

$$\|v\|_{C_{1+\lambda}^{2,\lambda}(Z_{\bar{R}_0}(p))} \leq CR^{\lambda'-\lambda}$$

We have $B_{(H_v^{-1})^*t^+}(t^+ + w_R) = 0$ in $Z_{R_1}(p)$, which yields

$$0 = (H_v)^*B_{(H_v^{-1})^*t^+}(t^+ + w_R) = B_{t^+}(H_v)^*(t^+ + w_R)$$

Recall $g^+ = t^+ + w$ in $Z_{R/2}(p)$. Hence $H_v^*g^+$ solves local gauge for gauged Einstein equation in $Z_{R/2}(p)$, that is, (i) is proved. The proof of (ii) is given in [14, Lemma4.4]. Finally, the result is proved. \square

4.3. ε -regularity. In this part, we want to prove high order regularity up to a diffeomorphism. We establish first the uniform bound for the linearized operator D_1F and its inverse.

Lemma 4.5. *Under the above the assumptions, there exists positive constant C independent of $p \in \partial X$ (and the sequence of the metrics) such that*

$$\|D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+)\| + \|(D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+))^{-1}\| \leq C$$

where $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+) : C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \rightarrow C_{2+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$ (or $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+) : C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \rightarrow C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$). Moreover, such estimates hold also for $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+) : C_{3+\lambda}^{3,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \rightarrow C_{3+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$ (or $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+) : C_{1+\lambda}^{3,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \rightarrow C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$)

Proof. We state the sectional curvature of t_{p,\bar{R}_0}^+ is negative in $\tilde{Z}_{\bar{R}_0}(p)$. It is known (see [26, Proof of Theorem A]) the L^2 kernel of the operator $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+)$ is trivial. Hence by [26, Theorem C], $D_1F(t_{p,\bar{R}_0}^+, t_{p,\bar{R}_0}^+) : C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \rightarrow C_{2+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$ is an isomorphism since $2 + \lambda \in (0, d)$. Recall the family of t is compact in $C^{2,\lambda}$ for all $\lambda \in (0, 1)$ Cheeger-Gromov topology (even $C^{3,\lambda}$). By the same arguments in the proof of Lemma 4.3, the desired results follow. The proof in the high order Hölder spaces is same. We finish the proof. \square

Now we can prove the ε -regularity result.

Lemma 4.6. *Under the above the assumptions, there exists positive constant C and small positive constant ε independent of $p \in \partial X$ (and the sequence of the metrics) such that if for all $R < \min(R_1/2, 1)$ we have*

$$\|H_v^*g^+ - t^+\|_{C_\lambda^{0,\lambda}(Z_R(p))} \leq \varepsilon \text{ and } \|H_v^*g^+ - t^+\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))} \leq C,$$

then we have

$$\|H_v^*g^+ - t^+\|_{C_{2+\lambda}^{2,\lambda}(Z_{R/2}(p))} \leq \frac{C}{R},$$

moreover, there holds

$$\|H_v^*g^+ - t^+\|_{C_{3+\lambda}^{3,\lambda}(Z_{R/4}(p))} \leq \frac{C}{R^2},$$

Proof. We consider the following equation

$$E[u] := F(t^+ + u, t^+).$$

where u is a symmetric 2-tensor fields. By Lemma 4.3, $u = \tilde{g}^+ - t^+ := (H_v)^*g^+ - t^+$ is a solution of $E[u] = 0$ in $Z_{R_1/2}(p)$. It is a quasilinear uniformly degenerate equation with its linearized operator at 0, $DE[0] = \frac{1}{2}(\Delta_L + 2(d-1)) := P$, which is of course, a

geometric elliptic operator. Recall u solves $E(u) = 0$ in $Z_R(p)$. On the other hand, a direct calculation leads to (see [19])

$$E(0) = Ric[t^+] + (d-1)t^+ \in C_{(1+\lambda)}^{3,\lambda}(\overline{Z_R(p)}) \subset C_{(\lambda)}^{2,\lambda}(\overline{Z_R(p)})$$

so that by Lemma 2.12

$$\|E(0)\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))} \leq C$$

Here the bound C is independent of p and the sequence. Let $G[u] = E[u] - E[0] - DE[0]u$ be the quadratic polynomials and higher degree in u . Hence we can estimate for small u

$$\|G[u]\|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} \leq C\|G[u]\|_{C_{2+2\lambda}^{0,\lambda}(Z_R(p))} \leq C(\|u\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))}\|u\|_{C_{\lambda}^{0,\lambda}(Z_R(p))} + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2)$$

where C is independent of p and the sequence. As t is C^4 , we can choose $\varphi_R(x) = \varphi(d_t(x,p)/R)$ be the C^3 cut-off function in $Z_{\bar{R}_0}(p)$ such that

$$\text{supp}\varphi_R \in Z_R(p), \quad \varphi_R \equiv 1 \text{ on } Z_{\frac{R}{2}}(p),$$

$$\|\nabla^k \varphi_R(x)\| \leq C_0 R^{-k}, \quad \forall 0 \leq k \leq 3.$$

We have

$$\varphi_R G[u] = \varphi_R(E[u] - E[0] - DE[0]u) = -\varphi_R E[0] - P(\varphi_R u) + [\varphi_R, P]u$$

We note

$$[\varphi_R, P]u = -\nabla^*[t^+]u\nabla[t^+]\varphi_R - u\nabla^*[t^+]\nabla[t^+]\varphi_R$$

so that

$$\|[\varphi_R, P]u\|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} \leq C\left(\frac{\|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}}{R} + \frac{\|u\|_{C_{1+\lambda}^{0,\lambda}(Z_R(p))}}{R}\right)$$

