

COMPACTNESS OF CONFORMALLY COMPACT EINSTEIN 4-MANIFOLDS II

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ABSTRACT. In this paper, we establish a number of compactness results for some class of conformally compact Einstein 4-manifolds. In the first part of the paper, we improve the earlier compactness results obtained by Chang-Ge [16]. In the second part of the paper, as applications, we derive some further compactness results under the perturbation condition when the L^2 norm of the Weyl curvature is small. We also derive the global uniqueness of conformally compact Einstein metrics on the 4-Ball constructed in the earlier work of Graham-Lee [32].

1. INTRODUCTION

1.1. Statement of Improved results. Let X^4 be a smooth 4-manifold with boundary ∂X . A smooth conformally compact metric g^+ on X is a Riemannian metric such that $g = r^2 g^+$ extends smoothly to the closure \bar{X} for some defining function r of the boundary ∂X in X . A defining function r is a smooth nonnegative function on the closure \bar{X} such that $\partial X = \{r = 0\}$ and $dr \neq 0$ on ∂X . A conformally compact metric g^+ on X is said to be conformally compact Einstein (CCE) if, in addition,

$$\text{Ric}[g^+] = -ng^+,$$

where $\text{Ric}[\cdot]$ denotes the Ricci tensor of the Riemannian metric (sometimes we use also indice to indicate the dependence on the metric). The most significant feature of CCE manifolds (X, g^+) is that the metric g^+ is “canonically” associated with the conformal structure $[\hat{g}]$ on the boundary at infinity ∂X , where $\hat{g} = g|_{T\partial X}$. $(\partial X, [\hat{g}])$ is called the conformal infinity of a conformally compact manifold (X, g^+) . It is of great interest in both mathematics and theoretic physics to understand the correspondences between conformally compact Einstein manifolds (X, g^+) and their conformal infinities $(\partial X, [\hat{g}])$, especially due to the AdS/CFT correspondence in theoretic physics (cf. Maldacena [39], [40] and Witten [46]).

The project we work on is to address the compactness issue of given a sequence of CCE 4-manifolds $(X^4, M^3, \{g_i^+\})$ with $M = \partial X$ and $\{g_i\} = \{r_i^2 g_i^+\}$ a sequence of compactified metrics, denote $\hat{g}_i = g_i|_M$, assume $\{\hat{g}_i\}$ forms a compact family of metrics in M , when is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M\} = \{\hat{g}_i\}$ also forms a compact family of metrics in X ? We remark that, for a CCE manifold, given any conformal infinity, a special defining function which we call geodesic defining function r exists so that $|\nabla[\bar{g}]r| \equiv 1$ in an asymptotic neighbor $M \times [0, \epsilon)$ of M . We also remark that the eventual goal to study

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the compactness problem is to show existence of conformal filling in for some classes of Riemannian manifolds as conformal infinity.

One of the difficulty to address the compactness problem is due to the existence of some “non-local” term. To see this, we look at the asymptotic behavior of the compactified metric g of CCE 4-manifold (X^4, M^3, g^+) with conformal infinity (M^3, \hat{g}) ([30], [28]) which takes the form

$$g := r^2 g^+ = h + g^{(2)} r^2 + g^{(3)} r^3 + g^{(4)} r^4 + \dots$$

on an asymptotic neighborhood of $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g . It turns out $g^{(2)} = -\frac{1}{2}A[\hat{g}]$, where $A[\hat{g}] := Ric[\hat{g}] - \frac{1}{6}R[\hat{g}]\hat{g}$ denotes the Schouten tensor, $Ric[\hat{g}]$ the Ricci tensor and $R[\hat{g}]$ the scalar curvature respectively for the metric \hat{g} . Thus $g^{(2)}$ is determined by \hat{g} (we call such terms local terms), $Tr_{\hat{g}}g^{(3)} = 0$, while

$$g_{\alpha,\beta}^{(3)} = -\frac{1}{3}\partial_{\mathbf{n}}(Ric[g])_{\alpha,\beta}$$

where α, β denote the tangential coordinate on M and $\partial_{\mathbf{n}}$ is the normal derivative with respect to the outward unit normal \mathbf{n} of the boundary under the metric g , is a non-local term which is not determined by the boundary metric \hat{g} . We remark that \hat{g} together with $g^{(3)}$ determine the asymptotic behavior of g ([28], [9]).

In an earlier work of Chang-Ge [16, Lemma 2.1], for a CCE manifold (X^4, M^3, g^+) , we introduce the notion of 2-tensor S which on a 3-manifold M^3

$$(S[g])_{\alpha,\beta} := \nabla^i(W[g])_{i\alpha n\beta} + \nabla^i(W[g])_{i\beta n\alpha} - \nabla^{\mathbf{n}}(W[g])_{n\alpha n\beta} - \frac{4}{3}H[g](W[g])_{\alpha n\beta}^{\mathbf{n}}$$

where $W[g]$ denotes the Weyl tensor, $H[g]$ the mean curvature on the boundary M , letter i is full indice and Greek indices α, β represent the tangential indices. When the compactified metric g has totally geodesic boundary, it takes the form:

$$(S[g])_{\alpha,\beta} = \frac{1}{2}\partial_{\mathbf{n}} Ric[g]_{\alpha,\beta} - \frac{1}{12}\partial_{\mathbf{n}} R[g] \hat{g}_{\alpha,\beta}.$$

The 2-tensor S is conformally invariant in the sense that

$$S[r^2 g] = r^{-1} S[g].$$

The connection of the S tensor to that of $g^{(3)}$ is that (see [16, Remark 2.2, (2.7)]): Under any compactification by a geodesic defining function r , $g = r^2 g^+$ has $\partial_{\mathbf{n}} R[g] = 0$ on M , thus

$$(1.1) \quad (S[g])_{\alpha,\beta} = -\frac{3}{2}g_{\alpha,\beta}^{(3)}.$$

This shows that $g^{(3)}$ is also a local conformal invariant, which has been stated by Graham [30].

In [16], we have also considered a special choice of compactification g^* , which we named the Fefferman-Graham’s (FG) compactification, defined by solving the PDE:

$$(1.2) \quad -\Delta_{g^+} w = 3 \quad \text{on } X^4.$$

$g^* := e^{2w}g^+$ with $g^*|_M = g^Y$, the Yamabe metric on the conformal infinity of (X^4, g^+) .

We now state the first result of this paper.

Theorem 1.1. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X . Assume the following conditions:*

- (1) *The set $\{\hat{g}_i\}$ of Yamabe metrics that represent the conformal infinities lies in a given set \mathcal{C} of metrics that is of positive Yamabe type and compact in $C^{k,\alpha}$ Cheeger-Gromov topology with $k \geq 3$ and with some $\alpha \in (0, 1)$.*
- (2) *The FG compactifications $\{g_i^* = \rho_i^2 g_i^+\}$ associated with the Yamabe representatives $\{\hat{g}_i\}$ on the boundary satisfies:*

$$\limsup_{r \rightarrow 0} \sup_i \sup_{x \in \partial X} \int_{B(x,r)} |S_i|[g_i^*] d\text{vol}[\hat{g}_i] = 0$$

- (3) $H_2(X, \mathbb{Z}) = 0$.

Then, the set $\{g_i^\}$ of FG compactifications (after diffeomorphisms that fix the boundary) forms a compact family in the $C^{k,\alpha}$ Cheeger-Gromov topology.*

We now explain the connection of the S tensor to other scalar curvature invariants for the metric g^* , which plays a key role in the results in [16] and in this paper. In the following, if there is no confusion, we drop the argument g for the various curvature tensors Ric, R, W , etc.

On a 4-manifold (X^4, g) , a 4-th order curvature called the Q -curvature is defined as:

$$(1.3) \quad Q[g] := -\frac{1}{6}\Delta R - \frac{1}{2}|Ric|^2 + \frac{1}{6}R^2.$$

Q curvature is naturally associated with a 4th-order differential operator:

$$P[g] := (\Delta)^2 - \text{div} \left[\left(\frac{2}{3}Rg - 2Ric \right) \nabla \right]$$

called Paneitz operator [42, 12]. We remark that Paneitz operator is a special case of the family of GJMS ([31]) operators. The relation of the pair $\{Q, P\}$ in 4 dimensions is like that of the well known pair $\{K, -\Delta\}$ in 2 dimensions, where K denotes the Gaussian curvature:

$$\begin{aligned} -\Delta[g] + K[g] &= K[e^{2w}g]e^{2w} \quad \text{on } X^2, \\ P[g]w + Q[g] &= Q[e^{2w}g]e^{4w} \quad \text{on } X^4 \end{aligned}$$

for conformal changes of the metric. For a 4-manifold (X^4, g) with boundary, in the earlier works of Chang-Qing [17, 18], in connection with the 4th order Q curvature, a 3rd order "non-local" boundary curvature T was introduced on ∂X to study the boundary behavior of g . The relation between the pair (Q, T) is a generalization of that of the Dirichlet-Neumann pair $(-\Delta, \partial_n)$. The expression of T curvature is in general complicated, but in the special case when g is totally geodesic, the expression T take the simple form:

$$(1.4) \quad T[g] := \frac{1}{12}\partial_n R.$$

We now state the second result of our paper.

Theorem 1.2. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X . Assume the same conditions (1) and (3) as in Theorem 1.1 and*

(2') *For the associated Fefferman-Graham's compactifications $\{g_i^* = e^{2w_i} g_i^+\}$ with the Yamabe representatives $\{\hat{g}_i\}$ on the boundary,*

$$\liminf_{r \rightarrow 0} \inf_i \inf_{x \in \partial X} \oint_{B(x,r)} T[g_i^*] d\text{vol}[\hat{g}_i] \geq 0.$$

Then, the set $\{g_i^\}$ is compact in $C^{k,\alpha}$ norm up to diffeomorphisms that fix the boundary, provided $k \geq 7$.*

In the earlier work of Chang-Ge [16], both Theorem 1.1 and Theorem 1.2 were established under the extra conditions on the uniform bounds of the first and second Yamabe constants for manifolds with boundary. The conditions are:

(4) There exists some positive constant $C_5 > 0$ such that the first Yamabe constant for the compactified metric $g_i := \rho_i^2 g_i^+$ is bounded uniformly from below by C_5 i.e.

$$Y(X, \partial X, [g_i]) := \inf_{U \in C^1} \frac{\int_X (|\nabla U|^2 + \frac{1}{6} R[g_i] U^2) d\text{vol}[g_i]}{\left(\int_X U^4 d\text{vol}[g_i] \right)^{\frac{1}{2}}} \geq C_5.$$

(5) There exists some positive constant $C_6 > 0$ such that the second Yamabe constant for the metric g_i is bounded uniformly from below by C_6 , i.e.

$$Y_b(X, \partial X, [g_i]) := \inf_{U \in C^1} \frac{\int_X (|\nabla U|^2 + \frac{1}{6} R[g_i] U^2) d\text{vol}[g_i]}{\left(\oint_{\partial X} U^3 d\text{vol}[\hat{g}_i] \right)^{\frac{2}{3}}} \geq C_6,$$

where $\hat{g}_i = g_i|_{T\partial X}$.

