IMPROVED MOSER-TRUDINGER-ONOFRI INEQUALITY UNDER CONSTRAINTS

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Abstract. A classical result of Aubin states that the constant in Moser-Trudinger-Onofri inequality on $\mathbb{S}^2$ can be imporved for functions with zero first order moments of the area element. We generalize it to higher order moments case. These new inequalities bear similarity to a sequence of Lebedev-Milin type inequalities on $\mathbb{S}^1$ coming from the work of Grenander-Szego on Toeplitz determinants (as pointed out by Widom). We also discuss the related sharp inequality by a perturbation method.

1. Introduction

Let (M,g) be a smooth compact Riemann surface without boundary. For an integrable function u on M, we denote

$$\overline{u}=\frac{1}{\mu(M)}\int_M u\,d\mu. \tag{1.1}$$

Here μ is the measure associated with the Riemannian metric g.

The classical Moser-Trudinger inequality (see [ChY2, F, M]) tells us that for every $u\in H^1(M)\setminus\{0\}$ with $\overline{u}=0$, we have

$$\int_M e^{4\pi\frac{u^2}{\|\nabla u\|^2_{L^2(M)}}}d\mu\le c(M,g). \tag{1.2}$$

Here $c(M,g)$ is a positive constant independent of u.

A direct consequence of (1.2) is the following Moser-Trudinger-Onofri inequality: for every $u\in H^1(M)$ with $\overline{u}=0$, we have

$$\log\int_M e^{2u}d\mu\le\frac{1}{4\pi}\|\nabla u\|^2_{L^2(M)}+c_1(M,g). \tag{1.3}$$

We remark that the inequality (1.3) has attracted more interest than the original inequality (1.2) due to its close relation to Gauss curvature equation and spectral geometry through the classical Polyakov formula (see for example [On, OsPS]).

On the standard sphere, it is found in [A, corollary 2 on p159] that for $u\in H^1(S^2)$ with $\overline{u}=0$ and $\int_{\mathbb{S}^2}x_ie^{2u(x)}d\mu(x)=0$ for $i=1,2,3$, the constant $\frac{1}{4\pi}$ in (1.3) can be lowered i.e. for any $\varepsilon>0$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2}e^{2u}d\mu\right)\le\left(\frac{1}{8\pi}+\varepsilon\right)\|\nabla u\|^2_{L^2}+c_\varepsilon. \tag{1.4}$$

Here c_ε is a constant depending on ε only.

A closely related question is to find the best constant in (1.3) and (1.4). In [On], the best constant $c_1(M,g)$ for (1.3) is found on the standard S^2. More precisely it is shown that for $u \in H^1\left(S^2\right)$ with $\overline{u}=0$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq \frac{1}{4\pi}\left\|\nabla u\right\|_{L^2}^2. \tag{1.5}$$

For (1.4), it is proved recently in [GuM] that the best constant c_ε is 0. In other words, for $u \in H^1\left(S^2\right)$ with $\overline{u}=0$ and $\int_{\mathbb{S}^2} x_i e^{2u(x)} d\mu(x) = 0$ for $i=1,2,3$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq \frac{1}{8\pi}\left\|\nabla u\right\|_{L^2}^2. \tag{1.6}$$

This confirms a conjecture in [ChY1].

To motivate our discussion, let us look at some research on S^1 which has similar spirit as above. For convenience we let D be the unit disk in R^2. For any $u \in H^1(D)$ with $\int_{\mathbb{S}^1} u d\theta = 0$, the Lebedev-Milin inequality (see [D, chapter 5]) tells us

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right) \leq \frac{1}{4\pi}\left\|\nabla u\right\|_{L^2(D)}^2. \tag{1.7}$$

This should be compared to (1.5).

On the other hand, as observed in [Wi], we have a sequence of Lebedev-Milin type inequalities following from the work of Grenander-Szego [GrS] on Toeplitz determinants. More precisely for any integer $m \geq 0$, $u \in H^1(D)$ with $\int_{\mathbb{S}^1} u d\theta = 0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0$ for $k = 1, \cdots, m$, we have

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right) \leq \frac{1}{4\pi(m+1)}\left\|\nabla u\right\|_{L^2(D)}^2. \tag{1.8}$$

For $m=0$, (1.8) is just (1.7). For $m=1$, (1.8) is proved in [OsPS, section 2]. These inequalities should be compared to (1.6). Note that $\cos k\theta$ and $\sin k\theta$ are eigenfunctions of $-\Delta_{\mathbb{S}^1}$ with eigenvalue k^2. So (1.8) actually tells us we can improve the coefficient of $\left\|\nabla u\right\|_{L^2(D)}^2$ further if e^u is perpendicular to more eigenfunctions of $-\Delta_{\mathbb{S}^1}$. For a while, people wonder whether we have similar improvements of (1.4) or (1.6) on S^2. The main aim of this note, as stated in Theorem 1.1 below, is to confirm this guess.

To state the main results, we need some notations. For any nonnegative integer k, we denote

$$\begin{aligned}
\mathcal{P}_k &= \left\{\text{all polynomials on } R^3 \text{ with degree at most } k\right\}; & (1.9)\\
\mathring{\mathcal{P}}_k &= \left\{p \in \mathcal{P}_k : \int_{\mathbb{S}^2} p d\mu = 0\right\}; & (1.10)\\
H_k &= \left\{\text{all degree } k \text{ homogeneous polynomials on } R^3\right\}; & (1.11)\\
\mathcal{H}_k &= \left\{h \in H_k : \Delta_{\mathbb{R}^3} h = 0\right\}. & (1.12)
\end{aligned}$$

It is known that

$$\left.\mathcal{H}_k\right|_{\mathbb{S}^2} = \left\{\left.h\right|_{\mathbb{S}^2} : h \in \mathcal{H}_k\right\} \tag{1.13}$$

is exactly the eigenspace of $-\Delta_{\mathbb{S}^2}$ associated with eigenvalue $k(k+1)$. Moreover

$$\left.\mathring{\mathcal{P}}_k\right|_{\mathbb{S}^2} = \bigoplus_{i=1}^k \left.\mathcal{H}_i\right|_{\mathbb{S}^2}. \tag{1.14}$$

We refer the reader to [SW, chapter IV] for these facts.

Definition 1.1. *Let $m \in N$, we denote*

$$\begin{aligned} & \mathcal{N}_m \qquad (1.15) \\ = \;& \Big\{ N \in N : \exists x_1, \cdots, x_N \in S^2 \text{ and } \nu_1, \cdots, \nu_N \in [0, \infty) \text{ s.t. } \nu_1 + \cdots + \nu_N = 1 \\ & \text{and for any } p \in \overset{\circ}{\mathcal{P}}_m,\ \nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0. \Big\} \\ = \;& \Big\{ N \in N : \exists x_1, \cdots, x_N \in S^2 \text{ and } \nu_1, \cdots, \nu_N \in [0, \infty) \text{ s.t. for any } p \in \mathcal{P}_m, \\ & \nu_1 p(x_1) + \cdots + \nu_N p(x_N) = \frac{1}{4\pi} \int_{\mathbb{S}^2} p \, d\mu. \Big\}. \end{aligned}$$

The smallest number in $\mathcal{N}_m$ is denoted as N_m i.e. $N_m = \min \mathcal{N}_m$.