For this purpose, let Φ_l be any Möbius chart around some point $p_l \in Z_R(p)$. We write

$$\Phi_l^*([\varphi_R, P]u) = -\nabla^*[\Phi_l^*t^+]\Phi_l^*u\nabla[\Phi_l^*t^+]\Phi_l^*\varphi_R - \Phi_l^*u\nabla^*[\Phi_l^*t^+]\nabla[\Phi_l^*t^+]\Phi_l^*\varphi_R$$

so that

$$\begin{aligned} \|\Phi_l^*([\varphi_R, P]u)\|_{0,\lambda;B_1} &\leq C(\|\nabla\Phi_l^*u\|_{0,\lambda;B_1}\|\nabla\Phi_l^*\varphi_R\|_{0,\lambda;B_1} + \|\Phi_l^*u\|_{0,\lambda;B_1}\|\nabla^2\Phi_l^*\varphi_R\|_{0,\lambda;B_1}) \\ &\leq C(\|\Phi_l^*u\|_{1,\lambda;B_1}\|\nabla\Phi_l^*\varphi_R\|_{0,\lambda;B_1} + \|\Phi_l^*u\|_{0,\lambda;B_1}\|\nabla^2\Phi_l^*\varphi_R\|_{0,\lambda;B_1}) \\ &\leq C\rho(p_l)^{2+\lambda}(\|u\|_{C_{1,\lambda}^{1,\lambda}}/R + \|u\|_{C_{1,\lambda}^{0,\lambda}}/R) \end{aligned}$$

where C is some positive constant independent of p and the sequence. Thus, the desired estimate follows. Now we estimate

$$\|\varphi_R E(0)\|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} \leq C\|\varphi_R\|_{C^{0,\lambda}(Z_R(p))}\|E(0)\|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} \leq C$$

Similarly

$$\begin{aligned} \|\varphi_R G[u]\|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} &\leq C\|\varphi_R G[u]\|_{C_{2+2\lambda}^{0,\lambda}(Z_R(p))} \\ &\leq C(\|\varphi_R u\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))}\|u\|_{C_{\lambda}^{0,\lambda}(Z_R(p))} + \|\varphi_R\|_{C^{0,\lambda}(Z_R(p))}\|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2) \\ &\leq C(\|\varphi_R u\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))}\|u\|_{C_{\lambda}^{0,\lambda}(Z_R(p))} + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2) \end{aligned}$$

Gathering the above estimates, we infer

$$\begin{aligned} & \| -\varphi_R G[u] - \varphi_R E[0] + [\varphi_R, P]u \|_{C_{2+\lambda}^{0,\lambda}(Z_R(p))} \\ & \leq C(\|\varphi_R u\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))} \|u\|_{C_\lambda^{0,\lambda}(Z_R(p))} + (1 + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2)/R) \end{aligned}$$

provided $R < 1$. Now we write

$$P(\varphi_R u) = -\varphi_R G[u] - \varphi_R E[0] + [\varphi_R, P]u$$

Given a section w on $Z_R(p)$, let us denote $\tilde{w} := \Psi_{p, \bar{R}_0}^* w$ and $\tilde{P}, \tilde{E}, \tilde{G}$ the pull back by Ψ_{p, \bar{R}_0} of P, L, G . It is clear

$$\|w\|_{k,\lambda;\delta} = (\bar{R}_0)^\delta \|\tilde{w}\|_{k,\lambda;\delta}$$

Hence

$$\begin{aligned} & \| -\widetilde{\varphi_R G[u]} - \widetilde{\varphi_R E[0]} + \widetilde{[\varphi_R, P]u} \|_{C_{2+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \\ & \leq C(\|\widetilde{\varphi_R u}\|_{C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \|u\|_{C_\lambda^{0,\lambda}(Z_R(p))} + (1 + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2)(\bar{R}_0)^{2+\lambda}/R) \end{aligned}$$

We know $\widetilde{\varphi_R u} \in C_{1+\lambda}^{1,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$ and $\tilde{P}(\widetilde{\varphi_R u}) \in C_{2+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \subset C_{1+\lambda}^{0,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$ which implies by [26, Lemma 4.8] $\widetilde{\varphi_R u} \in C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$. Therefore, applying Lemma 4.5, we can write

$$\widetilde{\varphi_R u} = \tilde{P}^{-1}(-\widetilde{\varphi_R G[u]} - \widetilde{\varphi_R E[0]} + \widetilde{[\varphi_R, P]u}) \in C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p)) \subset C_{1+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))$$

Again from Lemma 4.5, we can obtain

$$\|\widetilde{\varphi_R u}\|_{C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \leq C(\|\widetilde{\varphi_R u}\|_{C_{2+\lambda}^{2,\lambda}(\tilde{Z}_{\bar{R}_0}(p))} \|u\|_{C_\lambda^{0,\lambda}(Z_R(p))} + (1 + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2)(\bar{R}_0)^{2+\lambda}/R)$$

so that

$$(1 - C\|u\|_{C_\lambda^{0,\lambda}(Z_R(p))})\|\varphi_R u\|_{C_{2+\lambda}^{2,\lambda}(Z_R(p))} \leq C(1 + \|u\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))}^2)/R$$

Now, we take $1 - C\|u\|_{C_{1+\lambda}^{0,\lambda}(Z_R(p))} \leq 1/2$, and the desired result yields.