We remark Condition (1) in the earlier work [16, Theorem 1.1] is stated slightly weaker, that is, the Yamabe constant $Y(\partial X, [\hat{g}_i])$ is assumed to be non-negative.

In the current paper we managed to drop both conditions (4) and (5) from the statements of Theorems 1.1 and 1.2, which is accomplished by applying another round of blow-up analysis to reduce the situation to the earlier version of the theorems in [16]. We will present the proof in section 3 of this paper. Once the curvature of metric g_i^* is bounded, we will prove the diameter is uniformly bounded in section 4 by some new arguments.

1.2. Statement of new results. Due to the nature of the problem in the CCE setting, natural conditions to imply the compactness of the solutions should be conformally invariant conditions, condition (1) in the statements of Theorems 1.1 and 1.2 is conformally invariant, condition (3) is topologically invariant hence a conformal invariant, but unfortunately the condition (2) in Theorem 1.1 and condition (2') in Theorem 1.2 are not. It is in this direction we now have new results where compactness is reached under some conformally invariant conditions; as a consequence we also establish some "uniqueness" result of conformal filling in for a special class of CCE manifolds with given conformal infinity.

Theorem 1.3. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X . Assume the same condition (1) in Theorem 1.1. Then there is $\delta_0 > 0$ such that if either*

$$(2'') \quad \int_{X^4} (|W|^2 dvol)[g_i^+] < \delta_0, \quad \text{or}$$

$$(2''') \quad Y(\partial X, [\hat{g}_i]) \geq Y(\mathbb{S}^3, [g_S]) - \delta_0,$$

then the set $\{g_i^\}$ of the FG compactifications (after diffeomorphisms that fix the boundary) is compact in $C^{k,\alpha}$ Cheeger-Gromov topology for some $k \geq 3$.*

In fact, for $\epsilon > 0$ and $\alpha \in (0, 1)$, there is $\delta > 0$, if g^+ is a conformally compact Einstein metric on X^4 with the conformal infinity $(\partial X, [\hat{g}])$ and g^ is the FG compactification associated with the Yamabe representative that belongs the set \mathcal{C} in (1) in Theorem 1.1, and if (2'') (or (2''')) holds for some $\delta \leq \delta_0$, then there is a diffeomorphism*

$$\phi_i : \bar{X}^4 \rightarrow \bar{B}^4 \text{ and } \phi_i = Id : \partial B^4 = \mathbb{S}^3 \rightarrow \partial X^4 = \mathbb{S}^3$$

satisfying

$$(1.5) \quad \|\phi_i^* g_i^* - g_{FG}\|_{C^{k,\alpha}(\bar{B}^4)} < \epsilon$$

where g_{FG} is the FG compactification of the hyperbolic metric associated with a round metric on \mathbb{S}^3 .

We will now relate the condition (2'') in Theorem 1.3 to some other natural geometric conformal invariant, namely the "renormalized volume" in the CCE setting. Although the renormalized volume can be defined on CCE manifolds $(X^{n+1}, \partial X, g^+)$ for any dimension n , we will here mainly recall some basic facts on CCE manifolds $(X^4, \partial X, g^+)$ when $n = 3$.

The concept of "renormalized volume" in the CCE setting was introduced by Maldacena [39] (see also the works of Witten [46], Henningson-Skenderis [34] and Graham [30]). On CCE manifolds (X^4, M^3, g^+) with geodesic defining function r ,

For n odd,

$$\text{Vol}_{g^+}(\{r > \epsilon\}) = c_0 \epsilon^{-3} + c_2 \epsilon^{-1} + V + o(1).$$

We call the zero order term V the renormalized volume. V is independent of $g^+ \in [g^+]$, and hence are conformal invariants.

We now recall Gauss-Bonnet-Chern formula on compact 4-manifolds $(X^4, \partial X, g)$ with totally geodesic boundary.

$$(1.6) \quad 8\pi^2\chi(X) = \frac{1}{4} \int_X |W[g]|^2 d\text{vol}[g] + 4 \int_X \sigma_2(A[g]) d\text{vol}[g],$$

where $\sigma_2(A[g]) := \frac{1}{4}(\frac{1}{6}R[g]^2 - \frac{1}{2}|Ric[g]|^2)$ is the second elementary symmetric function of the Schouten tensor $A[g] := \frac{1}{2}(Ric[g] - \frac{1}{6}R[g]g)$. We also recall an earlier result:

Proposition 1.4. (*M. Anderson [3], Chang-Qing-Yang [20], [21]*)
On conformal compact Einstein manifold (X^4, M^3, g^+) , we have

$$V = \frac{2}{3} \int_{X^4} \sigma_2(A[g]) d\text{vol}[g]$$

for any compactified metric g with totally geodesic boundary. Thus

$$(1.7) \quad 8\pi^2\chi(X^4, M^3) = \frac{1}{4} \int |W[g]|^2 d\text{vol}[g] + 6V.$$

We briefly recall the proof of above Proposition in Chang-Qing-Yang [20], as this is the crucial point that leads us to adopt the Fefferman-Graham's compactification to study the compactness problem of CCE manifolds.

Sketch proof of Proposition 1.4.

Lemma 1.5. (*Fefferman-Graham [27]*) *Suppose $(X^4, \partial X, g^+)$ is conformally compact Einstein with conformal infinity $(\partial X, [\hat{g}])$, fix $\hat{g} \in [\hat{g}]$ and r its corresponding geodesic defining function. Consider the solution w to (1.2), then w has the asymptotic behavior*

$$w = \log r + A + Br^3$$

near ∂X , where A, B are functions even in r , $A|_{\partial X} = 0$, then

$$V = \int_{\partial X} B|_{\partial X}.$$

Lemma 1.6. (*Chang-Qing-Yang [20]*) *With the same notation as in Lemma 1.5, Consider the metric $g^* = g_w = e^{2w}g^+$, then g^* is totally geodesic on boundary with (1) $Q[g^*] \equiv 0$, (2) $B|_{\partial X} = \frac{1}{36}\partial_{\mathbf{n}}R[g^*] = \frac{1}{3}T[g^*]$.*

Proof. Recall we have g^+ is Einstein with $Ric_{g^+} = -3g^+$, thus

$$P_{g^+} = (-\Delta_{g^+}) \circ (-\Delta_{g^+} - 2)$$

and $Q[g^+] = 6$. Therefore

$$P_{g^+}w + Q[g^+] = 0 = e^{2w}Q[g^*].$$

Assertion (2) follows from a straight forward computation using the scalar curvature equation and the asymptotic behavior of w . \square

Applying Lemmas 1.5 and 1.6, we get

$$\begin{aligned} 6V &= 6 \oint_{\partial X} B|_{\partial X} d\text{vol}[\hat{g}] = \frac{1}{6} \oint_{\partial X} \partial_{\mathbf{n}} R[g^*] d\text{vol}[\hat{g}] \\ &= \int_X Q[g^*] d\text{vol}[g^*] + 2 \oint_{\partial X} T[g^*] d\text{vol}[\hat{g}] = 4 \int_X \sigma_2(A[g^*]) d\text{vol}[g^*]. \end{aligned}$$

For any other compactified metric g with totally geodesic boundary, $\int_X \sigma_2(A[g]) d\text{vol}[g]$ is a conformal invariant, and V is a conformal invariant, thus the result holds once for g^* , holds for any such g in the same conformal class, which establishes Proposition 1.4. \square

We also recall some well known fact that (cf. [19, 20]). for a conformally compact Einstein 4-manifold with the conformal infinity of positive Yamabe type,

$$(1.8) \quad V(X^4, g^+) \leq V(\mathbb{H}^4, g_{\mathbb{H}}) = \frac{4\pi^2}{3}$$

where the equality holds if and only if (X^4, g^+) is isometric to $(\mathbb{H}^4, g_{\mathbb{H}})$.

We now restrict our attention to class of CCE manifolds (B^4, S^3, g^+) , in this class, for the model case when $g^+ = g_{\mathbb{H}}$, formulas for the specific FG g^* metric can be computed straight forwardly.

Lemma 1.7. *On $(B^4, S^3, g_{\mathbb{H}})$,*

$$\begin{aligned} g^* &= e^{(1-|x|^2)} |dx|^2 \text{ on } B^4 \\ Q[g^*] &\equiv 0, \quad T[g^*] \equiv 2 \text{ on } S^3 \\ (g^*)^{(3)} &\equiv 0 \end{aligned}$$

and

$$\int_{B^4} \sigma_2(A[g^*]) d\text{vol}[g^*] = 2\pi^2.$$

On (B^4, S^3, g) , for a compact metric g with totally geodesic boundary, Gauss-Bonnet-Chern formula takes the form:

$$8\pi^2 \chi(B^4, S^3) = 8\pi^2 = \int_{B^4} \left(\frac{1}{4} |W[g]|^2 + 4\sigma_2(A[g]) \right) d\text{vol}[g],$$

Thus we reached the following corollary of Theorem 1.3:

Corollary 1.8. *Let $\{X = B^4, M = \partial X = S^3, g^+\}$ be a 4-dimensional oriented CCE on X with boundary ∂X . Assume the boundary Yamabe metric $h = h^Y$ in the conformal infinity of positive type and $Y(S^3, [h]) > c_1$ for some fixed $c_1 > 0$ and h is compact in $C^{k,\alpha}$ norm with $k \geq 3$. Let g^* be the corresponding FG compactification. Then the following properties are equivalent:*

(1) *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X \sigma_2(A[g^*]) d\text{vol}[g^*] \geq 2\pi^2 - \varepsilon.$$

(2) *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X |W[g^+]|^2 d\text{vol}[g^+] \leq 4\varepsilon.$$

(3) *There exists some small positive number $\varepsilon_1 > 0$ such that*

$$Y(S^3, [g_c]) \geq Y(S^3, [h]) > Y(S^3, [g_c]) - \varepsilon_1$$

where g_c is the standard metric on S^3 .