The importance of N_m lies in the following theorem, which is the main result of this paper.

Theorem 1.1. *Assume $u \in H^1(S^2)$ such that $\int_{\mathbb{S}^2} u \, d\mu = 0$ (here μ is the standard measure on S^2) and for every $p \in \overset{\circ}{\mathcal{P}}_m$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, then for any $\varepsilon > 0$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq \left(\frac{1}{4\pi N_m} + \varepsilon \right) \|\nabla u\|_{L^2}^2 + c_\varepsilon. \qquad (1.16)$$

It is worth pointing out that the coefficient $\frac{1}{4\pi N_m} + \varepsilon$ is almost optimal (see Lemma 3.1). On the other hand, in view of (1.6) and (1.8), it would be very interesting to determine the best possible constant c_ε in (1.16) for $m \geq 2$.

The condition in (1.15) is the same as saying the cubature formula (a more familiar name of cubature formula is quadrature formula)

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} f \, d\mu \approx \nu_1 f(x_1) + \cdots + \nu_N f(x_N) \qquad (1.17)$$

for functions f on S^2 has nonnegative weights and degree of precision m (here we use the terminology in [HSW]). Various cubature formulas are of great practical importance in scientific computing and have been extensively studied in the literature (see the review articles [Co, HSW] and the references therein). In particular, the size of N_m is discussed in [HSW, section 4.6]. It follows from [Co, theorem 7.1] or [HSW, theorem 4] that

$$N_m \geq \left(\left[\frac{m}{2} \right] + 1 \right)^2. \qquad (1.18)$$

Here $[t]$ denotes the largest integer less than or equal to t. In our case when all the weights ν_i's are nonnegative, a simple proof of (1.18) is given on [HSW, p1203]. In general, finding the exact values of N_m for all m's is still an open problem.

On the other hand, it is straightforward to see that $N_1 = 2$ (see Example 4.1). Hence (1.4) follows from Theorem 1.1. It is also well known in numerical analysis community that $N_2 = 4$ (we provide an elementary proof of this fact in Lemma 4.1 for reader's convenience). As a consequence, we have

Corollary 1.1. *Assume $u \in H^1(S^2)$ such that $\int_{\mathbb{S}^2} u \, d\mu = 0$ and for every $p \in \overset{\circ}{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, then for any $\varepsilon > 0$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq \left(\frac{1}{16\pi} + \varepsilon \right) \|\nabla u\|_{L^2}^2 + c_\varepsilon. \qquad (1.19)$$

At last we want to point out that our analysis of H^1 on surfaces depends heavily on the Hilbert space structure of H^1, and closely follows [L, p197]. For similar discussion of $W^{1,n}$ $(n \geq 3)$ on a Riemannian manifold of dimension n, [L, p197] has to use special symmetrization process to gain the pointwise convergence of the gradient of functions considered. In [H], by adapting the approach in this paper, we are able to avoid the symmetrization process and generalize the analysis to dimensions at least 3 as well as higher order Sobolev spaces. We also remark that in a forthcoming paper [ChG] we discuss an inequality on S^2 which is the counterpart of the second inequality in the Szego limit theorem of the Toeplitz determinants on the unit circle.

In Section 2, we will derive some extensions of the concentration compactness principle in dimension 2. These refinements will be used in Section 3 to prove our main theorem. In Section 4, we discuss some elementary facts about N_m. In particular we will show $N_2 = 4$. In Section 5, we will make a first effort toward related sharp inequalities generalizing (1.6). In Section 6, we will show our approach gives a new way to prove the sequence of Lebedev-Milin type inequalities on the unit circle.

2. Refinements of concentration compactness principle in dimension 2

In this section, we will extend the concentration compactness principle in dimension 2 developed in [L, section I.7]. These extensions will be crucial in the derivation of Theorem 1.1.

We start from a basic consequence of Moser-Trudinger inequality (1.2).

Lemma 2.1. *For any $u \in H^1(M)$ and $a > 0$, we have*

$$\int_M e^{au^2} d\mu < \infty. \tag{2.1}$$

Proof. Without losing of generality, we can assume u is nonnegative and unbounded. For $b > 0$, let $v = (u-b)^+$, then

$$\|\nabla v\|_{L^2}^2 = \int_{u>b} |\nabla u|^2 d\mu \to 0$$

as $b \to \infty$. Let $w = v - \overline{v}$, then

$$0 \leq u \leq v + b = w + \overline{v} + b.$$

Hence

$$u^2 \leq 2w^2 + 2(\overline{v} + b)^2.$$

We have

$$e^{au^2} \leq e^{2a(\overline{v}+b)^2} e^{2aw^2} \leq e^{2a(\overline{v}+b)^2} e^{4\pi \frac{w^2}{\|\nabla w\|_{L^2}^2}}$$

when b is large enough. It follows that

$$\int_M e^{au^2} d\mu \leq ce^{2a(\overline{v}+b)^2} < \infty.$$

Next we prove a localized version of [L, Theorem I.6].

Lemma 2.2. *Assume $u_i \in H^1(M)$ such that $\overline{u_i} = 0$ and $\|\nabla u_i\|_{L^2} \leq 1$. We also assume $u_i \rightharpoonup u$ weakly in $H^1(M)$, $u_i \to u$ a.e. and*

$$|\nabla u_i|^2 \, d\mu \to |\nabla u|^2 \, d\mu + \sigma \tag{2.2}$$

in measure. If $K \subset M$ is a compact subset with $\sigma(K) < 1$, then for any $1 \leq p < \frac{1}{\sigma(K)}$, we have $e^{4\pi u_i^2}$ is bounded in $L^p(K)$ i.e.