For the high order regularity, we state first that

$$E(0) = Ric[t^+] + (d-1)t^+ \in C_{(1+\lambda)}^{3,\lambda}(\overline{Z_R(p)})$$

so that by Lemma 2.12

$$\|E(0)\|_{C_{3+\lambda}^{3,\lambda}(Z_R(p))} \leq C$$

The proof for the rest is similar. Therefore, we finish the proof. \square

Now, we could prove the high order regularity in a neighborhood of conformal infinity up to a diffeomorphism. Namely, we have

Lemma 4.7. *Under the above the assumptions, there exists positive constant C and small positive constant $\bar{R}_1 < \min(R_1, 1)$ independent of $p \in \partial X$ (and the sequence of the metrics) such that*

$$\|H_v^* g^+ - t^+\|_{C_{2+\lambda}^{2,\lambda}(Z_{\bar{R}_1}(p))} \leq \frac{C}{\bar{R}_1}.$$

Moreover, we have

$$\|H_v^* g^+ - t^+\|_{C_{3+\lambda}^{3,\lambda}(Z_{\bar{R}_1}(p))} \leq \frac{C}{\bar{R}_1^2}.$$

Proof. We claim $\|H_v^*g^+ - t^+\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))} \leq CR^{\lambda'-\lambda}$ with $0 < \lambda < \lambda' < 1$. We write

$$H_v^*g^+ - t^+ = H_v^*t^+ - t^+ + H_v^*w$$

Thanks of Lemmas 4.3 and 4.4 , we estimate

$$\|H_v^*t^+ - t^+\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))} \leq CR^{\lambda'-\lambda}$$

On the other hand, for sufficiently small $v \in C_{1+\lambda}^{2,\lambda}(Z_{\bar{R}_0}(p); TX)$, $H_v : Z_{R_1}(p) \rightarrow Z_{2R_1}(p)$ is a diffeomorphism. As in the proof of Lemma 4.3, set $A(x) = H(x) - x$ in Möbius chart around some point $p_l \in Z_{\bar{R}}(p)$. Therefore, we have

$$\|A(x)\|_{2,\lambda;\bar{B}_2} \leq C\|\Phi_i^*v\|_{2,\lambda;\bar{B}_2} \leq C\rho(p_i)^{1+\lambda}\|v\|_{C_{1+\lambda}^{2,\lambda}(Z_{\bar{R}_0}(p))}$$

Here C is some positive constant independent of the sequence and $p \in \partial X$. We write $w = g^+ - t^+$ so that

$$\begin{aligned} \Phi_i^*((H_v)^*w) &= ((\Phi_i^*w)_{jk}(H(x))dx^j \otimes dx^k + 2(\Phi_i^*w)_{jk}(H(x))\frac{\partial A^k}{\partial x^q}dx^j \otimes dx^q \\ &\quad + (\Phi_i^*w)_{jk}(H(x))\frac{\partial A^j}{\partial x^m}\frac{\partial A^k}{\partial x^q}dx^m \otimes dx^q \end{aligned}$$

In view of Lemma 4.2, we can estimate

$$\|\Phi_i^*((H_v)^*w)\|_{1,\lambda;\bar{B}_1} \leq C\|\Phi_i^*(w)\|_{1,\lambda;\bar{B}_2} \leq C\rho(p_l)^{1+\lambda'} \leq C\rho(p_l)^{1+\lambda}R^{\lambda'-\lambda}$$

As a consequence, we infer

$$\|H_v^*w\|_{C_{1+\lambda}^{1,\lambda}(Z_R(p))} \leq CR^{\lambda'-\lambda}$$

Therefore, we prove the claim. Therefore, we choose small \bar{R}_1 such that $C(4\bar{R}_1)^{\lambda'-\lambda} < \varepsilon$. The result yields. We finish the proof. \square

4.4. Regularity of g^* . In this part, we want to get the regularity of adopted metric g^* . For this purpose, our key observation is to obtain first the regularity result for the Cotton tensor (or the Bach tensor).

Lemma 4.8. *Under the above the assumptions, we have in $Z_{\bar{R}_1}(p)$*

$$\|W[g^*]\|_{C^{1,\lambda}(Z_{\bar{R}_1}(p))} \leq C$$

so that

$$\|\mathcal{C}[g^*]\|_{C^{0,\lambda}(Z_{\bar{R}_1}(p))} \leq C$$

where C is some positive constants independent of the sequence and of p (depend on λ and \bar{R}_1).

Proof. We write $g^* = \rho^2g^+ = (H_v^{-1})^*(\frac{(\rho \circ H_v)^2}{\rho^2}\rho^2H_v^*g^+)$ and $g_1 = \frac{(\rho \circ H_v)^2}{\rho^2}\rho^2H_v^*g^+ = H_v^*g^*$. It follows from Lemmas 4.2 to 4.7, we obtain

$$\|H_v^*g^+\|_{C_{3+\lambda}^{3,\lambda}(Z_{\bar{R}_1}(p))} \leq C$$

so that the compactified metric verifies

$$\|\rho^2H_v^*g^+\|_{C^{3,\lambda}(\overline{Z_{\bar{R}_1}(p)})} \leq C$$

that is, $\rho^2 H_v^* g^+$ has the curvature in $C^{1,\lambda}$ for all $\lambda \in (0, 1)$. We recall the Weyl tensor is local conformal invariant. As a $(3, 1)$ tensor, we have