(4) *There exists some small positive number $\varepsilon_2 > 0$ such that for all metrics g^* with boundary metric h same volume as the standard metric g_c on S^3 , we have*

$$T[g^*] \geq 2 - \varepsilon_2.$$

(5) *There exists some small positive number $\varepsilon_3 > 0$ such that*

$$|(g^*)^{(3)}| \leq \varepsilon_3.$$

Where all the ε_i ($i = 1, 2, 3$) tends to zero when ε tends to zero and vice versa for each i .

As an application of Theorem 1.3, we are able to establish the global uniqueness for the conformally compact Einstein metrics on \mathbb{B}^4 with prescribed conformal infinities that very close to the conformal round 3-sphere (cf. [32, 37, 38]). Namely,

Theorem 1.9. *For a given conformal 3-sphere $(\mathbb{S}^3, [\hat{g}])$ that is sufficiently close to the round one in $C^{3,\alpha}$ topology with $\alpha \in (0, 1)$, there is exactly one conformally compact Einstein metric g^+ on \mathbb{B}^4 whose conformal infinity is the prescribed conformal 3-sphere $(\mathbb{S}^3, [\hat{g}])$.*

Remark 1.10. *We remark*

- *First, in Theorems 1.1 and 1.2, if we assume more assumptions on the topology in (3) $H_1(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = 0$, we can drop the assumption on the boundary $\partial X = \mathbb{S}^3$ (see [16]).*
- *Second, in Theorems 1.3, we do not need the boundary condition $\partial X = \mathbb{S}^3$ for the compactness result in the first part; for the second part, when the boundary condition $\partial X = \mathbb{S}^3$ holds and δ_0 is small enough, we have the topology of X is the ball \mathbb{B}^4 . Therefore, Theorem 1.9 holds even without the assumption $X = \mathbb{B}^4$.*
- *Third, in Theorems 1.1, 1.2 and 1.3, we have the compactness results in $C^{2,\beta}$ topology for any $\beta \in (0, 1)$ if we assume the compactness (or more generally boundedness) of the boundary metric \hat{g} in C^3 topology (or even in $C^{2,\beta}$ topology). Hence, we could expect the uniqueness result in Theorem 1.9 holds in the C^3 topology (or even in $C^{2,\beta}$ topology) (thanks to [29, Theorem 8.29]).*

The rest of this paper is organized as follows: In Section 2, we recall some basic ingredients in the proofs and list some of their key properties. In Section 3, we prove the injectivity radius estimates which is the major technical step in the blow-up analysis in Riemannian geometry. In Section 4, we establish various compactness results for Fefferman-Graham's compactifications and prove Theorems 1.1, 1.2 and 1.3. Finally, in Section 5, we prove Theorem 1.9 to obtain the global uniqueness for the conformally compact Einstein metrics on \mathbb{B}^4 constructed earlier in [32, 37].

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2. PRELIMINARIES

2.1. Fefferman-Graham's compactifications. Suppose that X is a smooth 4-manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X . Let $g^* = \rho^2 g^+$ be the Fefferman-Graham's compactification, that is, $w := \log \rho$ satisfies the equation (1.2). The function ρ was first introduced in [27] then in [20] to study the renormalized volume. We recall some basic calculations for curvatures under conformal changes. Write $g^+ = r^{-2}g$ for some defining function r and calculate

$$\text{Ric}[g^+] = \text{Ric}[g] + 2r^{-1}\nabla^2 r + (r^{-1}\Delta r - 3r^{-2}|\nabla r|^2)g.$$

Then one has

$$R[g^+] = r^2(R[g] + 6r^{-1}\Delta r - 12r^{-2}|\nabla r|^2).$$

Here the covariant derivatives is calculated with respect to the metric g (or Fefferman-Graham's compactification g^* in the following). Therefore, for a Fefferman-Graham's compactification g^* of a conformally compact Einstein metric g^+ , one has

$$(2.1) \quad R[g^*] = 6\rho^{-2}(1 - |\nabla \rho|^2),$$

which in turn gives

$$(2.2) \quad \text{Ric}[g^*] = -2\rho^{-1}\nabla^2 \rho$$

and

$$(2.3) \quad R[g^*] = -2\rho^{-1}\Delta \rho.$$

Now we recall

Lemma 2.1. ([15], [16, Lemma 4.2]) *Suppose that X is a smooth 4-manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X with the conformal infinity $(\partial X, [\hat{g}])$ of nonnegative Yamabe type. Let $g^* = \rho^2 g^+$ be the Fefferman-Graham's compactification associated with the Yamabe metric \hat{g} of conformal infinity. Then the scalar curvature $R[g^*]$ is positive in X . In particular,*

$$(2.4) \quad \|\nabla \rho\| [g^*] \leq 1.$$

2.2. Elliptic estimates for Bach-flat and Q-flat metrics. Next we recall from [16] the ε -estimates for Fefferman-Graham's compactifications g^* of conformally compact Einstein metrics g^+ . We will continue to use the 2-tensor S when deriving estimates for Fefferman-Graham's compactifications, which are Bach-flat and Q -flat metrics. We first recall Bach equations in 4 dimensions for Bach-flat metrics:

$$(2.5) \quad \Delta A_{ij} - \frac{1}{6}R_{,ij} + R_{ikjl}A^{kl} - R_{ik}A^k{}_j + \frac{1}{2}W_{ikjl}A^{kl} = 0,$$

where

$$A_{ij} = \frac{1}{2}(R_{ij} - \frac{1}{6}Rg_{ij})$$

is the Schouten tensor, R_{ikjl} and W_{ikjl} are Riemann and Weyl curvature tensors respectively, and Q -flat equation:

$$(2.6) \quad \Delta R = 3|\text{Ric}|^2 - R^2$$

the light of (1.3). One may use Bach equations coupled with Q -flat equation to derive estimates for the Schouten tensor. To see Bach equations coupled with Q -flat equation also provide estimates of Weyl curvature, one may rewrite Bach equation as follows:

$$(2.7) \quad \Delta W_{ijkl} + \nabla_l C_{kji} + \nabla_k C_{lij} + \nabla_i C_{jkl} + \nabla_j C_{ilk} := K_{ijkl},$$

where $C_{ijk} = A_{ij,k} - A_{ik,j}$ is the Cotton tensor and $K = W * Rm + g^* * W * A$ is a quadratic of curvatures (cf. [16, (2.5)]). Finally, to get estimates for the full Riemann curvature tensor Rm , one recalls that

$$Rm = W + g^* \otimes A.$$

The most important analytic tools for elliptic estimates here are Sobolev inequalities. The conditions (4) and (5) in Section 1 (cf. [16, Theorem 1.1]) essentially provide the following *Sobolev inequality* and *trace Sobolev inequality* respectively:

$$(2.8) \quad \left(\int_X |u|^6 d\text{vol}[g^*] \right)^{\frac{1}{3}} \leq C_s \int_X (|\nabla u|^2) d\text{vol}[g^*]$$

and

$$(2.9) \quad \left(\oint_{\partial X} |u|^4 d\text{vol}[\hat{g}] \right)^{\frac{1}{2}} \leq C_b \int_X (|\nabla u|^2) d\text{vol}[g^*]$$

for all $u \in C_0^1(B(p, r_0))$, where p is any point in \bar{X} and $r_0 > 0$ is fixed. Moreover, a global *trace Sobolev inequality* holds

$$(2.10) \quad \left(\oint_{\partial X} |u|^4 d\text{vol}[\hat{g}] \right)^{\frac{1}{2}} \leq C'_b \int_X (|\nabla u|^2 + |u|^2) d\text{vol}[g^*]$$

for all $u \in C_0^1(X)$.

Lemma 2.2. ([16, Theorem 3.4]) *Suppose that X is a smooth 4-manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X with the conformal infinity of positive Yamabe type. Let $g^* = \rho^2 g^+$ be the Fefferman-Graham's compactification associated with the Yamabe metric of the conformal infinity. Assume the Sobolev inequality (2.8) holds for the Fefferman-Graham's compactification g^* . Then there exists constants $\varepsilon > 0$ and $C_k > 0$ such that if*

$$\|Rm\|_{L^2(B(p,r))} \leq \varepsilon$$

for a geodesic ball $B(p, r) \subset X$, then, for each $k = 0, 1, 2, \dots$,

$$(2.11) \quad \sup_{B(p,r/2)} |\nabla^k Rm| \leq \frac{C_k}{r^{k+2}} \left(\int_{B(p,r)} |Rm|^2 d\text{vol}[g^*] \right)^{\frac{1}{2}}.$$

Lemma 2.3. ([16, Theorem 3.1 and Theorem 3.2]) *Suppose that X is a smooth 4-manifold with boundary ∂X and g^+ is a conformally compact Einstein metric on X with the conformal infinity of positive Yamabe type. Let $g^* = \rho^2 g^+$ be the Fefferman-Graham's compactification associated with the Yamabe metric of the conformal infinity. Assume the Sobolev inequalities (2.8) and (2.9) hold for the Fefferman-Graham's compactification g^* . Then there exists constants $\varepsilon > 0$ and $C_k > 0$ such that if*

$$\|Rm\|_{L^2(B(p,r))} \leq \varepsilon$$

for a geodesic ball $B(p, r) \subset \bar{X}$, then, for each $k = 0, 1, 2, \dots$,

$$(2.12) \quad \sup_{B(p, r/2)} |\nabla^k Rm| \leq \frac{C_k}{r^{k+2}} \left(\int_{B(p, r)} |Rm|^2 d\text{vol}[g^*] + \oint_{B(p, r) \cap \partial X} |S| d\text{vol}[\hat{g}] + \text{vol}(B(p, r)) \right)^{\frac{1}{2}}.$$

2.3. Sobolev inequalities. In the above estimates, Sobolev inequalities are essential. We need to control the constants C_s and C_b in Sobolev inequalities (2.8) and (2.9) in terms of other geometric quantities in Riemannian geometry. For the convenience of readers, we recall the following notions of injectivity radii for a Riemannian manifold (X, g) with boundary ∂X . For any interior point $p \in X$, let $i_{\text{int}}(p, g)$ be the supremum of r such that the normal geodesic $\gamma(t)$ from p is minimizing for any $t \in [0, \min\{r, t_\gamma\}]$, where t_γ is the first intersection of γ with the boundary ∂X . Then the interior injectivity radius is defined by

$$i_{\text{int}}(X, g) = \inf\{i_{\text{int}}(p, g) : p \in X\}.$$

For $p \in \partial X$, let $i_\partial(p, g)$ be the supremum of r such that the normal geodesic γ from p in the inward unit normal direction ν_p is minimizing for any $t \in [0, r]$. Then the boundary injectivity radius is defined by

$$i_\partial(X, g) = \inf\{i_\partial(p, g) : p \in \partial X\}.$$

The other equivalent definition for the boundary injectivity radius is that $i_\partial(X, g)$ is the supremum of the height h of the Fermi coordinates from the boundary ∂X in X :

$$\exp_p(s\nu_p) : \partial X \times [0, h] \rightarrow X$$

for $p \in \partial X$ and $s \in [0, h]$ (cf. [35] [22, Section 3.6]).