$$\sup_i \int_K e^{4\pi p u_i^2} d\mu < \infty. \tag{2.3}$$

Proof. For basics about measure theory we refer the readers to [EG]. Let $v_i = u_i - u$, then $v_i \rightharpoonup 0$ weakly in $H^1(M)$, $v_i \to 0$ in $L^2(M)$. For any $\varphi \in C^\infty(M)$, we have

$$\begin{aligned}
& \|\nabla(\varphi v_i)\|_{L^2}^2 \\
= & \int_M \left(|\nabla \varphi|^2 v_i^2 + 2\varphi v_i \nabla\varphi \cdot \nabla v_i + \varphi^2 |\nabla v_i|^2 \right) d\mu \\
= & \int_M |\nabla \varphi|^2 v_i^2 d\mu + 2\int_M \varphi v_i \nabla \varphi \cdot \nabla v_i d\mu \\
& + \int_M \left(\varphi^2 |\nabla u_i|^2 - 2\varphi^2 \nabla u \cdot \nabla u_i + \varphi^2 |\nabla u|^2 \right) d\mu \\
\to & \int_M \varphi^2 d\sigma
\end{aligned}$$

as $i \to \infty$. Assume $1 \leq p_1 < \frac{1}{\sigma(K)}$, then $\sigma(K) < \frac{1}{p_1}$. Hence there exists $\varphi \in C^\infty(M)$ such that $\varphi|_K = 1$ and $\int_M \varphi^2 d\sigma < \frac{1}{p_1}$. It follows that for i large enough,

$$\|\nabla(\varphi v_i)\|_{L^2}^2 < \frac{1}{p_1}.$$

Hence

$$\begin{aligned}
\int_K e^{4\pi p_1 (v_i - \overline{\varphi v_i})^2} d\mu & \leq \int_M e^{4\pi p_1 (\varphi v_i - \overline{\varphi v_i})^2} d\mu \\
& \leq \int_M e^{4\pi \frac{(\varphi v_i - \overline{\varphi v_i})^2}{\|\nabla(\varphi v_i)\|_{L^2}^2}} d\mu \\
& \leq c(M, g).
\end{aligned}$$

To continue, we observe that for any $\varepsilon > 0$,

$$\begin{aligned}
u_i^2 &= \left((v_i - \overline{\varphi v_i}) + u + \overline{\varphi v_i} \right)^2 \\
&= (v_i - \overline{\varphi v_i})^2 + 2(v_i - \overline{\varphi v_i})(u + \overline{\varphi v_i}) + (u + \overline{\varphi v_i})^2 \\
&\leq (1+\varepsilon)(v_i - \overline{\varphi v_i})^2 + \left(1 + \varepsilon^{-1}\right)(u + \overline{\varphi v_i})^2 \\
&\leq (1+\varepsilon)(v_i - \overline{\varphi v_i})^2 + 2\left(1+\varepsilon^{-1}\right) u^2 + 2\left(1+\varepsilon^{-1}\right) \overline{\varphi v_i}^2.
\end{aligned}$$

Hence

$$e^{4\pi u_i^2} \leq e^{4\pi(1+\varepsilon)(v_i - \overline{\varphi v_i})^2} e^{8\pi\left(1+\varepsilon^{-1}\right)u^2} e^{8\pi\left(1+\varepsilon^{-1}\right)\overline{\varphi v_i}^2}.$$

Given $1 \leq p < \frac{1}{\sigma(K)}$, we can choose a $p_1 \in \left(p, \frac{1}{\sigma(K)}\right)$. There exists a $\varepsilon > 0$ such that $\frac{p_1}{1+\varepsilon} > p$. Note that $e^{4\pi(1+\varepsilon)(v_i - \overline{\varphi v_i})^2}$ is bounded in $L^{\frac{p_1}{1+\varepsilon}}(K)$, $e^{8\pi\left(1+\varepsilon^{-1}\right)u^2} \in L^q(K)$ for any $q < \infty$ (by Lemma 2.1) and $e^{8\pi\left(1+\varepsilon^{-1}\right)\overline{\varphi v_i}^2} \to 1$ as $i \to \infty$, it follows from Holder's inequality that $e^{4\pi u_i^2}$ is bounded in $L^p(K)$.

Corollary 2.1. *With the same assumption as in Lemma 2.2, let*

$$\kappa=\max_{x\in M}\sigma\left(\{x\}\right)\leq 1. \tag{2.4}$$

(1) *If $\kappa<1$, then for any $1\leq p<\frac{1}{\kappa}$, $e^{4\pi u_i^2}$ is bounded in $L^p(M)$. In particular, $e^{4\pi u_i^2}\to e^{4\pi u^2}$ in $L^1(M)$.*
(2) *If $\kappa=1$, then $\sigma=\delta_{x_0}$ for some $x_0\in M$, $u=0$ and after passing to a subsequence,*

$$e^{4\pi u_i^2}\to 1+c_0\delta_{x_0} \tag{2.5}$$

in measure for some $c_0\geq 0$.

Proof. First we assume $\kappa<1$. Let $1\leq p<\frac{1}{\kappa}$, then for any $x\in M$, $\sigma(x)<\frac{1}{p}$. Hence for some $r_x>0$ small, we have $\sigma\left(\overline{B_{r_x}(x)}\right)<\frac{1}{p}$. By the compactness of M, we see

$$M=\bigcup_{i=1}^{N}B_{r_i}(x_i).$$

Here $r_i=r_{x_i}$. Then

$$M=\bigcup_{i=1}^{N}\overline{B_{r_i}(x_i)}.$$

It follows from the Lemma 2.2 that

$$\sup_j\int_{\overline{B_{r_i}(x_i)}}e^{4\pi p u_j^2}d\mu<\infty.$$

Summing up, we get

$$\sup_j\int_M e^{4\pi p u_j^2}d\mu<\infty.$$

Next we assume $\kappa=1$. Since

$$\int_M|\nabla u|^2\,d\mu+\sigma(M)\leq 1,$$

and $\overline{u}=0$, we see $u=0$ and $\sigma=\delta_{x_0}$ for some $x_0\in M$. For $r>0$ small, we know $e^{4\pi u_i^2}$ is bounded in $L^q(M\backslash B_r(x_0))$ for any $q<\infty$, hence $e^{4\pi u_i^2}\to 1$ in $L^1(M\backslash B_r(x_0))$. It follows that after passing to a subsequence, $e^{4\pi u_i^2}\to 1+c_0\delta_{x_0}$ in measure for some $c_0\geq 0$.

Now we are ready to derive the main refinement of the earlier concentration compactness principle.

Proposition 2.1. *Assume $\alpha>0$, $m_i>0$, $m_i\to\infty$, $u_i\in H^1(M)$ such that $\overline{u_i}=0$, $\|\nabla u_i\|_{L^2}=1$ and*

$$\log\int_M e^{2m_iu_i}d\mu\geq\alpha m_i^2. \tag{2.6}$$

We also assume $u_i\rightharpoonup u$ weakly in $H^1(M)$, $|\nabla u_i|^2\,d\mu\to|\nabla u|^2\,d\mu+\sigma$ in measure and

$$\frac{e^{2m_iu_i}}{\int_M e^{2m_iu_i}d\mu}\to\nu \tag{2.7}$$

in measure. Let

$$\{x\in M:\sigma(x)\geq 4\pi\alpha\}=\{x_1,\cdots,x_N\}, \tag{2.8}$$

then

$$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}, \tag{2.9}$$

here $\nu_i \geq 0$ *and* $\sum_{i=1}^{N} \nu_i = 1$.