$$W[\rho^2 H_v^* g^+] = W[H_v^* g^+] = H_v^* W[g^+] = H_v^* W[g^*]$$

that is, $((H_v)^{-1})^* W[\rho^2 H_v^* g^+] = W[g^*]$. Recall H_v is a $C^{2,\lambda}$ diffeomorphism so that

$$\|W[g^*]\|_{C^{1,\lambda}(\bar{Z}_{\bar{R}_1}(p))} \leq C$$

It is known that

$$\mathcal{C}[g^*]_{ijk} = \frac{1}{d-3} W[g^*]_{jkip},^p$$

Therefore, we infer

$$\|\mathcal{C}[g^*]\|_{C^{0,\lambda}(Z_{\bar{R}_1}(p))} \leq C$$

Hence, we prove the desired result. \square

Lemma 4.9. *Under the above the assumptions, we have in $Z_{\bar{R}_1/2}(p)$*

$$\|Rm_{g^*}\|_{C^{1,\lambda}} \leq C$$

where C is some positive constant independent of the sequence and of p (depend on λ and \bar{R}_1).

Proof. It is known the Bach tensor can be written

$$B_{ij} = \nabla^k \mathcal{C}_{jki} + A^{kl} W_{kijl}$$

Using the equation (2.9), we can write by Lemma 4.8

$$\Delta R_{ij} = \partial_k f_k + g$$

where $f_k \in C^{0,\lambda}$ and $g \in L^\infty$. Assume that h is C^4 on ∂X so that $Ric|_{\partial X}$ is C^2 on the boundary. By the classical regularity theory, for example [18, theorem 8.33], there holds

$$\|Ric[g^*]\|_{C^{1,\lambda}(\bar{Z}_{\bar{R}_1/2}(p))} \leq C$$

Finally, by the decomposition of Riemann curvature tensor, we prove the desired result. \square

5. PROOF OF THEOREMS 1.1 AND 1.2

Based on the preparation in the previous sections we are ready to establish the compactness of adopted metrics on conformally compact Einstein d -dimensional manifolds which were stated in the introduction. The approach follows closely from the corresponding results in [10, 11]. The proof in even dimensions and in all dimensions are quite similar. The only difference relies on the fact that we have high order regularity for the metrics in even dimensions. However, it is not the case in odd dimensions because of the non-vanishing obstruction tensors in general. We give the proof of Theorems 1.1. For the proof of Theorems 1.2, we give the different curvature regularity result and leave the details for the interested readers.

5.1. Proof of Theorems 1.1. To begin the proof, we will first establish some bounded curvature estimates.

Lemma 5.1. *Suppose that $\{(X_i^d, g_i^+)\}$ is a sequence of conformally compact Einstein even d -dimensional manifolds satisfying the assumptions in Theorem 1.1. Then there is a positive constant K_0 such that, for the adopted metrics $\{(X_i^d, g_i^*)\}$ associated with a compact family of boundary metrics h_i –a representative of the conformal infinity $(\partial X_i^d, [h_i])$*

$$(5.1) \quad \max_{X_i^d} \sup_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_i^*}|^{\frac{2}{k+2}} \leq K_0$$

for all i .

Suppose otherwise that there is a subsequence $\{(X_i^d, g_i^+)\}$ satisfying

$$K_i = \max_{X_i^d} \sup_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_i^*}|^{\frac{2}{k+2}} \rightarrow \infty.$$

and either

$$(5.2) \quad \int_{X^d} (|W_{g_i^+}|^{d/2} dvol)[g_i^+] \rightarrow 0$$

or

$$(5.3) \quad Y(\partial X, [h_i]) \rightarrow Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}}]).$$

Let

$$K_i = K_i(p_i) = \max_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_i^*}|^{\frac{2}{k+2}}(p_i)$$

for some $p_i \in \overline{X_i}$. Then we consider the rescaling

$$(X_i^d, \bar{g}_i = K_i g_i^*, p_i).$$

In view of Lemmas 2.9 and 2.11, we have the uniform lower bound of the intrinsic injectivity radius $i_{\text{int}}(X, \bar{g}_i)$ and of the boundary injectivity radius $i_{\partial}(X, \bar{g}_i)$. Together with the assumption on the conformal infinity, we know the intrinsic injectivity radius $i(\partial X, \hat{g}_i := \bar{g}_i|_M)$ on the boundary is also uniformly bounded from below. Thus, for given $M > 1$, the harmonic radius $r^{1,\gamma}(M)$ is uniformly bounded from below for the family of metrics \bar{g}_i . Applying Lemmas 3.1 and 2.8, we have the compactness result in $C^{d-2,\gamma'}$ Cheeger-Gromov topology with base points for the metrics \bar{g}_i with $\gamma' < \gamma$, provided that the conformal infinity is bounded in $C^{d-2,\gamma}$ norm.

Lemma 5.2. *Under the above assumptions, there is no blow-up near the boundary.*

Proof. We argue by the contradiction. Let us first consider the cases where

$$\text{dist}_{\bar{g}_i}(p_i, \partial X_i) < \infty.$$

For the pointed manifolds (X_i, \bar{g}_i, p_i) with boundary, in the light of all the preparations in the previous sections, we have Cheeger-Gromov convergence

$$(X_i^d, \bar{g}_i, p_i) \rightarrow (X_{\infty}^d, g_{\infty}, p_{\infty})$$

in $C^{d-2,\gamma'}$ Cheeger-Gromov topology (for a subsequence if necessary), where the limit space is a complete Obstruction tensor flat and Q -flat manifold with a totally geodesic

boundary ∂X_∞ ; the boundary $(\partial X_\infty, h_\infty)$ is simply the Euclidean space \mathbb{R}^{n-1} because $i(\partial X_i) \geq i_0 > 0$; and

$$\max_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_\infty}|^{\frac{2}{k+2}}(p_\infty) = 1.$$