Lemma 2.4. *Let (X^4, g) be a complete Riemannian 4-manifolds with totally geodesic boundary. Suppose that $|Rm| \leq k$ and that*

$$i_{\text{int}}(X, g) \geq i_0, \quad i_\partial(X, g) \geq i_0, \quad \text{and} \quad i(\partial X) \geq i_0.$$

for a positive constant i_0 , where $i(\partial X)$ is the intrinsic injectivity radius of the boundary. Then the Sobolev inequalities (2.8) and (2.9) (resp. (2.10)) hold for uniform constants C_s and C_b (resp. C'_b).

Proof. We consider the doubling $\tilde{X} = X \cup_{\partial X} X$: the union of two copies of X along the boundary ∂X where the second X is the reflexion of X . It is easy to see that $i(\tilde{X}, \tilde{g}) \geq i_0$. Then, (2.8) (local and global) simply follows from [14, Theorem 1] (see also [7] and other related results [33, Theorem 3.14 and Lemma 3.17]).

For the trace Sobolev inequality (2.9), one may first use [35, Theorem A] to find uniform Lipschitz boundary local coordinate system in which the trace Sobolev inequality (2.9) is valid with uniform constant C_b at least for the local version.

To prove that (2.10) holds globally, we work with a partition of unity associated with a countable coordinate chart covering $\{B(x_i, \delta/2)\}$, where (x_i) be a sequence of points in \bar{X} , such that

$$\bar{X} = \cup_i B(x_i, \delta/2) \text{ and } B(x_i, \delta/4) \cap B(x_j, \delta/4) = \emptyset \text{ if } i \neq j.$$

Then there exists $N = N(n, k, i_0)$, depending on n, k, v , such that each point of \tilde{X} has a neighborhood which intersects at most N of the balls $B(x_i, \delta)$'s. This comes from Gromov-Bishop volume comparison theorem. Meanwhile, if let K be the total number of $B(x_i, \delta/2)$ that intersects with $B(p, r_0) \cap \partial X$, then K depends only of r_0 and δ .

Let ξ be some non-negative cut-off function such that $\xi(t) = 1$ on $[0, \delta/2]$ and $\xi(t) = 0$ on $[3\delta/4, +\infty)$, and it satisfies $|\xi'| \leq C/\delta$ on $[0, +\infty)$. Let $\alpha_i(x) = \xi(d(x, x_i))$ and $\eta_i = \alpha_i / \sum_m \alpha_m$. Let $u \in C^1(X)$. We can estimate

$$\begin{aligned} \left(\int_{\partial X} |u|^4 d\text{vol}[g|_{\partial X}] \right)^{\frac{1}{2}} &\leq \left(\sum_i \left(\int_{\partial X} |\eta_i u|^4 d\text{vol}[g|_{\partial X}] \right)^{\frac{1}{4}} \right)^2 \\ &\leq \left(\sum_i \left(\int_X |\nabla(\eta_i u)|^2 d\text{vol}[g] \right)^{\frac{1}{2}} \right)^2 \\ &\leq CKN \int_X (|\nabla u|^2 + u^2) d\text{vol}[g] \end{aligned}$$

Thus the proof is complete. \square

Remark 2.5. *In the recent paper [25], the authors have established the remarkable inequality that*

$$Y_b(X, \partial X, [g^*])^2 \geq 6Y(\partial X, [\hat{g}])$$

for any conformally compact Einstein manifold (X, g^+) with its conformal infinity of positive Yamabe type. We remark that as a direct consequence one can drop the assumption (5) in the statements of the main theorems in [16]. In other words, the global trace-Sobolev inequality (5) (therefore (2.10)) is always available for any conformally compact Einstein manifold (X, g^+) with its conformal infinity of positive Yamabe type. Thus the effort in the current paper is to drop the assumption (4), although the same procedure also works to drop assumption (5) in the statements of both Theorem 1.1 and Theorem 1.2.

2.4. Cheeger-Gromov convergences for manifolds with boundary. Our approach to establish the compactness of conformally compact Einstein 4-manifolds is to prove by contradiction. We will analyze and eliminate the causes of possible non-compactness by the method of blow-up. This method has been essential and powerful in many compactness problems in geometric analysis, particularly in Riemannian geometry. The fundamental tool in the context of Riemannian geometry is the so-called Cheeger-Gromov convergences of Riemannian manifolds developed from Gromov-Hausdorff convergences (see, for example, [23, 2], for Cheeger-Gromov convergences of Riemannian manifolds without boundary). In this subsection, for later uses in our paper, we will present the Cheeger-Gromov convergences for manifolds with boundary. Good references in the subject are for examples in [43, 35, 36, 48, 6].

Let us first recall the definition of harmonic radius for a Riemannian manifold with boundary (cf. [43]). Assume (X, g) is a complete Riemannian 4-manifold with the boundary ∂X . A local coordinates

$$(x_0, x_1, x_2, x_3) : B(p, r) \rightarrow \Omega \subset \mathbb{R}^4$$

is said to be harmonic if,

- $\Delta x_i = 0$ for all $0 \leq i \leq 3$ in $B(p, r) \subset X$, when $p \in X$ is in the interior;
- $\Delta x_i = 0$ for all $0 \leq i \leq 3$ in $B(p, r) \cap X$ and, on the boundary $B(p, r) \cap \partial X$, (x_1, x_2, x_3) is a harmonic coordinate in ∂X at p while $x_0 = 0$, when $p \in \partial X$ is on the boundary.

For $\alpha \in (0, 1)$ and $M \in (1, 2)$, we define the harmonic radius $r^{1,\alpha}(M)$ to be the biggest number r satisfying the following properties:

- If $\text{dist}(p, \partial X) > r$, there is a harmonic coordinate chart on $B(p, r)$ such that

$$(2.13) \quad M^{-2}\delta_{jk} \leq g_{jk}(x) \leq M^2\delta_{jk}$$

and

$$(2.14) \quad r^{1+\alpha} \sup |x - y|^{-\alpha} |\partial g_{jk}(x) - \partial g_{jk}(y)| \leq M - 1$$

in $\overline{B(p, \frac{r}{2})}$.

- If $p \in \partial X$, there is a boundary harmonic coordinate chart on $B(p, 4r)$ such that (2.13) and (2.14) hold in $\overline{B(p, 2r)}$.

The following is the extension of the $C^{1,\alpha}$ convergence theorem of Anderson [23, 2] to manifolds with boundary (cf. [35, 6]).

Lemma 2.6. ([6, Theorem 3.1]) *Suppose that $\mathcal{M}(R_0, i_0, h_0, d_0)$ is the set of all compact Riemannian manifolds (X, g) with boundary such that*

$$\begin{aligned} |Ric_X| &\leq R_0, & |Ric_{\partial X}| &\leq R_0 \\ i_{int}(X) &\geq i_0, & i_{\partial}(X) &\geq 2i_0, & i(\partial X) &\geq i_0, \\ Diam(X) &\leq d_0, & \|H\|_{Lip(\partial X)} &\leq h_0, \end{aligned}$$

where Ric_X (resp. $Ric_{\partial X}$) is the Ricci curvature of X (resp. the boundary), $i(\partial X)$ is the injectivity radius of the boundary, and H is the mean curvature of the boundary. Then $\mathcal{M}(R_0, i_0, h_0, d_0)$ is pre-compact in the $C^{1,\alpha}$ Cheeger-Gromov topology for any $\alpha \in (0, 1)$.

Remark 2.7. *We remark*

- First, in [6], it is showed that the harmonic radius $r^{1,\alpha}(M)$ is uniformly bounded from below in $\mathcal{M}(R_0, i_0, h_0, d_0)$ (cf. [6, Theorem 3.2.1]).
- Second, it is easy to see that, after having harmonic coordinate charts with the uniform size, one has the pre-compactness in $C^{k+2,\alpha}$ Cheeger-Gromov topology if the Ricci curvatures are bounded in $C^{k,\alpha}$ norm and the boundaries are all totally geodesic (even for $k = 0$), which is the convergence theorem that is useful to us later (see [16]).
- Third, one may have the pre-compactness in the Cheeger-Gromov topology with base points if dropping the assumption on the diameter $Diam(X)$.

3. INJECTIVITY RADII: BLOW-UP BEFORE BLOW-UP

Our main results in this section concern the injectivity radius estimates for manifolds with boundary. For our purpose we may always assume that the geometry of the boundary

is compact in Cheeger-Gromov sense. The following is an easy consequence from [6, Theorem 3.1], which was mentioned as Lemma 2.6 in the previous Section 2.

Lemma 3.1. *Suppose that (X^4, g^+) is a conformally compact Einstein 4-manifold with the conformal infinity of Yamabe constant $Y(\partial X, [\hat{g}]) \geq Y_0 > 0$. And suppose that (X^4, g^*) is the Fefferman-Graham's compactification associated with the Yamabe metric \hat{g} on the boundary such that the intrinsic injectivity radius $i(\partial X, \hat{g}) \geq i_0 > 0$, and that $i_\partial(X, g^*) \leq i_{int}(X, g^*)$. Then there is a constant $C_\partial > 0$, depending of i_0 , such that*

$$(3.1) \quad \max_X |Rm|(i_\partial(X, g^*))^2 + i_\partial(X, g^*) \geq C_\partial$$

where Rm is Riemann curvature of g^* .