Proof. First we claim that if K is a compact subset of M with $\sigma(K) < 4\pi\alpha$, then $\nu(K) = 0$. Indeed, we can find another compact set K_1 such that $K \subset \operatorname{int} K_1$ and $\sigma(K_1) < 4\pi\alpha$. Fix a number p such that

$$\frac{1}{4\pi\alpha} < p < \frac{1}{\sigma(K_1)},$$

then Lemma 2.2 tells us

$$\int_{K_1} e^{4\pi p u_i^2} d\mu \leq c,$$

here c is a constant independent of i. Using

$$2m_i u_i \leq 4\pi p u_i^2 + \frac{m_i^2}{4\pi p},$$

we see

$$\int_{K_1} e^{2m_i u_i} d\mu \leq c e^{\frac{m_i^2}{4\pi p}}.$$

It follows that

$$\frac{\int_{K_1} e^{2m_i u_i} d\mu}{\int_M e^{2m_i u_i} d\mu} \leq c e^{\left(\frac{1}{4\pi p} - \alpha\right) m_i^2}.$$

Hence

$$\nu(K) \leq \nu(\operatorname{int} K_1) \leq \liminf_{i\to\infty} \frac{\int_{K_1} e^{2m_i u_i} d\mu}{\int_M e^{2m_i u_i} d\mu} = 0.$$

It follows that $\nu(K) = 0$.

If $\sigma(x) < 4\pi\alpha$, then for some $r_x > 0$ small, we have $\sigma\left(\overline{B_{r_x}(x)}\right) < 4\pi\alpha$. It follows from the claim that $\nu\left(\overline{B_{r_x}(x)}\right) = 0$. Hence

$$\nu(M \backslash \{x_1, \cdots, x_N\}) = 0.$$

In another word, $\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}$ with $\nu_i \geq 0$ and $\sum_{i=1}^{N} \nu_i = 1$.

3. Proof of Theorem 1.1

Let $f_1, \cdots, f_L \in C(M)$ and $\alpha > 0$ be given. Here is our strategy to show for any $u \in H^1(M)$ with $\overline{u} = 0$ and $\int_M f_i e^{2u} d\mu = 0$ for $1 \leq i \leq L$, we have

$$\log \int_M e^{2u} d\mu \leq \alpha \|\nabla u\|_{L^2}^2 + c. \tag{3.1}$$

This will be proven by contradiction argument. If it is not the case, then there exists $v_i \in H^1(M)$, $\overline{v_i} = 0$, $\int_M f_j e^{2v_i} d\mu = 0$ for $1 \leq j \leq L$, such that

$$\log \int_M e^{2v_i} d\mu - \alpha \|\nabla v_i\|_{L^2}^2 \to \infty \tag{3.2}$$

as $i \to \infty$. Then $\log \int_M e^{2v_i} d\mu \to \infty$. Since

$$\log \int_M e^{2v_i} d\mu \leq \frac{1}{4\pi} \|\nabla v_i\|_{L^2}^2 + c(M, g), \tag{3.3}$$

we see $\|\nabla v_i\|_{L^2} \to \infty$. Let $m_i = \|\nabla v_i\|_{L^2}$ and $u_i = \frac{v_i}{m_i}$, then $m_i \to \infty$, $\|\nabla u_i\|_{L^2} = 1$, $\overline{u_i} = 0$. After passing to a subsequence, we have

$$\begin{aligned} u_i &\rightharpoonup u \text{ weakly in } H^1(M); \\ \log \int_M e^{2m_i u_i} d\mu - \alpha m_i^2 &\to \infty, \\ |\nabla u_i|^2 d\mu &\to |\nabla u|^2 d\mu + \sigma \text{ in measure,} \\ \frac{e^{2m_i u_i}}{\int_M e^{2m_i u_i} d\mu} &\to \nu \text{ in measure.} \end{aligned}$$

Let

$$\{x \in M : \sigma(x) \geq 4\pi\alpha\} = \{x_1, \cdots, x_N\}, \tag{3.4}$$

then it follows from Proposition 2.1 that

$$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}, \tag{3.5}$$

here $\nu_i \geq 0$ and $\sum_{i=1}^{N} \nu_i = 1$. On the other hand we have

$$\int_M f_j d\nu = 0$$

for $1 \leq j \leq L$. In another word, we have

$$4\pi\alpha N \leq 1; \tag{3.6}$$

$$\sum_{i=1}^{N} \nu_i f_j(x_i) = 0 \tag{3.7}$$

for $1 \leq j \leq L$. We hope to get contradiction from these inequalities.

Proof of Theorem 1.1. Let $\alpha = \frac{1}{4\pi N_m} + \varepsilon$. If (1.16) is not true, then the above discussion gives us $x_1, \cdots, x_N \in S^2$, $\nu_1, \cdots, \nu_N \geq 0$ such that $\sum_{i=1}^{N} \nu_i = 1$ and for any $p \in \overset{\circ}{\mathcal{P}}_m$, $\nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0$. Moreover $4\pi\alpha N \leq 1$. In particular, $N \in \mathcal{N}_m$ and hence $N \geq N_m$. It follows that

$$\alpha \leq \frac{1}{4\pi N} \leq \frac{1}{4\pi N_m}.$$

This contradicts with the choice of α.

Next we want to show the constant $\frac{1}{4\pi N_m} + \varepsilon$ in (1.16) is almost sharp.

Lemma 3.1. *Assume $m \in N$. If $a \geq 0$ and $c \in R$ such that for any $u \in H^1(S^2)$ with $\overline{u} = 0$ and $\int_{\mathbb{S}^2} pe^{2u} d\mu = 0$ for every $p \in \overset{\circ}{\mathcal{P}}_m$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq a \|\nabla u\|_{L^2}^2 + c, \tag{3.8}$$

then $a \geq \frac{1}{4\pi N_m}$.

Proof. First we note that we can rewrite the assumption as for any $u \in H^1(S^2)$ with $\int_{\mathbb{S}^2} pe^{2u} d\mu = 0$ for every $p \in \overset{\circ}{\mathcal{P}}_m$, we have

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq a \|\nabla u\|_{L^2}^2 + 2\overline{u} + c. \tag{3.9}$$

Assume $N \in \boldsymbol{N}$, $x_1, \cdots, x_N \in S^2$ and $\nu_1, \cdots, \nu_N \in [0, \infty)$ s.t. $\nu_1 + \cdots + \nu_N = 1$ and for any $p \in \overset{\circ}{\mathcal{P}}_m$, $\nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0$. We will prove $a \geq \frac{1}{4\pi N}$. Lemma 3.1 follows. Without losing of generality we can assume $\nu_i > 0$ for $1 \leq i \leq N$ and $x_i \neq x_j$ for $1 \leq i < j \leq N$.

To continue let us fix some notations. For $x, y \in S^2$, we denote $\overline{xy}$ as the geodesic distance between x and y on S^2. For $r > 0$ and $x \in S^2$, we denote $B_r(x)$ as the geodesic ball with radius r and center x i.e. $B_r(x) = \left\{ y \in S^2 : \overline{xy} < r \right\}$.