Now, clearly, to finish the proof is to show that the limit space $(X_\infty^d, g_\infty, p_\infty)$ is a locally Euclidean space. For the convenience of readers, we very briefly sketch the proof from [10, 11]. One first needs to show that $\bar{\rho}_i \rightarrow \rho_\infty$ where ρ_∞ satisfies

- $g_\infty^+ = \rho_\infty^{-2} g_\infty$ is a (partially) conformally compact Einstein metric on X_∞^d whose conformal infinity is the Euclidean space \mathbb{R}^{d-1} ;
- $v_\infty = \rho_\infty^{\frac{d-4}{2}}$ solves $-\Delta_{g_\infty^+} v_\infty - \frac{(d-1)^2-9}{4} v_\infty = 0$.

Then, by Condition (5.2), one shows that g_∞^+ is Weyl free and is locally hyperbolic space metric.

Now we assume Condition (5.3). We choose $q_i \in X$ such that $d(q_i, \partial X) \geq 1$ and $d(p_i, q_i)$ is bounded so that $(X_i^d, g_i^+, q_i) \rightarrow (X_\infty^d, g_\infty^+, q_\infty)$ in $C^{d-2, \gamma'}$ Cheeger-Gromov topology with based points. It follows from Lemma 2.10 that for any $r > 0$

$$1 = \frac{\text{vol}_{g_\infty^+}(B(q_\infty, r))}{\text{vol}_{g_{\mathbb{H}^d}}(B(r))}$$

so that g_∞^+ is locally hyperbolic space metric by the Bishop-Gromov's volume comparison.

The following is quite similar to the proof of [10, Proposition 4.8]. We sketch the proof and indicate the difference. We work with the limit metric. For simplicity, we omit the index ∞ . We denote \tilde{g}^+ standard hyperbolic space with the upper half space model. As $\tilde{g}^+ = g^+$ in a neighborhood of the boundary $\{x_1 = 0\}$, we can extend this local isometry to a covering map $\pi : \tilde{g}^+ \rightarrow g^+$. We write

$$g_1 = x_1^2 \tilde{g}^+ \text{ and } g_2 = \rho^2 g^+$$

where g_1 is the standard euclidean metric and g_2 the limit adopted metric. With the help of the covering map π , we have $\pi^* g_2 = \tilde{\rho}^2 g^+$ where $\tilde{\rho} = \rho \circ \pi$. We have

$$(5.4) \quad -\Delta_{\tilde{g}^+} \tilde{\rho}^{\frac{d-4}{2}} - \frac{(d-1)^2-9}{4} \tilde{\rho}^{\frac{d-4}{2}} = 0$$

Also, it is evident

$$-\Delta_{\tilde{g}^+} x_1^{\frac{d-4}{2}} - \frac{(d-1)^2-9}{4} x_1^{\frac{d-4}{2}} = 0.$$

Remind x_1 is the geodesic defining function w.r.t. the flat boundary metric. We write $\pi^* g_2 = \tilde{\rho}^2 g^+ = (\frac{\tilde{\rho}}{x_1})^2 g_1 := u^{\frac{4}{d-4}} g_1$ where $u = (\frac{\tilde{\rho}}{x_1})^{\frac{d-4}{2}}$. The semi-compactified metric g_2 (or $\pi^* g_2$) has flat Q_4 and the boundary metric of g_2 is the $(d-1)$ -dimensional euclidean space and totally geometric. Thus u satisfies the following conditions

$$(5.5) \quad \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}_+^d \\ -\frac{\Delta u}{u} - \frac{2}{d-4} \frac{|\nabla u|^2}{u^2} \geq 0 & \text{in } \mathbb{R}_+^d \\ u = 1 & \text{on } \partial \mathbb{R}_+^d \\ \nabla u = \Delta u = 0 & \text{on } \partial \mathbb{R}_+^d \end{cases}$$

The first equation comes from the flat Q_4 curvature and second one from the non-negative scalar curvature. As g_2 on the boundary is euclidean, u on the boundary is a constant 1. On the other hand, we know both g_1 and g_2 have the totally geodesic boundary. Hence on the boundary, $\partial_1 u = 0$ so that $\nabla u = 0$. On the other hand, it follows from Lemma 2.3 the restriction of the scalar curvature vanishes on the boundary so that $-\Delta u - \frac{2}{d-4}|\nabla u|^2 = 0$. This yields $\Delta u = 0$ on the boundary. On the other hand, we know that $-\Delta u \geq 0$ in \mathbb{R}_+^d . Using a result due to H.P.Boas and R.P. Boas [8], there exists some $a \geq 0$

$$(5.6) \quad -\Delta u = ax_1$$

We denote $w := \widetilde{\rho}^{\frac{d-4}{2}}$. Then, equation (5.4) is equivalent to the following one

$$\Delta w + \frac{2-d}{x_1} \partial_1 w = -\frac{(d-1)^2 - 9}{4x_1^2} w$$

so that

$$\Delta u = \frac{\Delta w}{x_1^{\frac{d-4}{2}}} - \frac{d-4}{x_1^{\frac{d-2}{2}}} \partial_1 w + \frac{(d-2)(d-4)}{4x_1^{\frac{d}{2}}} w = \frac{2}{x_1^{\frac{d-2}{2}}} \partial_1 w + \frac{4-d}{x_1^{\frac{d}{2}}} w$$