Proof. We show by contradiction. Suppose otherwise there are a sequence of Fefferman-Graham's compactified Riemannian manifolds (X_i, g_i^*) such that

$$\max_{X_i} |Rm[g_i^*]|(i_\partial(X_i, g_i^*))^2 + i_\partial(X_i, g_i^*) \rightarrow 0$$

and $i(\partial X_i) \geq i_0$ for some fixed positive number i_0 . We then rescale the metrics as follows:

$$\bar{g}_i = K_i^{-2} g_i^*$$

where $K_i = i_\partial(p_i) = i_\partial(X_i, g_i^*)$ for some $p_i \in \partial X_i$. Here we use the fact that the boundary injectivity radius $i_\partial(\cdot)$ is a continuous function on the boundary since the limit of minimizing geodesics is still minimizing geodesic. Recall the boundary of g_i^* is totally geodesic. On the other hand, because the curvature

$$\max_{X_i} |Rm[\bar{g}_i]| = \max_{X_i} |Rm[g_i^*]|(i_\partial(X_i, g_i^*))^2 \rightarrow 0,$$

by [35, Lemma 6.3], there is a normal geodesic γ of length 2 such that γ is orthogonal to boundary ∂X_i at $\gamma(0) = p_i$ and $\gamma(2) = p'_i$.

In light of Lemma 2.6, we may extract a subsequence (we will always use the same index for subsequences for convenience in this paper) (X_i, \bar{g}_i, p_i) that converges to $(X_\infty^4, g_\infty, p_\infty)$ in $C^{1,\alpha}$ Cheeger-Gromov topology. From the assumptions, it is easily seen that $(X_\infty^4, g_\infty, p_\infty)$ is a complete flat metric manifold with the totally geodesic complete flat boundary (it is smooth in harmonic coordinates as demonstrated in [2, 6]). First, each connected component of the boundary $(\partial X_\infty, \hat{g}_\infty)$ is the Euclidean space \mathbb{R}^3 because of $i(\partial X_i) \geq i_0$. Second, by the Cheeger and Gromoll's splitting theorem, the complete metric g_∞ is a product metric on $\mathbb{R}^3 \times (0, \infty)$ or $\mathbb{R}^3 \times (0, 2)$. We claim the later case does not appear, that is, $X_\infty = \mathbb{R}^3 \times (0, \infty)$. Let $\bar{\rho}_i$ be the Fefferman-Graham's defining function related to the metric \bar{g}_i . As in the proof of [16, Lemma 4.3], we know, up to a subsequence, $\bar{\rho}_i \rightarrow \rho_\infty$ and the hessian of ρ_∞ vanishes since g_∞ is flat. Moreover, $\rho_\infty(x) \geq C \text{dist}_{g_\infty}(x, \partial X_\infty)$ for some positive constant $C > 0$. In view of the boundary condition, ρ_∞ vanishes on $\mathbb{R}^3 \times \{0\}$. Hence, $\rho_\infty(x_0, x_1, x_2, x_3) = \alpha x_0$ with some positive constant $\alpha > 0$ since $\nabla^2 \rho_\infty = 0$. When ∂X_∞ has more than 2 connected components, this contradicts the fact ρ_∞ vanishes on ∂X_∞ . Therefore, we prove the claim.

Now, on the other hand, there is a geodesic of length 2 in (X_∞, g_∞) which are orthogonal to the boundary ∂X_∞ . This is a contradiction. Therefore, we have established the Lemma. \square

Next we would like to get the lower bound estimates for the interior injectivity radius i_{int} of a compact Riemannian manifold with boundary. The real reason for having no interior collapsing follows from the following recent work in [38].

Lemma 3.2. (*Li-Qing-Shi* [38, Theorem 1.3]) *Suppose that (X^4, g^+) is a conformally compact Einstein manifold with the conformal infinity of Yamabe constant $Y(\partial X, [\hat{g}]) > 0$. Then, for any $p \in X^4$,*

$$(3.2) \quad \frac{\text{vol}_{g^+}(B(p, r))}{\text{vol}_{g_{\mathbb{H}^4}}(B(r))} \geq \left(\frac{Y(\partial X, [\hat{g}])}{Y(\mathbb{S}^3, [g_{\mathbb{S}}])} \right)^{\frac{3}{2}}$$

As a consequence, we have

Lemma 3.3. *Suppose that (X^4, g^+) is a conformally compact Einstein 4-manifold with the conformal infinity of Yamabe constant $Y(\partial X, [\hat{g}]) \geq Y_0 > 0$. And suppose that (X^4, g^*) is the Fefferman-Graham's compactification associated with the Yamabe metric \hat{g} on the boundary such that the intrinsic injectivity radius $i(\partial X, \hat{g}) \geq i_0 > 0$, and that $i_{\partial}(X, g^*) \geq i_{\text{int}}(X, g^*)$. Then there is a constant $C_{\text{int}} > 0$, depending of Y_0 and i_0 , such that*

$$(3.3) \quad \max_X |Rm|(i_{\text{int}}(X, g^*))^2 + i_{\text{int}}(X, g^*) \geq C_{\text{int}}$$

where Rm is the Riemann curvature of g^* .

Proof. Again, we will prove this lemma by contradiction. Assume otherwise, then there is a sequence of conformally compact Einstein 4-manifolds (X_i^4, g_i^+) with the conformal infinity of Yamabe constants $Y(\partial X_i, [\hat{g}_i]) \geq Y_0 > 0$, whose Fefferman-Graham's compactifications (X_i^4, g_i^*) associated with the Yamabe metrics \hat{g}_i on the boundary are compact 4-manifolds with totally geodesic boundary and satisfy

$$\max_{X_i} |Rm[g_i]|(i_{\text{int}}(X_i, g_i^*))^2 + i_{\text{int}}(X_i, g_i^*) \rightarrow 0$$

and

$$i(\partial X_i, \hat{g}_i) \geq i_0.$$

Let us consider the rescaling

$$\bar{g}_i = K_i^{-2} g_i^*$$

for $K_i = i_{\text{int}}(X_i, g_i^*)$. Using [35, Lemma 6.4] to the almost flat metrics \bar{g}_i , one sees that $1 = i(X_i, \bar{g}_i) = i_{\text{int}}(p_i, \bar{g}_i)$ for some $p_i \in X_i$ in the interior. Now, if $K_i^{-1} \text{dist}_{g_i^*}(p_i, \partial X_i) < \infty$, we are in the same situation as in the proof of Lemma 3.1 and derive the contradiction by [35, Lemma 6.4] (there would be a closed geodesic of length 2 in the Euclidean half space).

Therefore we may assume that $K_i^{-1} \text{dist}_{g_i^*}(p_i, \partial X_i) = \text{dist}_{\bar{g}_i}(p_i, \partial X_i) \rightarrow \infty$. Thus the limit space $(X_{\infty}^4, g_{\infty}, p_{\infty})$ is a complete flat manifold with no boundary, but, with a simple closed geodesic of length 2. We claim that $(X_{\infty}^4, g_{\infty}, p_{\infty})$ is of Euclidean volume growth in dimensions 4. This would be a contradiction, since such flat manifold would be a product of a circle and a flat manifold of dimension 3, which would not be able to support the Euclidean volume growth in dimensions 4.

To finish the proof of the claim that $(X_\infty^4, g_\infty, p_\infty)$ is of Euclidean volume growth in dimensions 4, that is,

$$(3.4) \quad \text{vol}[g_\infty](B^{g_\infty}(p_\infty, r)) \geq c_v r^4$$

for some fixed c_v and any $r > 0$. First let us prove the following claim.

Claim: $\bar{\rho}_i(p_i) \rightarrow \infty$, where $\bar{\rho}_i = K_i^{-1} \rho_i$ and $K_i = i_{\text{int}}(X_i, g_i^*)$.

Proof. Assume otherwise that there is a constant $\bar{\rho}_0 > 0$ such that $\bar{\rho}_i(p_i) \leq \bar{\rho}_0$ for all i . Due to (2.1) at the beginning of the Section 2.1, we have

$$1 - |\nabla[\bar{g}_i]\bar{\rho}_i|^2 = \frac{1}{6}R[\bar{g}_i]\bar{\rho}_i^2 \geq 0,$$

where the covariant derivatives is calculated with respect to the background metric \bar{g}_i . Let us denote

$$\epsilon_i = \max\{|Rm[\bar{g}_i]|, (\text{dist}_{\bar{g}_i}(p_i, \partial X_i))^{-1}\} \rightarrow 0.$$

Then we obtain

$$\bar{\rho}_i(x) \leq \frac{1}{2}\epsilon_i^{-\frac{1}{2}} + \bar{\rho}_i(p_i)$$

for all $x \in B^{\bar{g}_i}(p_i, \frac{1}{2}\epsilon_i^{-1}) \subset X_i$ since $|\nabla[\bar{g}_i]\bar{\rho}_i| \leq 1$. This in turn implies

$$1 \geq |\nabla[\bar{g}_i]\bar{\rho}_i| \geq \frac{1}{2}$$

for all $x \in B^{\bar{g}_i}(p_i, \frac{1}{2}\epsilon_i^{-1}) \subset X_i$, at least for i sufficiently large. Therefore, along the integral curve $\gamma(t)$ of the gradient $\nabla[\bar{g}_i]\bar{\rho}_i$ from p_i , we may derive

$$\bar{\rho}_i(\gamma(t)) \leq \bar{\rho}_0 - \frac{t}{2} < 0$$

when $t > 2\bar{\rho}_0$, which is a contradiction since $\gamma(t) \in X$ for any $t \in (2\bar{\rho}_0, \frac{1}{2}\epsilon_i^{-1})$. So the proof of this claim is complete. \square

Now let

$$s_i = \min\{\text{dist}_{\bar{g}_i}(p_i, \partial X_i), \bar{\rho}_i(p_i)\} \rightarrow \infty.$$

We find, for each $x \in B^{\bar{g}_i}(p_i, \frac{s_i}{2})$,

$$|\bar{\rho}_i(x) - \bar{\rho}_i(p_i)| \leq \frac{1}{2}\bar{\rho}_i(p_i),$$

which implies,

$$\frac{1}{2} \leq \frac{\bar{\rho}_i(x)}{\bar{\rho}_i(p_i)} \leq \frac{3}{2}.$$

Notice that $\bar{g}_i = \bar{\rho}_i^2 g_i^+$. Now, applying Lemma 3.2 (cf. [38, Theorem 1.3]), we deduce, for $r < \frac{s_i}{2}$,

$$\begin{aligned} \text{vol}[\bar{g}_i](B^{\bar{g}_i}(p_i, r)) &\geq \text{vol}[\bar{g}_i](B^{g_i^+}(p_i, \frac{2}{3}(\bar{\rho}_i(p_i))^{-1}r)) \\ &\geq \frac{1}{(2(\bar{\rho}_i(p_i))^{-1})^4} \text{vol}[g_i^+](B^{g_i^+}(p_i, \frac{2}{3}(\bar{\rho}_i(p_i))^{-1}r)) \\ &\geq C \frac{1}{(2(\bar{\rho}_i(p_i))^{-1})^4} \text{vol}[g_{\mathbb{H}^4}](B^{g_{\mathbb{H}^4}}(\frac{2}{3}(\bar{\rho}_i(p_i))^{-1}r)) \\ &\geq c_v r^4 \end{aligned}$$

for a fixed constant c_v that is independent of i . Passing to the limit as $i \rightarrow \infty$, we get the desired inequality (3.4) on the limit space $(X_\infty^4, g_\infty, p_\infty)$. So the proof is complete. \square

4. ON COMPACTNESS OF FEFFERMAN-GRAHAM'S COMPACTIFICATIONS

Based on the preparation in the previous sections we are ready to establish the compactness of Fefferman-Graham's compactifications on conformally compact Einstein 4-manifolds which were stated in the introduction. The approach follows closely from the corresponding results in [16]. The difference from [16] is that, in light of the injectivity radius estimates in the previous section, Sobolev inequality and trace Sobolev inequality are all available for the rescaled metrics with bounded curvature, while Sobolev inequality and trace Sobolev inequality are parts of the assumptions in the main compactness theorem in [16]. Readers are referred to [16] for more details. To begin the proof, we will first establish some bounded curvature estimates.