Let $\delta > 0$ be small enough such that for $1 \leq i < j \leq N$, $\overline{B_{2\delta}(x_i)} \cap \overline{B_{2\delta}(x_j)} = \emptyset$. For $0 < \varepsilon < \delta$, we let

$$\phi_\varepsilon(t) = \begin{cases} 2\log\frac{\delta}{\varepsilon}, & 0 < t < \varepsilon; \\ 2\log\frac{\delta}{t}, & \varepsilon < t < \delta; \\ 0, & t > \delta. \end{cases}$$

If $b \in \boldsymbol{R}$, then we write

$$\phi_{\varepsilon,b}(t) = \begin{cases} \phi_\varepsilon(t) + b, & 0 < t < \delta; \\ b\left(2 - \frac{t}{\delta}\right), & \delta < t < 2\delta; \\ 0, & t > 2\delta. \end{cases}$$

Let

$$v(x) = \sum_{i=1}^N \phi_{\varepsilon, \frac{1}{2}\log\nu_i}(\overline{xx_i}), \tag{3.10}$$

then

$$\begin{aligned} \int_{\mathbb{S}^2} e^{2v} d\mu &= \sum_{i=1}^N \int_{B_\delta(x_i)} e^{2\phi_\varepsilon(\overline{xx_i}) + \log\nu_i} d\mu + O(1) \\ &= 2\pi \int_0^\delta e^{2\phi_\varepsilon(r)} \sin r dr + O(1) \\ &= 2\pi\delta^4\varepsilon^{-2} + O\left(\log\frac{1}{\varepsilon}\right) \end{aligned} \tag{3.11}$$

as $\varepsilon \to 0^+$.

Note that since $\dim\left(\overset{\circ}{\mathcal{P}}_m\Big|_{\mathbb{S}^2}\right) = m^2 + 2m$, we can fix $p_1, \cdots, p_{m^2+2m} \in \overset{\circ}{\mathcal{P}}_m$ such that $p_1|_{\mathbb{S}^2}, \cdots, p_{m^2+2m}|_{\mathbb{S}^2}$ is a base for $\overset{\circ}{\mathcal{P}}_m\Big|_{\mathbb{S}^2}$. For $1 \leq j \leq m^2 + 2m$, we have

$$\int_{\mathbb{S}^2} e^{2v} p_j d\mu = O\left(\log\frac{1}{\varepsilon}\right) \tag{3.12}$$

as $\varepsilon \to 0^+$. Indeed,

$$\begin{aligned} &\int_{\mathbb{S}^2} e^{2v} p_j d\mu \\ =\ & \sum_{i=1}^N \nu_i \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} p_j(x)\, d\mu(x) + O(1) \\ =\ & \sum_{i=1}^N \left(\nu_i p_j(x_i) \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} d\mu(x) + \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} O\left(\overline{xx_i}^2\right) d\mu(x) \right) + O(1), \end{aligned}$$

here we have used the Talyor expansion of p_j near x_i and the vanishing of integral of first order terms by symmetry. Using

$$\sum_{i=1}^{N} \nu_i p_j (x_i) = 0,$$

we see

$$\int_{\mathbb{S}^2} e^{2v} p_j d\mu = O\left(\log \frac{1}{\varepsilon}\right).$$

To get a test function satisfying orthogornality condition, we need to do some corrections. We first claim that there exists $\psi_1, \cdots, \psi_{m^2+2m} \in C_c^\infty \left(S^2 \backslash \bigcup_{i=1}^{N} \overline{B_{2\delta}(x_i)} \right)$ such that the determinant

$$\det \left[\int_{\mathbb{S}^2} \psi_j p_k d\mu \right]_{1 \le j,k \le m^2+2m} \neq 0. \tag{3.13}$$

Indeed, here is one way to construct these functions. Fix a nonzero smooth function $\eta \in C_c^\infty \left(S^2 \backslash \bigcup_{i=1}^{N} \overline{B_{2\delta}(x_i)} \right)$, then $\eta p_1, \cdots, \eta p_{m^2+2m}$ are linearly independent. It follows that the matrix

$$\left[\int_{\mathbb{S}^2} \eta^2 p_j p_k d\mu \right]_{1 \le j,k \le m^2+2m}$$

is positive definite and has positive determinant. Then $\psi_j = \eta^2 p_j$ satisfies the claim.

It follows from (3.13) that we can find $\beta_1, \cdots, \beta_{m^2+2m} \in R$ such that

$$\int_{\mathbb{S}^2} \left(e^{2v} + \sum_{j=1}^{m^2+2m} \beta_j \psi_j \right) p_k d\mu = 0 \tag{3.14}$$

for $k = 1, \cdots, m^2 + 2m$. Moreover

$$\beta_j = O\left(\log \frac{1}{\varepsilon}\right) \tag{3.15}$$

as $\varepsilon \to 0^+$. As a consequence we can find a constant $c_1 > 0$ such that

$$\sum_{j=1}^{m^2+2m} \beta_j \psi_j + c_1 \log \frac{1}{\varepsilon} \ge \log \frac{1}{\varepsilon}. \tag{3.16}$$

We define u as

$$e^{2u} = e^{2v} + \sum_{j=1}^{m^2+2m} \beta_j \psi_j + c_1 \log \frac{1}{\varepsilon}. \tag{3.17}$$

Note this u will be the test function we use to prove Lemma 3.1.

It follows from (3.14) that $\int_{\mathbb{S}^2} e^{2u} p d\mu = 0$ for all $p \in \overset{\circ}{\mathcal{P}}_m$. Moreover using (3.11) and (3.15) we see

$$\int_{\mathbb{S}^2} e^{2u} d\mu = 2\pi\delta^4\varepsilon^{-2} + O\left(\log \frac{1}{\varepsilon}\right) = 2\pi\delta^4\varepsilon^{-2} (1 + o(1)), \tag{3.18}$$

hence

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu = 2 \log \frac{1}{\varepsilon} + O(1) \tag{3.19}$$

as $\varepsilon \to 0^+$. Calculation shows

$$\overline{u} = o\left(\log \frac{1}{\varepsilon}\right). \tag{3.20}$$

At last we claim

$$\int_{\mathbb{S}^2} |\nabla u|^2 d\mu = 8\pi N \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right). \tag{3.21}$$

Once this is known, we plug u into (3.9) and get

$$2 \log \frac{1}{\varepsilon} \leq 8\pi N a \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right).$$

Divide $\log \frac{1}{\varepsilon}$ on both sides and let $\varepsilon \to 0^+$, we see $a \geq \frac{1}{4\pi N}$.