Together with (5.6), we infer

$$\partial_1 w + \frac{4-d}{2x_1} w = -\frac{a}{2} x_1^{\frac{d}{2}}$$

Therefore, for fixed $(x_1^0, x_2^0, \dots, x_d^0)$ with $x_1^0 > 0$, we have for $t > 0$

$$t^{\frac{4-d}{2}} w(t, x_2^0, \dots, x_d^0) - (x_1^0)^{\frac{4-d}{2}} w(x_1^0, x_2^0, \dots, x_d^0) = -\frac{a}{6} (t^3 - (x_1^0)^3).$$

Taking $t \rightarrow +\infty$, we infer

$$\begin{aligned} -(x_1^0)^{\frac{4-d}{2}} w(x_1^0, x_2^0, \dots, x_d^0) &\leq \lim t^{\frac{4-d}{2}} w(t, x_2^0, \dots, x_d^0) - (x_1^0)^{\frac{4-d}{2}} w(x_1^0, x_2^0, \dots, x_d^0) \\ &= \lim -\frac{a}{6} (t^3 - (x_1^0)^3) = -\infty, \end{aligned}$$

provided $a > 0$. This gives also a contradiction provided $a > 0$. Hence $a = 0$. Finally, $-\frac{\Delta u}{u} - \frac{2}{d-4} \frac{|\nabla u|^2}{u^2} \geq 0$ implies $\nabla u \equiv 0$, that is, g_2 is flat. This contradiction yields that there is no boundary blow-up. \square

Lemma 5.3. *Under the above assumptions, there is no interior blow-up.*

Proof. We consider the rest cases when

$$\text{dist}_{\bar{g}_i}(p_i, \partial X_i) \rightarrow \infty$$

(at least for some subsequence). Notice that,

$$K_i = \max_{X_i} \max_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_i^*}|_{\frac{2}{k+2}} = \max_{k=(k_1, \dots, k_l), |k|:=l \leq d-4} |\nabla^k Rm_{g_i^*}|_{\frac{2}{k+2}}(p_i)$$

for some $p_i \in X$ in the interior. Proceeding as the above boundary cases, one has the Cheeger-Gromov convergence

$$(X_i^d, \bar{g}_i, p_i) \rightarrow (X_\infty^d, g_\infty, p_\infty)$$

in $C^{d-2,\gamma'}$ Cheeger-Gromov topology. The proof in these cases follows from [10]. We again very briefly sketch the proof that is more or less from [10]. One first derives from (2.1) that

$$R_{\bar{g}_i} = 2(d-1)\bar{\rho}_i^{-2}(1 - |d\bar{\rho}_i|_{\bar{g}_i}^2)$$

and shows that

- $\bar{\rho}_i(x) \geq C \text{dist}_{\bar{g}_i}(x, \partial X_i)$. (cf. Step 2 in the proof of [10, Lemma 4.9]).

Then, consequently,

- $R_\infty = 0$, and
- g_∞ is Ricci-flat from being Q -flat and scalar flat in the light of the Q -curvature equation (2.10). (cf. Step 3 of the proof of [10, Lemma 4.9]).

Thus, (X_∞, g_∞) is a complete Ricci-flat d -dimensional manifold with no boundary. As same arguments as in the previous part, we have (X_∞, g_∞) is locally conformally flat, so that (X_∞, g_∞) is flat because of the decomposition of the curvature tensor. Therefore, we obtain the desired contradiction. For more details see [10] section 4.3. \square

Proof of Lemma 5.1. . It is a direct consequence of Lemmas 5.2 and 5.3 \square

We now begin the proof of Theorem 1.1. For this purpose, we need to prove the following diameter bound.

Lemma 5.4. *Under the assumptions in Theorem 1.1, the diameters of the adopted metrics g_i^* are uniformly bounded.*

Proof. We use the similar strategy as in [11, Section 4: *The proof of Lemma 4.2*]. We sketch the proof.

We have already proved the family of metrics g_i^* has the bounded curvature in $C^{d-4,\gamma'}$. In view of Lemmas 2.9 and 2.11, the boundary radius and the interior one are uniformly bounded from below. Therefore, for all i , for all $x \in \bar{X}$, we have $\text{vol}(B^{g_i^*}(x, 1)) \geq C > 0$ for some constant $C > 0$ independent of i, x , that is, there is non-collapse. We prove the diameter is uniformly bounded above by contradiction. Suppose that the diameter $\text{diam}(g_i^*)$ tends to the infinity. By Cheeger-Gromov-Hausdorff compactness theory, up to diffeomorphisms fixing the boundary, (X_i, g_i^*) converges to some complete non-compact manifold (X_∞, g_∞) with the boundary.

Step 1. As in the same way as in [10, Section 4: *the proof of Lemma 4.4*], there exists some $C > 0$ such that $\rho_i \geq C$ provided $d_{g_i^*}(x, \partial X) \geq 1$ and $d_{g_i^*}(x, \partial X) \leq C\rho_i(x)$ provided $0 \leq d_{g_i^*}(x, \partial X) \leq 1$. Thus the limit metric is conformal to an asymptotic hyperbolic Einstein manifold. Without loss of generality, assume the boundary injectivity radius is bigger than 1.