Lemma 4.1. *Suppose that $\{(X_i^4, g_i^+)\}$ is a sequence of conformally compact Einstein 4-manifolds satisfying the assumptions (1), (2), and (3) in Theorem 1.1. Then there is a positive constant K_0 such that, for the Fefferman-Graham's compactifications $\{(X_i^4, g_i^*)\}$ associated with the Yamabe metric \hat{g}_i of the conformal infinity $(\partial X_i, [\hat{g}_i])$*

$$(4.1) \quad \max_{X_i} |Rm[g_i^*]| \leq K_0$$

for all i .

Proof. Suppose otherwise that there is a subsequence $\{(X_i^4, g_i^+)\}$ satisfying

$$K_i = \max_{X_i} |Rm[g_i^*]| \rightarrow \infty.$$

Let

$$K_i = K_i(p_i) = |Rm[g_i^*]|(p_i)$$

for some $p_i \in \bar{X}_i$. Then we consider the rescaling

$$(X_i^4, \bar{g}_i = K_i g_i^*, p_i).$$

We claim for the metrics \bar{g}_i the derivatives of curvature $\|\nabla Rm[\bar{g}_i^*]\|$ is uniformly bounded. For this purpose, we recall the curvature tensor satisfies some elliptic PDE with the

Dirichlet boundary conditions [16, section 4]

$$(4.2) \quad \begin{cases} \Delta R = R^2 - 3|Ric|^2 & \text{in } X \\ R = 3\hat{R} & \text{on } \partial X \\ \Delta A - \frac{1}{6}\nabla^2 R = Rm * A & \text{in } X \\ A_{\alpha\beta} = \hat{A}_{\alpha\beta}, A_{\alpha n} = 0, A_{\alpha n}, A_{nn} = \frac{\hat{R}}{4} & \text{on } \partial X \\ \Delta W = Rm * W + g * W * A + L(Ric) & \text{in } X \\ W = 0 & \text{on } \partial X, \end{cases}$$

where L is a linear differential operator of second order. In view of [29, Theorem 8.33] we have the boundedness of the curvature tensor in $C^{1,\alpha}$ topology since the curvature tensor on the boundary $\hat{R}m$ is bounded in $C^{1,\alpha}$ topology.

No boundary Blow-up: Let us first consider the cases where

$$\text{dist}_{\bar{g}_i}(p_i, \partial X_i) < \infty.$$

For the pointed manifolds (X_i, \bar{g}_i, p_i) with boundary, in the light of all the preparations in the previous sections, particularly Lemma 2.6 and Remark 2.7, we have Cheeger-Gromov convergence

$$(X_i^4, \bar{g}_i, p_i) \rightarrow (X_\infty^4, g_\infty, p_\infty)$$

in $C^{k,\alpha}$ Cheeger-Gromov topology (for a subsequence if necessary), where the limit space is a complete Bach-flat and Q -flat manifold with a totally geodesic boundary ∂X_∞ ; the boundary $(\partial X_\infty, \hat{g}_\infty)$ is simply the Euclidean space \mathbb{R}^3 because $i(\partial X_i) \geq i_0 > 0$; and

$$|Rm[g_\infty]|(p_\infty) = 1.$$

To derive the a priori estimates for Cheeger-Gromov convergence, one applies the ϵ -estimates in Lemma 2.2 and Lemma 2.3, where Sobolev inequality (2.8) and trace Sobolev inequality (2.9) are established in Lemma 2.4. The injectivity radii estimates that are needed for \bar{g}_i to satisfy Sobolev and trace Sobolev are given in Lemma 3.1 and Lemma 3.3.

Now, clearly, to finish the proof is to show that the limit space $(X_\infty^4, g_\infty, p_\infty)$ is the Euclidean half space. For the convenience of readers, we very briefly sketch the proof from [16]. One first needs to show that $\bar{\rho}_i \rightarrow \rho_\infty$ where ρ_∞ satisfies

- $g_\infty^+ = \rho_\infty^{-2}g_\infty$ is a (partially) conformally compact Einstein metric on X_∞^4 whose conformal infinity is the Euclidean space \mathbb{R}^3 ;
- $-\Delta_{g_\infty^+} \log \rho_\infty = 3$.

Then, by Condition (2) in Theorem 1.1, one shows that g_∞^+ is locally hyperbolic space metric nearby the infinity $\partial X_\infty^4 = \mathbb{R}^3$ based on the unique continuation theorem in [9, 11]. Finally one concludes that $\rho_\infty = x_0$, since (X_∞^4, g_∞^+) is hyperbolic space in half space model $(\mathbb{R}_+^4, \frac{|dx|^2}{x_0^2})$ for $\mathbb{R}_+^4 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 > 0\}$, which implies that (X_∞^4, g_∞) is the Euclidean half space (please see the details in the proof of [16, Proposition 4.8]).

No interior blow-up: Next we consider the rest cases when

$$\text{dist}_{\bar{g}_i}(p_i, \partial X_i) \rightarrow \infty$$

(at least for some subsequence). Notice that,

$$K_i = \max_{X_i} |Rm[g_i^*]| = |Rm[g_i^*]|(p_i)$$

for some $p_i \in X$ in the interior. Proceeding as the above boundary cases, one has the Cheeger-Gromov convergence

$$(X_i^4, \bar{g}_i, p_i) \rightarrow (X_\infty^4, g_\infty, p_\infty)$$

in $C^{k,\alpha}$ Cheeger-Gromov topology. The proof in these cases follows from [16]. We again very briefly sketch the proof that is more or less from [16]. One first derives from (2.1) that

$$R[\bar{g}_i] = \bar{\rho}_i^{-2}(1 - |\nabla[\bar{g}_i]\bar{\rho}_i|^2)$$

and shows that

- $\bar{\rho}_i(x) \geq C \text{dist}_{\bar{g}_i}(x, \partial X_i)$. (cf. Step 2 in the proof of [16, Lemma 4.9]).

Then, consequently,

- $R_\infty = 0$, and
- g_∞ is Ricci-flat from being Q -flat and scalar flat in the light of the Q -curvature equation (2.6). (cf. Step 3 of the proof of [16, Lemma 4.9]).

Thus, (X_∞, g_∞) is a complete Ricci-flat 4-manifold with no boundary. At this point, as argued in [16], first, due to the recent work in [24], one concludes that (X_∞, g_∞) is a complete ALE Ricci flat 4-manifold. By the assumptions, the doubling of X is a homological sphere. By a topological result due to Crisp-Hillman ([26, Theorem 2.2]), (X_∞, g_∞) at the infinity is asymptotic to \mathbb{S}^3/Γ with $\Gamma = \{1\}$ or $\Gamma = Q_8$ (quaternion group) or Γ the perfect group (that is, \mathbb{S}^3/Γ is a homology 3-sphere). By the Chern-Gauss-Bonnet formula and the signature formula, we obtain the desired contradiction. For more details see [16] section 4.3. \square

We now begin the proof of Theorem 1.1.

With the curvature bound (4.1), the injectivity radius estimates in Lemma 3.1 and Lemma 3.3, the ϵ -regularities Lemma 2.2 and Lemma 2.3, one last piece that is needed to apply the Cheeger-Gromov convergences for manifolds with boundary in Lemma 2.6 and Remark 2.7 to finish the proof of Theorem 1.1 is the following diameter bound.

Lemma 4.2. *Under the assumptions in Theorem 1.1, the diameters of the Fefferman-Graham's compactifications g_i^* are uniformly bounded.*

Proof. The proof of this lemma under the assumption that the first Yamabe invariant is uniformly bounded from below is obtained in [16, Section 5: *The proof of Theorem 1.1*]. However, we do not know if one has the suitable Euclidean Sobolev type inequality in actual setting. This makes the problem is more delicate. Here we give a different approach to overcome the difficulty.

We have already proved the family of metrics g_i^* has the bounded curvature in C^1 so that the arguments given in [16, Section 4.4: *The proof of Theorem 4.4*] yields the bound in C^{k+1} norm. In view of Lemmas 3.1 and 3.3, the boundary radius and the interior one are uniformly bounded from below. Therefore, for all i , for all $x \in \bar{X}$, we have

$\text{vol}(B^{g_i^*}(x, 1)) \geq C > 0$ for some constant $C > 0$ independent of i, x , that is, there is non-collapse. We prove the diameter is uniformly bounded from above by contradiction. Suppose that the diameter $\text{diam}(g_i^*)$ tends to the infinity. By Cheeger-Gromov-Hausdorff compactness theory, up to diffeomorphisms fixing the boundary, (X_i, g_i^*) converges to some complete non-compact manifold (X_∞, g_∞) with the boundary. We divide the proof in 5 steps.

Step 1. There exists some $C > 0$ such that $\rho_i \geq C$ provided $d_{g_i^*}(x, \partial X) \geq 1$ and $d_{g_i^*}(x, \partial X) \leq C\rho_i(x)$ provided $0 \leq d_{g_i^*}(x, \partial X) \leq 1$. Thus the limit metric is conformal to an asymptotic hyperbolic Einstein manifold. Moreover, there exists some constant $C_1 > 0$ independent of i such that $\int |Rm[g_i^*]|^2 d\text{vol}[g_i^*] \leq C_1$.