To derive (3.21), we note that on $S^2 \backslash \bigcup_{i=1}^{N} \overline{B_{2\delta}(x_i)}$, $|\nabla u| = O(1)$ (here we need to use (3.15) and (3.16)), hence

$$\begin{aligned}
\int_{\mathbb{S}^2} |\nabla u|^2 d\mu &= \sum_{i=1}^{N} \int_{B_{2\delta}(x_i)} |\nabla u|^2 d\mu + O(1) \\
&= \sum_{i=1}^{N} \int_{B_{\delta}(x_i)} |\nabla u|^2 d\mu + O(1) \\
&= \sum_{i=1}^{N} 8\pi \int_{\varepsilon}^{\delta} \frac{r^{-10} \sin r}{\left(\frac{c_1 \log \frac{1}{\varepsilon}}{\nu_i \delta^4} + r^{-4}\right)^2} dr + O(1) \\
&= 8\pi N \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right).
\end{aligned}$$

4. The number N_m

We start with the following basic observation.

Example 4.1. $N_1 = 2$. *It is clear that* $N_1 \geq 2$*, on the other hand, by setting* $\nu_1 = \nu_2 = \frac{1}{2}$ *and* $x_2 = -x_1$*, we see* $N_1 \leq 2$*. Hence* $N_1 = 2$.

Lemma 4.1. $N_2 = 4$.

Proof. Indeed it follows from (1.18) that $N_2 \geq 4$. Here we give a direct proof. Note that $N_2 \geq N_1 = 2$.

If $N_2 = 2$, then we have $\nu_1 x_1 + \nu_2 x_2 = 0$. It implies $\nu_1 = \nu_2 = \frac{1}{2}$. Hence $x_2 = -x_1$. By rotation, we assume $x_1 = (0, 0, 1)$. Let $p(y) = y_1^2$, then

$$\nu_1 p(x_1) + \nu_2 p(x_2) = 0 \neq \frac{1}{4\pi} \int_{\mathbb{S}^2} p d\mu.$$

We get a contradiction.

If $N_2 = 3$, then we have $\nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 = 0$. It follows that x_1, x_2, x_3 must lie in a plane. By rotation we can assume that plane is the horizontal plane. Let $p = y_3^2$, then

$$\nu_1 p(x_1) + \nu_2 p(x_2) + \nu_3 p(x_3) = 0 \neq \frac{1}{4\pi} \int_{\mathbb{S}^2} p d\mu.$$

This gives us a contradiction.

Hence we only need to find $x_1, x_2, x_3, x_4 \in S^2$, $\nu_1, \nu_2, \nu_3, \nu_4 \geq 0$ with $\nu_1 + \nu_2 + \nu_3 + \nu_4 = 1$ such that for any $p \in \overset{\circ}{\mathcal{P}}_2$, we have

$$\nu_1 p(x_1) + \nu_2 p(x_2) + \nu_3 p(x_3) + \nu_4 p(x_4) = 0. \tag{4.1}$$

We claim the four vortices of a regular tetrahedron inside the unit sphere with $\nu_i = \frac{1}{4}$ for $1 \leq i \leq 4$ would satisfy the property. Indeed, let

$$\begin{aligned} x_1 &= (0, 0, 1); \\ x_2 &= \left(0, \frac{2\sqrt{2}}{3}, -\frac{1}{3}\right); \\ x_3 &= \left(\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}\right); \\ x_4 &= \left(-\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}\right). \end{aligned}$$

Then we have

$$x_1 + x_2 + x_3 + x_4 = 0.$$

Moreover using

$$\mathcal{H}_2 = \operatorname{span}\left\{ y_1^2 - \frac{|y|^2}{3}, y_2^2 - \frac{|y|^2}{3}, y_1 y_2, y_1 y_3, y_2 y_3 \right\},$$

checking (4.1) for each p in the base verifies the identity.

It remains an interesting question to find N_m for all m's.

5. A sharp inequality by perturbation

In this section we prove a sharp inequality by the perturbation method in the same spirit as [ChY1].

Theorem 5.1. *There exists an $a_0 < \frac{1}{8\pi}$ such that for all $u \in H^1(S^2)$ satisfying $\int_{\mathbb{S}^2} u d\mu = 0$ and for every $p \in \overset{\circ}{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, we have*

$$\log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq a_0 \|\nabla u\|_{L^2}^2. \tag{5.1}$$

For convenience we denote

$$\mathcal{S}_2 = \left\{ u \in H^1(S^2) : \overline{u} = 0, \int_{\mathbb{S}^2} p e^{2u} d\mu = 0 \text{ for all } p \in \overset{\circ}{\mathcal{P}}_2 \right\}. \tag{5.2}$$

For a given number $a \in \left(\frac{1}{16\pi}, \frac{1}{8\pi}\right)$, it follows from Corollary 1.1 that for every $u \in \mathcal{S}_2$,

$$\log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq a \|\nabla u\|_{L^2}^2 + c_a. \tag{5.3}$$

Let

$$s = s_a = \inf_{u \in \mathcal{S}_2} \left[a \|\nabla u\|_{L^2}^2 - \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \right]. \tag{5.4}$$

We claim s is achieved. Indeed if $u_i \in \mathcal{S}_2$ is a minimizing sequence, then

$$a \|\nabla u_i\|_{L^2}^2 - \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u_i} d\mu \right) \leq c.$$

Here c is a constant independent of i. Choose a number ε with $0 < \varepsilon < a - \frac{1}{16\pi}$. Using Corollary 1.1 we have

$$a \|\nabla u_i\|_{L^2}^2 \leq \log\left(\frac{1}{4\pi} \int_{S^2} e^{2u_i} d\mu \right) + c \leq \left(\frac{1}{16\pi} + \varepsilon \right) \|\nabla u_i\|_{L^2}^2 + c.$$

It follows that

$$\|\nabla u_i\|_{L^2} \leq c.$$

After passing to a subsequence we can find $u \in H^1\left(S^2\right)$ such that $u_i \rightharpoonup u$ weakly in $H^1\left(S^2\right)$. Hence $u_i \to u$ in $L^2\left(S^2\right)$ and we can also assume $u_i \to u$ a.e. For any $b > 0$, we have

$$2bu_i \leq 4\pi \frac{u_i^2}{\|\nabla u_i\|_{L^2}^2} + \frac{b^2 \|\nabla u_i\|_{L^2}^2}{4\pi}.$$

Hence

$$\int_{\mathbb{S}^2} e^{2bu_i} d\mu \leq c e^{\frac{b^2 \|\nabla u_i\|_{L^2}^2}{4\pi}} \leq c.$$

It follows that $e^{2u_i} \to e^{2u}$ in $L^1\left(S^2\right)$. Hence for any $p \in \overset{\circ}{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$. It follows that $u \in \mathcal{S}_2$.

$$\begin{aligned} s &\leq a \|\nabla u\|_{L^2}^2 - \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \\ &\leq \liminf_{i \to \infty} \left[a \|\nabla u_i\|_{L^2}^2 - \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u_i} d\mu \right) \right] \\ &= s. \end{aligned}$$

Hence u is a minimizer.