Step 2. There exists some constant $C_2 > 0$ independent of i such that

$$(5.7) \quad \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \rho_i^{-3/2}(x) \leq C_2.$$

Thanks of (2.1) and (2.3), we infer

$$-\Delta\sqrt{\rho_i} = \frac{(d+2)R_i\rho_i^{1/2}}{8(d-1)} + \frac{|\nabla\rho_i|^2}{4\rho_i^{3/2}} = \frac{(d+2)(1-|\nabla\rho_i|^2)}{4\rho_i^{3/2}} + \frac{|\nabla\rho_i|^2}{4\rho_i^{3/2}}$$

Integrating on the set $\{x, d_{g_i^*}(x, \partial X) \geq 1\}$, we obtain

$$\int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \frac{(d+2)(1-|\nabla\rho_i|^2)}{4\rho_i^{3/2}} + \frac{|\nabla\rho_i|^2}{4\rho_i^{3/2}} = \left| \oint_{\{x, d_{g_i^*}(x, \partial X) = 1\}} \frac{1}{2\sqrt{\rho_i}} \langle \nabla\rho_i, \nu \rangle \right|$$

where ν is the outside normal vector on the boundary $\{x, d_{g_i^*}(x, \partial X) = 1\}$. By Step 1, we know ρ_i is uniformly bounded from below on the set $\{x, d_{g_i^*}(x, \partial X) = 1\}$. Together the facts the curvature of g_i^* is bounded and the boundary $(\partial X_i, h_i)$ is compact, we infer for some positive constant $C > 0$

$$\int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \frac{(1-|\nabla\rho_i|^2)}{\rho_i^{3/2}} \leq C, \text{ and } \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \frac{|\nabla\rho_i|^2}{\rho_i^{3/2}} \leq C$$

since $|\nabla\rho_i| \leq 1$. Combining these estimates, the desired claim yields.

Step 3. We have

$$\lim_{x \rightarrow \infty} \rho_\infty(x) = +\infty$$

Letting $i \rightarrow \infty$ in (5.7), we get

$$(5.8) \quad \int_{\{x, d_{g_\infty}(x, \partial X) \geq 1\}} \rho_\infty^{-3/2}(x) \leq \lim_i \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \rho_i^{-3/2}(x) \leq C_2.$$

For all $\varepsilon > 0$, there exists $A > 0$ such that

$$\int_{\{x, d_{g_\infty}(x, \partial X) \geq A\}} \rho_\infty^{-3/2}(x) \leq \varepsilon$$

Therefore, for any y with $d_{g_\infty}(y, \partial X) \geq A + 1$, we can estimate

$$\int_{B^{g_\infty}(y, 1)} \rho_\infty^{-3/2}(x) \leq \int_{\{x, d_{g_\infty}(x, \partial X) \geq A\}} \rho_\infty^{-3/2}(x) \leq \varepsilon$$

so that

$$\left(\sup_{B^{g_\infty}(y, 1)} \rho_\infty \right)^{-3/2} \text{Vol}(B^{g_\infty}(y, 1)) \leq \varepsilon$$

that is,

$$\sup_{B^{g_\infty}(y, 1)} \rho_\infty \geq C\varepsilon^{-2/3}$$

Together with Lemma 2.3, we deduce

$$\inf_{B^{g_\infty}(y, 1)} \rho_\infty \geq \sup_{B^{g_\infty}(y, 1)} \rho_\infty - 1 \geq C\varepsilon^{-2/3} - 1$$

Finally, we prove Step 3.

Step 4. We claim that there exists some $c_v > 0$ such that for any $p \in X_\infty$ and for any $r < \frac{1}{2}\rho_\infty(p)$

$$(5.9) \quad \text{Vol}(B^{g_\infty}(p, r)) \geq c_v r^d$$

Let $p_i \in X_i$ such that $p_i \rightarrow p$. First we remark that $\text{dist}_{g_i^*}(p_i, \partial X_i) \geq \rho_i(p_i)$ because of Lemma 2.3. Again by Lemma 2.3 and together with Lemma 2.10(cf. [11, Section 3: *the end of the proof of Lemma 3.3*]), we have

$$\text{Vol}(B^{g_i^*}(p_i, r)) \geq c_v r^d,$$

where c_v is some positive constant independent of i . Letting $i \rightarrow \infty$, the claim is proved.

Step 5. A contradiction.

On choose $p \in X_\infty$ such that $\rho_\infty(p)$ is sufficiently large. We fix $r = (\rho_\infty(p))^{3/4}$. Using the results in Steps 2 and 4, we get $\rho_\infty(p)$

$$(5.10) \quad \left(\sup_{B^{g_\infty}(p, r)} \rho_\infty \right)^{-3/2} \text{Vol}(B^{g_\infty}(p, r)) \leq \int_{B^{g_\infty}(p, r)} \rho_\infty^{-3/2}(x) \leq C_2$$

so that for some positive constant $C > 0$ there holds

$$(5.11) \quad \sup_{B^{g_\infty}(p, r)} \rho_\infty \geq C r^{2d/3} = C(\rho_\infty(p))^{d/2}$$

On the other hand, it follows from Lemma 2.3, we deduce

$$(5.12) \quad \inf_{B^{g_\infty}(p, r)} \rho_\infty \geq \sup_{B^{g_\infty}(p, r)} \rho_\infty - r$$

so that

$$\rho_\infty(p) + (\rho_\infty(p))^{3/4} = \rho_\infty(p) + r \geq C(\rho_\infty(p))^{d/2}.$$

This yields that $\rho_\infty(p)$ is bounded. This contradicts the claim in Step 3.