The first part of the claim can be proved in the same way as in [16, Section 4: *the proof of Lemma 4.4*]. The second part is proved in [16, Section 5: *the step 2 of the proof of Theorem 1.1*]. Without loss of generality, assume the boundary injectivity radius is bigger than 1.

Step 2. There exists some constant $C_2 > 0$ independent of i such that

$$(4.3) \quad \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \rho_i^{-2}(x) d\text{vol}[g_i^*] \leq C_2.$$

Thanks of (2.1) and (2.3), we infer

$$-\Delta \log \rho_i = \frac{R_i}{2} + \frac{|\nabla \rho_i|^2}{\rho_i^2} = 3 \frac{(1 - |\nabla \rho_i|^2)}{\rho_i^2} + \frac{|\nabla \rho_i|^2}{\rho_i^2}$$

Integrating on the set $\{x, d_{g_i^*}(x, \partial X) \geq 1\}$, we obtain

$$\int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \left(3 \frac{(1 - |\nabla \rho_i|^2)}{\rho_i^2} + \frac{|\nabla \rho_i|^2}{\rho_i^2} \right) d\text{vol}[g_i^*] = \left| \int_{\{x, d_{g_i^*}(x, \partial X) = 1\}} \frac{1}{\rho_i} \langle \nabla \rho_i, \nu \rangle \right|$$

where ν is the outside normal vector on the boundary $\{x, d_{g_i^*}(x, \partial X) = 1\}$. By Step 1, we know ρ_i is uniformly bounded from below on the set $\{x, d_{g_i^*}(x, \partial X) = 1\}$. Together the facts the curvature of g_i^* is bounded and the boundary $(\partial X_i, \hat{g}_i)$ is compact, we infer for some positive constant $C > 0$

$$\int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} 3 \frac{(1 - |\nabla \rho_i|^2)}{\rho_i^2} d\text{vol}[g_i^*] \leq C, \quad \text{and} \quad \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \frac{|\nabla \rho_i|^2}{\rho_i^2} d\text{vol}[g_i^*] \leq C$$

since $|\nabla \rho_i| \leq 1$. Combining these estimates, the desired claim yields.

Step 3. We have

$$\lim_{x \rightarrow \infty} \rho_\infty(x) = +\infty$$

Letting $i \rightarrow \infty$ in (4.3), we get

$$(4.4) \quad \int_{\{x, d_{g_\infty}(x, \partial X) \geq 1\}} \rho_\infty^{-2}(x) d\text{vol}[g_\infty] \leq \lim_i \int_{\{x, d_{g_i^*}(x, \partial X) \geq 1\}} \rho_i^{-2}(x) d\text{vol}[g_i^*] \leq C_2.$$

For all $\varepsilon > 0$, there exists $A > 0$ such that

$$\int_{\{x, d_{g_\infty}(x, \partial X) \geq A\}} \rho_\infty^{-2}(x) d\text{vol}[g_\infty] \leq \varepsilon$$

Therefore, for any y with $d_{g_\infty}(y, \partial X) \geq A + 1$, we can estimate

$$\int_{B^{g_\infty}(y, 1)} \rho_\infty^{-2}(x) d\text{vol}[g_\infty] \leq \int_{\{x, d_{g_\infty}(x, \partial X) \geq A\}} \rho_\infty^{-2}(x) d\text{vol}[g_\infty] \leq \varepsilon$$

so that

$$\left(\sup_{B^{g_\infty}(y, 1)} \rho_\infty \right)^{-2} \text{Vol}(B^{g_\infty}(y, 1)) \leq \varepsilon$$

that is,

$$\sup_{B^{g_\infty}(y, 1)} \rho_\infty \geq C\varepsilon^{-1/2}$$

Together with Lemma 2.1, we deduce

$$\inf_{B^{g_\infty}(y, 1)} \rho_\infty \geq \sup_{B^{g_\infty}(y, 1)} \rho_\infty - 1 \geq C\varepsilon^{-1/2} - 1$$

Finally, we prove Step 3.

Step 4. We claim that there exists some $c_v > 0$ such that for any $p \in X_\infty$ and for any $r < \frac{1}{2}\rho_\infty(p)$

$$(4.5) \quad \text{Vol}[g_\infty](\text{dist}(B^{g_\infty}(p, r))) \geq c_v r^4$$

Let $p_i \in X_i$ such that $p_i \rightarrow p$. First we remark that $\text{dist}_{g_i^*}(p_i, \partial X_i) \geq \rho_i(p_i)$ because of Lemma 2.1. As in the proof of the end of Section 3, we have

$$\text{Vol}[g_i^*](\text{dist}(B^{g_i^*}(p_i, r))) \geq c_v r^4,$$

where c_v is some positive constant independent of i . Letting $i \rightarrow \infty$, the claim is proved.

Step 5. A contradiction.

On choose $p \in X_\infty$ such that $\rho_\infty(p)$ is sufficiently large. We fix $r = (\rho_\infty(p))^{2/3}$. Using the results in Steps 2 and 4, we get $\rho_\infty(p)$

$$(4.6) \quad \left(\sup_{B^{g_\infty}(p, r)} \rho_\infty \right)^{-2} \text{Vol}[g_\infty](B^{g_\infty}(p, r)) \leq \int_{B^{g_\infty}(p, r)} \rho_\infty^{-2}(x) d\text{vol}[g_\infty] \leq C_2$$

so that for some positive constant $C > 0$ there holds

$$(4.7) \quad \sup_{B^{g_\infty}(p, r)} \rho_\infty \geq Cr^2 = C(\rho_\infty(p))^{4/3}$$

On the other hand, it follows from Lemma 2.1, we deduce

$$(4.8) \quad \inf_{B^{g_\infty}(p, r)} \rho_\infty \geq \sup_{B^{g_\infty}(p, r)} \rho_\infty - r$$

so that

$$\rho_\infty(p) + (\rho_\infty(p))^{2/3} = \rho_\infty(p) + r \geq C(\rho_\infty(p))^{4/3}.$$

This yields that $\rho_\infty(p)$ is bounded. This contradicts the claim in Step 3.

Thus we have finished the proof of Lemma 4.2, hence the proof of Theorem 1.1. \square

The proof of Theorem 1.2 is quite similar to the corresponding theorem in [16] albeit the removing of conditions (4) and (5). We leave the details to the readers. The proof of Theorem 1.3 is different as we have no informations on the S -tensor or T curvature in the statement of the theorem. Here we will present the proof.

Proof of Theorem 1.3. Again we will prove by contradiction. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X^4 and $\{g_i^*\}$ corresponding Fefferman-Graham's compactifications associated with the Yamabe metrics $\hat{g}_i \in \mathcal{C}$, where \mathcal{C} is compact in $C^{k,\alpha}$ Cheeger-Gromov topology as given in (1) in Theorem 1.1. Assume that

$$(2^*) \quad \text{either } Y(\partial X, [\hat{g}_i]) \rightarrow Y(\mathbb{S}^3, [g_S]) \text{ or } \int_X (|W|^2 d\text{vol})[g_i^+] \rightarrow 0,$$

but (X, g_i^*) does not converges in $C^{k,\alpha}$ Cheeger-Gromov topology. If the interior blow-up were to happen, then, it is easily seen that it would be a contradiction with the fact that any possibly limit space is flat due to (2^*) , in light of the rigidity in Gromov-Bishop's volume comparison principle or simply the limit metric is both Ricci flat and locally conformally flat. If the boundary blow-up were to happen, then it is again easily seen that it would be a contradiction with the fact that any possibly limit space would be with g_∞^+ being hyperbolic. Therefore, by the proof of Theorem 1.1, one concludes that (X, g_i^*) converges to the Fefferman-Graham's compactification of hyperbolic space in $C^{k,\alpha}$ Cheeger-Gromov topology for some $\alpha \in (0, 1)$, from which we reached a contradiction. \square

Before ending this section, we turn to an important fact which is a consequence of the compactness result in Theorem 1.1, and in fact is an improved statement of the compactness for conformally compact Einstein metrics with the same conformal infinity. We will later use this fact (Theorem 4.4) to establish the uniqueness result in section 5. To state the result, we will use the notion of weighted spaces of tensors, which we refer the readers to [37] (see also [32]). We first recall the following expansions in terms of the Fermi coordinate from the boundary, which we have stated in the introduction of this paper in terms of geodesic coordinates. The expansion in this form is motivated by an observation in [41].

Lemma 4.3. *Let (X^4, g) be a Bach-flat and Q -flat 4-manifold with the totally geodesic boundary. Then, in the Fermi coordinate from the boundary, one has $g = dr^2 + g_r$ and the expansion*

$$(4.9) \quad g_r = \hat{g} + g^{(2)}r^2 + g^{(3)}r^3 + \dots,$$

where $g^{(2)}$ is a curvature of $\hat{g} = g|_{T\partial X}$ and $g^{(3)}$ is not local.

Theorem 4.4. *Suppose that, for two sequences of conformally compact Einstein metrics g_i^+ and h_i^+ that have the same conformal infinity of positive Yamabe type and satisfy the assumption in Theorem 1.1 (or the assumptions in Theorem 1.3). Then, for a weight $\delta \in (0, 3)$, there are subsequence (possibly after diffeomorphisms ψ_i and ϕ_i that fix the*

boundary) that $\psi_i^* g_i^* - \phi_i^* h_i^*$ converges in weighted $C_\delta^{2,\alpha}$ topology, where g_i^* (resp. h_i^*) denotes Fefferman-Graham's compactification of g_i^+ (resp. h_i^+).

Proof. For each Fefferman-Graham's compactification g_i^* (resp. h_i^*), we first set the Fermi coordinate from the boundary. By the lower bound of the boundary injectivity radius, we know that the heights of these Fermi coordinates are bounded from the below. The necessary diffeomorphisms that fix the boundary one needs to use is to make sure that each of these Fefferman-Graham compactification g_i^* share the same distance function r to the boundary ∂X at least within the focal loci of g_i^* (resp. h_i^*).