Let u_a be a minimizer for (5.4). When no confusion would happen, we simply write u instead of u_a. We will show that if a is close enough to $\frac{1}{8\pi}$, the minimizer u must be identically zero. This would imply Theorem 5.1.

To achieve this aim, we can assume $\frac{5}{48\pi} < a < \frac{1}{8\pi}$. Since u is a minimizer, we see

$$a \|\nabla u\|_{L^2}^2 - \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \leq 0.$$

Hence applying Corollary 1.1 we get

$$a \|\nabla u\|_{L^2}^2 \leq \log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \leq \frac{1}{12\pi} \|\nabla u\|_{L^2}^2 + c.$$

It implies $\|\nabla u\|_{L^2}^2 \leq c$, a constant independent of a.

Next we claim that as $a \to \frac{1}{8\pi}$, $u_a \rightharpoonup 0$ weakly in $H^1\left(S^2\right)$. Indeed if this is not the case, then we can find a sequence $a_i \to \frac{1}{8\pi}$, $u_i = u_{a_i}$ such that $u_i \rightharpoonup w$ weakly in $H^1\left(S^2\right)$ and $w \neq 0$. We can also assume $u_i \to w$ a.e. It follows from classical

Moser-Trudinger inequality (see (1.2)) that $e^{2u_i} \to e^{2w}$ in $L^1\left(S^2\right)$. Hence $w \in \mathcal{S}_2$. Since

$$a_i \left\|\nabla u_i\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u_i} d\mu\right),$$

taking a limit we get

$$\frac{1}{8\pi}\left\|\nabla w\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2w} d\mu\right).$$

It follows from equality case of (1.4) (see [GuM]) that $w = 0$. This gives us a contradiction.

Applying the Moser-Trudinger inequality (1.2) again we see for any $b > 0$, $e^{2bu_a} \to 1$ in $L^q\left(S^2\right)$ for any $q \in [1, \infty)$ as $a \to \frac{1}{8\pi}$. Hence

$$a\left\|\nabla u_a\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u_a} d\mu\right) \to 0.$$

It follows that $\left\|\nabla u_a\right\|_{L^2} = o(1)$ as $a \to \frac{1}{8\pi}$.

To continue we observe that since

$$\left.\mathring{\mathcal{P}}_2\right|_{\mathbb{S}^2} = \left.\mathcal{H}_1\right|_{\mathbb{S}^2} \oplus \left.\mathcal{H}_2\right|_{\mathbb{S}^2} = \left.\left(\mathcal{H}_1 + \mathcal{H}_2\right)\right|_{\mathbb{S}^2},$$

u satisfies the Euler-Langrage equation

$$-a\Delta u - \frac{e^{2u}}{\int_{\mathbb{S}^2} e^{2u} d\mu} = -\frac{1}{4\pi} + \ell e^{2u} + h e^{2u} \tag{5.5}$$

for some $\ell = \ell_a \in \mathcal{H}_1$ and $h = h_a \in \mathcal{H}_2$.

Since $\mathcal{H}_1 + \mathcal{H}_2$ is a finite dimensional vector space, any two norms on it are equivalent. Hence we fix an arbitrary norm on $\mathcal{H}_1 + \mathcal{H}_2$ from now on. We claim that $\ell_a \to 0$ and $h_a \to 0$ as $a \to \frac{1}{8\pi}$. For convenience we write

$$\lambda = \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu.$$

Note that $\lambda = 1 + o(1)$. The equation becomes

$$-a\Delta u + \frac{1}{4\pi} = e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right). \tag{5.6}$$

Multiplying $\frac{1}{4\pi\lambda} + \ell + h$ and integrating on S^2, we see

$$\int_{\mathbb{S}^2}\left(-a\Delta u + \frac{1}{4\pi}\right)\left(\frac{1}{4\pi\lambda} + \ell + h\right) d\mu = \int_{\mathbb{S}^2} e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right)^2 d\mu.$$

Using the fact $u \in \mathcal{S}_2$ it becomes

$$\begin{aligned} & a\int_{\mathbb{S}^2} u\,(2\ell + 6h)\, d\mu \\ = & \int_{\mathbb{S}^2} e^{2u}(\ell + h)^2 d\mu \\ = & \int_{\mathbb{S}^2}\left(e^{2u} - 1\right)(\ell + h)^2 d\mu + \int_{\mathbb{S}^2} \ell^2 d\mu + \int_{\mathbb{S}^2} h^2 d\mu. \end{aligned}$$

It follows that

$$o(\|\ell\| + \|h\|) = \int_{\mathbb{S}^2} \ell^2 d\mu + \int_{\mathbb{S}^2} h^2 d\mu + o\left(\|\ell\|^2 + \|h\|^2\right).$$

Hence

$$\|\ell\|^2 + \|h\|^2 = o\left(\|\ell\| + \|h\|\right).$$

We get $\|\ell\| + \|h\| = o(1)$.

Now we claim that $\|u_a\|_{L^\infty} = o(1)$. Indeed since

$$\begin{aligned} & \left\| e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right) - \frac{1}{4\pi}\right\|_{L^2} \\ \leq & \left\| e^{2u}\left(\frac{1}{4\pi\lambda} - \frac{1}{4\pi}\right)\right\|_{L^2} + \frac{1}{4\pi}\left\|e^{2u} - 1\right\|_{L^2} + \left\|e^{2u}(\ell + h)\right\|_{L^2} \\ = & \; o(1), \end{aligned}$$

it follows from (5.6) and standard elliptic theory that $\|u_a\|_{W^{2,2}} = o(1)$. Sobolev embedding theorem tells us $\|u_a\|_{L^\infty} = o(1)$.

At last we observe that $e^{2u} - \lambda$ is perpendicular to R, $\mathcal{H}_1$ and $\mathcal{H}_2$, hence

$$\begin{aligned} & 12\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu \\ \leq & \int_{\mathbb{S}^2}\left|\nabla e^{2u}\right|^2 d\mu \\ = & \; 4\int_{\mathbb{S}^2} e^{4u}\left|\nabla u\right|^2 d\mu \\ = & \int_{\mathbb{S}^2}\nabla u\cdot\nabla e^{4u} d\mu \\ = & \int_{\mathbb{S}^2}(-\Delta u)\, e^{4u} d\mu \\ = & \int_{\mathbb{S}^2}(-\Delta u)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \; \frac{1}{a}\int_{\mathbb{S}^2}\left[e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right) - \frac{1}{4\pi}\right]\left(e^{4u} - \lambda^2\right) d\mu \\ = & \; \frac{1 + o(1)}{2\pi a}\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \frac{1}{a}\int_{\mathbb{S}^2} e^{2u}(\ell + h)\left(e^{4u} - \lambda^2\right) d\mu. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{\mathbb{S}^2} e^{2u}(\ell + h)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)(\ell + h)\left(e^{4u} - \lambda^2\right) d\mu + \lambda\int_{\mathbb{S}^2}(\ell + h)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \; o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \lambda\int_{\mathbb{S}^2}(\ell + h)\left(e^{4u} - 2\lambda e^{2u} + \lambda^2\right) d\mu \\ = & \; o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \lambda\int_{\mathbb{S}^2}(\ell + h)\left(e^{2u} - \lambda\right)^2 d\mu \\ = & \; o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu. \end{aligned}$$

Here we have used the fact $u \in \mathcal{S}_2$. Plug this equality back we see

$$\left(12 - \frac{1}{2\pi a} + o(1)\right)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu \leq 0.$$

Since a is close to $\frac{1}{8\pi}$, we get $\int_{\mathbb{S}^2}\left(e^{2u}-\lambda\right)^2 d\mu=0$. Hence u must be constant function. In view of the fact $\overline{u}=0$, we get $u=0$. This finishes the proof of Theorem 5.1.