Thus we have finished the proof of Lemma 5.4. \square

Proof of Theorem 1.1. As in the step 4 of the proof of Lemma 5.4, there is no collapse for the sequence of the metrics g_i^* . Thanks of Lemmas 5.1 and 5.4, we use the Cheeger-Gromov compactness result to prove Theorem 1.2. Hence, we finish the proof. \square

5.2. Proof of Theorems 1.2. The proof in all dimensions is quite same as for even ones. We just point out the only below difference on the regularity of the curvature tensors, that is, with the ε -regularity, we could prove the curvature estimate in all dimensions

Lemma 5.5. *Suppose that $\{(X_i^d, g_i^+)\}$ is a sequence of conformally compact Einstein d -dimensional manifolds with all $d \geq 4$ satisfying the assumptions in Theorem 1.2. Then there is a positive constant K_0 such that, for the adopted metrics $\{(X_i^d, g_i^*)\}$ associated with a compact family of boundary metrics h_i – a representative of the conformal infinity $(\partial X_i^d, [h_i])$*

$$(5.13) \quad \max_{X_i^d} |Rm_{g_i^*}| \leq K_0$$

for all i .

6. UNIQUENESS OF GRAHAM-LEE SOLUTIONS IN HIGH DIMENSION AND GAP PHENOMENON

In this section we derive the global uniqueness result Theorem 1.3 and give a gap phenomenon Corollary 6.1 below, which are direct consequences of compactness result Theorem 1.2..

Proof of Theorem 1.3. The proof is almost as same as in dimension 4 [11, Section 5]. We sketch the proof here. We will prove this by contradiction. Assume otherwise there is a sequence of conformal $(d-1)$ -dimensional sphere $(\mathbb{S}^{d-1}, [h_i])$ that converges to the round sphere such that, for each i , there exist two non-isometric conformally compact Einstein metrics g_i^+ and \tilde{g}_i^+ .

Up to a subsequence, both g_i^+ and \tilde{g}_i^+ converge to the hyperbolic space in $C^{d-2, \gamma'}$ Cheeger-Gromov sense (in particular in $C^{2, \gamma'}$ Cheeger-Gromov sense) due to Theorem 1.2 and the uniqueness result when the conformal infinity is the standard sphere [33, 27].

The main facts are the following

- There exists a diffeomorphism φ_i of class $C^{2, \gamma}$ for any $\gamma \in (0, 1)$ (equal to the identity on the boundary) (see Lemma 4.4), such that

$$F(\varphi_i^* \tilde{g}_i^+, g_i^+) = 0$$

Moreover $\|\varphi_i(x) - x\|_{C^{2, \gamma}} \rightarrow 0$ and $\|\varphi_i^* \tilde{g}_i^+ - g_i^+\|_{C_{1+\gamma}^{1, \gamma}} \rightarrow 0$.

- Because of local uniqueness (see Lemma 4.6), for large i , we have

$$g_i^+ = \varphi_i^* \tilde{g}_i^+.$$

□

As a direction consequence of Theorem 1.2, we are able to prove some gap phenomenon. Given some large positive number $\Lambda > 0$ and number $d \geq 4$, let

$$\mathcal{A}_C := \{(\mathbb{S}^{d-1}, [h]) \mid \begin{array}{l} h \text{ could not be joint by a continuous path in the set of the metrics} \\ \text{with positive scalar curvature to the standard metric } g_{\mathbb{S}^{d-1}} \\ \text{in } C^6(\mathbb{S}^{d-1}) \text{ topology, } (\mathbb{S}^{d-1}, [h]) \text{ is the conformal infinity of CCE metrics,} \\ h \text{ has the constant positive scalar curvature and } \|h\|_{C^6(g_{\mathbb{S}^{d-1}})} \leq \Lambda \end{array}\}$$

denote the union of the path connected components of the metrics on the spheres with the constant positive scalar curvature which are not connected to the standard metric in C^3 topology. We have the following result:

Corollary 6.1. *Under the above assumptions and given $\Lambda > 0$ and number $d \geq 4$, there exists some small positive constants $\varepsilon > 0$ and $\varepsilon_1 > 0$ such that*

$$(1) \quad \sup_{h \in \mathcal{A}_\Lambda} Y(\mathbb{S}^{d-1}, [h]) \leq Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}^{d-1}}]) - \varepsilon$$

$$(2) \quad \text{Given any } h \in \mathcal{A}_\Lambda, \text{ let } (X, \partial X = \mathbb{S}^{d-1}, g^+) \text{ be some CCE metric with conformal infinity } [h] \text{ on sphere } \mathbb{S}^{d-1}. \text{ Then we have}$$

$$\int_{X^n} (|W|^{d/2} d\text{vol})[g^+] > \varepsilon_1.$$

Proof of Corollary 6.1. We will prove this by contradiction. Suppose there exists a sequence of CCE metrics (X, g_i^+) with $[h_i] \in \mathcal{A}_\Lambda$ such that :

Either

$$\int_{X^d} (|W|^{d/2} d\text{vol})[g_i^+] \rightarrow 0$$

or

$$Y(\partial X, [h_i]) \rightarrow Y(\mathbb{S}^{d-1}, [g_S])$$

In view of Theorem 1.2, up to a subsequence, h_i converges to the standard metric $h_{\mathbb{S}^{d-1}}$ in $C^{3,\alpha}$ topology for all $\alpha \in (0, 1)$ so that h_i should be in the same connected component of metrics on the sphere with positive scalar curvature. Thus, we get a desired contradiction of definition of \mathcal{A}_Λ . Hence, we finish the proof. \square

Remark 6. *In the above result, we can assume the metrics in the set \mathcal{A}_C in $C^{5,\gamma}$ Cheeger-Gromov topology.*

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