Suppose that g_i^* (resp. h_i^*) (a subsequence) converges in $C^{3,\alpha}$ Cheeger-Gromov topology. Now, let us align the distance functions to be the same for all g_i^* (resp. h_i^*) in this subsequence by diffeomorphisms ψ_i (resp. ϕ_i) that fix the boundary, and get

$$\psi_i^* g_i^* - \phi_i^* h_i^* = O(r^3).$$

for any i from Lemma 4.3. If necessary, extract a subsequence, for $\delta \in (0, 3)$ and any $\epsilon > 0$, there is an index N , for $i, j \geq N$,

$$\|(\psi_i^* g_i^* - \phi_i^* h_i^*) - (\psi_j^* g_j^* - \phi_j^* h_j^*)\|_{C^{2,\alpha}(\bar{X}_\infty)} < \frac{1}{2}\epsilon.$$

For any fixed $\delta < 3$, one gets

$$\|\psi_i^* g_i^* - \phi_i^* h_i^*\|_{C^{2,\alpha}} \leq Cr^3 \leq \epsilon r^\delta$$

over the region $\{r \leq r_\epsilon\}$ for some small $r_\epsilon > 0$ such that $Cr_\epsilon^{3-\delta} \leq \epsilon$ (C is independent of i due to the compactness in Theorem 1.1 (resp. Theorem 1.3) and Lemma 4.3, and the sizes of Fermi coordinates for g_i^* has a uniform lower bound again follows from Theorem 1.1 (resp. Theorem 1.3)); while one gets

$$\|(\psi_i^* g_i^* - \phi_i^* h_i^*) - (\psi_j^* g_j^* - \phi_j^* h_j^*)\|_{C^{2,\alpha}(\bar{X}_\infty)} \leq \epsilon r^\delta$$

over the rest $\{r \geq r_\epsilon\}$ by setting N larger, in the light of Theorem 1.1 (resp. Theorem 1.3). It is then easily seen that the corresponding $\psi_i^* g_i^+ - \phi_i^* h_i^+$ converges in the weighted space $C_\delta^{2,\alpha}(X_\infty)$ with $\delta \in (0, 3)$. This completes the proof of Theorem 4.4. \square

5. UNIQUENESS OF GRAHAM-LEE SOLUTIONS IN DIMENSION 4

In this section we derive the global uniqueness result Theorem 1.9 based on some result in the recent work [38].

Proof of Theorem 1.9. We will prove this by contradiction. Assume otherwise there is a sequence of conformal 3-sphere $(\mathbb{S}^3, [\hat{g}_i])$ that converges to the round sphere such that, for each i , there exist two non-isometric conformally compact Einstein metrics g_i^+ and h_i^+ .

We first claim that, for a subsequence, both g_i^+ and h_i^+ converge to the hyperbolic space in $C^{3,\alpha}$ Cheeger-Gromov sense due to Theorem 1.2 and the uniqueness result when the conformal infinity is the standard sphere [45, 38].

Next, according to the proof of Theorem 4.4, we actually can conclude that, after some diffeomorphism that fix the boundary ϕ_i and ψ_i ,

$$\|\psi_i^* g_i^+ - \phi_i^* h_i^+\|_{C_\delta^{2,\alpha}(\mathbb{H}^4)} \rightarrow 0$$

for any $\delta \in (0, 3)$ and some subsequence. In other words, in this subsequence, the two distinct conformally compact Einstein metrics g_i^+ and h_i^+ are arbitrarily close to each other in weighted spaces, as long i is sufficiently large. We will show this is not possible by applying the local uniqueness result via the implicit function theorem on weighted norm space established in [32, 37]. We now give more details.

We denote $\psi_i^* g_i^+$ by g_i^+ and $\phi_i^* h_i^+$ by h_i^+ . We denote Fefferman-Graham's compactification $g_i^* = \rho_i^2 g_i^+$ and $h_i^* = \tilde{\rho}_i^2 h_i^+$ where $\log \rho_i$ and $\log \tilde{\rho}_i$ solve (1.2). Let us consider the nonlinear functional on 4-dimensional ball B^4 introduced by Biquard [10] for two metrics g^+ and t^+ .

$$(5.1) \quad F(g^+, t^+) := Ric[g^+] + 3g^+ - \delta_{g^+}(B_{t^+}(g^+)),$$

where $B_{t^+}(g^+)$ is a linear condition, essentially the infinitesimal version of the previous harmonicity condition

$$B_{t^+}(g^+) := \delta_{t^+} g^+ + \frac{1}{2} d\text{tr}_{t^+}(g^+).$$

We have for any asymptotically hyperbolic Einstein metrics h^+

$$D_1 F(h^+, h^+) = \frac{1}{2}(\Delta_L + 6),$$

where D_1 denotes the differentiation of F with respect to its first variable, and where the Lichnerowicz Laplacian Δ_L on symmetric 2-tensors is given by

$$\Delta_L := \nabla^* \nabla[g^+] + 2\overset{\circ}{Ric}[g^+] - 2\overset{\circ}{Rm}[g^+];$$

where

$$\overset{\circ}{Ric}[g^+](u)_{ij} = \frac{1}{2}(R_{ik}[g^+]u_j^k + R_{jk}[g^+]u_i^k),$$

and

$$\overset{\circ}{Rm}[g^+](u)_{ij} = R_{ikjl}[g^+]u^{kl}.$$

It is clear

$$F(g_i^+, g_i^+) = 0$$

We now divide the proof in 2 steps.

*Step 1. **Claim.** We could find a diffeomorphism φ_i of class $C^{3,\alpha}$ (equal to the identity on the boundary), such that*

$$F(\varphi_i^* h_i^+, g_i^+) = 0$$

Moreover $\|\varphi_i(x) - x\|_{C^{3,\alpha}} \rightarrow 0$ and $\|\varphi_i^ h_i^+ - g_i^+\|_{C_\delta^{2,\alpha}} \rightarrow 0$ for some $\delta \in (2 + \alpha, 3)$.*

It is sufficient to check this infinitesimally: the diffeomorphism group acts infinitesimally on g_i^+ by taking the covector field X_i to the symmetrized covariant derivative $(\delta_{g_i^+})^* X_i$, so the problem to solve is

$$B_{g_i^+}((\delta_{g_i^+})^* X_i) = -B_{g_i^+}(h_i^+).$$

Recall

$$B_{g_i^+}(\delta_{g_i^+})^* = \frac{1}{2}((\nabla)^* \nabla [g_i^+] - Ric[g_i^+]),$$

On the other hand, a direct calculation leads to (Proposition 2.5 [32])

$$B_{g^+}(h^+) = \rho^{-1} \mathcal{E}^0(h, g) + \mathcal{E}^1(h, g)$$

where $h = r^2 h^+$ and $g = r^2 g^+$ for some defining function r , \mathcal{E}^m denotes any tensor whose components in any coordinate system smooth up to the boundary of the g, g^{-1}, h, h^{-1} and their partial derivatives such that in each term the total number of derivatives of g and h that appear is at most m . More precisely, we have

$$\begin{aligned} \mathcal{E}^0(h, g)_m &= -h^{jk}(g_{jk}r_m - 4h_{mj}r_k), \\ \mathcal{E}^1(h, g)_m &= -h^{jk}\partial_k g_{mj} - \Gamma(h)_{mk}^l g_{lj} - \Gamma(h)_{jk}^l g_{lm} + \frac{1}{2}\partial_m(h^{jk}g_{jk}) \end{aligned}$$

If there is no confusion, we drop the index i for the metrics $g_i^+, h_i^+, g_i^*, h_i^*$, the covector field X_i . In view of Theorem 4.4, we note $B_{g^+}(h^+) = B_{g^+}(h^+ - g^+) \in C_\delta^{1,\alpha}$ for all $\delta \in (0, 3)$. Moreover, $B_{g_i^+}(h_i^+) \rightarrow 0$ in $C_\delta^{1,\alpha}$. We consider a C^1 fully nonlinear operator Ψ for $\delta \in (2, 3)$

$$\begin{aligned} \Psi : C_\delta^{3,\alpha}(B^4; TB^4) &\rightarrow C_\delta^{1,\alpha}(B^4; TB^4) \\ \tilde{X} &\mapsto B_{g^+}(\exp(\tilde{X})^* h^+) \end{aligned}$$

where \exp is the exponential map and \tilde{B} is a vector field related to the one form B . We know $2d\Psi(0) = \nabla^* \nabla + 3$. It follows from Theorem C [37] that $d\Psi(0) : C_\delta^{3,\alpha}(B^4; TB^4) \rightarrow C_\delta^{1,\alpha}(B^4; TB^4)$ is an isomorphism provided $2 < \delta < 3$. Applying inverse functions theorem, for large i , we find $\tilde{X}_i \in C_\delta^{3,\alpha}$ such that

$$\Psi(\tilde{X}_i) = 0.$$

Again from Lemma 3.7 [37], we have $C_\delta^{3,\alpha}(B^4; TB^4) \subset C_{2+\alpha}^{3,\alpha}(B^4; TB^4) \subset C_{(0)}^{3,\alpha}(\bar{B}^4; T\bar{B}^4)$ provided $\delta > 2 + \alpha$. Thus, we find a diffeomorphism $\varphi_i = \exp(\tilde{X}_i)$ of class $C^{3,\alpha}$ (equal to the identity on the boundary), such that

$$F((\varphi_i)_* h_i^+, g_i^+) = 0.$$

Moreover, $\|\varphi_i(x) - x\|_{C^{3,\alpha}} \rightarrow 0$ and $\|\varphi_i^* h_i^+ - g_i^+\|_{C_\delta^{2,\alpha}} \rightarrow 0$ for $\delta \in (2 + \alpha, 3)$. Thus we proved the claim.

Step 2. Claim. For large i , we have

$$g_i^+ = \varphi_i^* h_i^+.$$

We know $F(\varphi_i^* h_i^+, g_i^+) = F(g_i^+, g_i^+) = 0$ and by step 1 $\|g_i^+ - \varphi_i^* h_i^+\|_{C_\delta^{2,\alpha}} \rightarrow 0$ for $\delta \in$

$(2 + \alpha, 3)$. On the other hand, using [37, Theorems C and D] and [8, Lemma 12.71], the linearized operator

$$D_1 F(g_i^+, g_i^+) : C_\delta^{2,\alpha}(B^4; \Sigma^2 B^4) \rightarrow C_\delta^{0,\alpha}(B^4; \Sigma^2 B^4)$$

is an isomorphism. Applying the implicit function theorem, we infer the claim. \square

Remark 5.1. *Lemmas 1.6, 2.1, 2.4, 2.6, 3.1-3.3, 4.2 can be generalized in all dimensions in a similar way. So is Theorem 1.9 once the compactness result is established.*

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