6. A revisit of Lebedev-Milin type inequalities on S^1

In this section we will show the above method on S^2 provides a variational approach for a sequence of Lebedev-Milin type inequalities on S^1. Let D be the unit disk in the plane and $S^1=\partial D$ be the unit circle. We use θ as the usual angle variable and identify R^2 as C.

Theorem 6.1. *For $m\in N$, $u\in H^1(D)$ with $\int_{\mathbb{S}^1} u d\theta=0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta=0$ for $k=1,\cdots,m$, we have*

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right)\leq\frac{1}{4\pi(m+1)}\|\nabla u\|_{L^2(D)}^2. \tag{6.1}$$

Moreover equality holds if and only if $u(z)=\log\frac{1}{|1-\xi z^{m+1}|^2}$ for some $\xi\in C$ with $|\xi|<1$.

For $m=1$, (6.1) is proved in [OsPS] by variational method. As observed in [Wi], (6.1) follows from the work of Grenander-Szego [GrS] on Toeplitz determinants.

On S^1, the Moser-Trudinger inequality (1.2) is replaced by the Beurling-Chang-Marshall inequality (see [ChM, corollary 2]): for $u\in H^1(D)\setminus\{0\}$ with $\int_{\mathbb{S}^1} u d\theta=0$, we have

$$\int_{\mathbb{S}^1} e^{\pi\frac{u^2}{\|\nabla u\|_{L^2(D)}^2}} d\theta\leq c. \tag{6.2}$$

Similar to (1.9)–(1.12), for any nonnegative integer k, we write

$$\begin{aligned}
\mathcal{P}_k &= \left\{\text{real polynomials on } R^2 \text{ with degree at most } k\right\}; &(6.3)\\
\mathring{\mathcal{P}}_k &= \left\{p\in\mathcal{P}_k:\int_{\mathbb{S}^1} p d\theta=0\right\}; &(6.4)\\
H_k &= \left\{\text{degree } k \text{ homogeneous real polynomials on } R^2\right\}; &(6.5)\\
\mathcal{H}_k &= \left\{h\in H_k:\Delta_{\mathbb{R}^2}h=0\right\}=span_{\mathbb{R}}\left\{\operatorname{Re}\left(z^k\right),\operatorname{Im}\left(z^k\right)\right\}. &(6.6)
\end{aligned}$$

Note that

$$\mathcal{H}_k|_{\mathbb{S}^1}=span_{\mathbb{R}}\{\cos k\theta,\sin k\theta\} \tag{6.7}$$

and

$$\left.\mathring{\mathcal{P}}_k\right|_{\mathbb{S}^1}=span_{\mathbb{R}}\{\cos j\theta,\sin j\theta: j\in N, j\leq k\}. \tag{6.8}$$

Corresponds to Definition 1.1, we have for $m\in N$,

$$\begin{aligned}
&\mathcal{N}_m\left(S^1\right) &(6.9)\\
=\ &\Big\{N\in N:\exists z_1,\cdots,z_N\in S^1 \text{ and } \nu_1,\cdots,\nu_N\in[0,\infty) \text{ s.t. for any } p\in\mathcal{P}_m,\\
&\nu_1 p(z_1)+\cdots+\nu_N p(z_N)=\frac{1}{2\pi}\int_{\mathbb{S}^1} p d\theta.\Big\}
\end{aligned}$$

and $N_m\left(S^1\right)=\min\mathcal{N}_m\left(S^1\right)$. Unlike the case on S^2, it is known that

$$N_m\left(S^1\right)=m+1. \tag{6.10}$$

Indeed if $N \in \mathcal{N}_m\left(S^1\right)$, we must have $N \geq m+1$. Otherwise, for the $z_1, \cdots, z_N \in S^1$ in (6.9), we let $f(z) = (z - z_1) \cdots (z - z_N)$, then $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{P}_m$. It follows that

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} f d\theta = \nu_1 f(z_1) + \cdots + \nu_N f(z_N) = 0.$$

On the other hand, we clearly have

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} f d\theta = (-1)^N z_1 \cdots z_N \neq 0.$$

This gives us a contradiction. Hence $N_m\left(S^1\right) \geq m+1$. On the other hand, for $1 \leq k \leq m+1$, we let $\nu_k = \frac{1}{m+1}$ and $z_k = e^{\frac{2k\pi}{m+1} i}$. It follows that $m+1 \in \mathcal{N}_m\left(S^1\right)$. Hence $N_m\left(S^1\right) = m+1$.

Now we are ready to state the analogue of Theorem 1.1 on S^1.

Lemma 6.1. *Assume $m \in N$, $u \in H^1(D)$ such that $\int_{\mathbb{S}^1} u d\theta = 0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0$ for $1 \leq k \leq m$, then for any $\varepsilon > 0$ we have*

$$\begin{aligned} \log \int_{\mathbb{S}^1} e^u d\theta &\leq \left(\frac{1}{4\pi N_m\left(S^1\right)} + \varepsilon\right) \|\nabla u\|_{L^2(D)}^2 + c_\varepsilon \\ &= \left(\frac{1}{4\pi(m+1)} + \varepsilon\right) \|\nabla u\|_{L^2(D)}^2 + c_\varepsilon. \end{aligned} \tag{6.11}$$

Note that for $m = 1$, Lemma 6.1 is treated in [OsPS, lemma 2.5]. We can prove Lemma 6.1 by replacing (1.2) with (6.2) and following the approach in Section 2 and Section 3. The detail is left to interested readers.

To continue we denote

$$\mathcal{S}_m = \left\{u \in H^1(D) : \int_{\mathbb{S}^1} u d\theta = 0, \int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0 \text{ for } k = 1, \cdots, m\right\}. \tag{6.12}$$

Let $a \in \left(\frac{1}{4\pi(m+1)}, \frac{1}{4\pi m}\right)$, then it follows from Lemma 6.1 that

$$\inf_{u \in \mathcal{S}_m} \left[a \|\nabla u\|_{L^2(D)}^2 - \log\left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u d\theta\right)\right] \tag{6.13}$$

is achieved.

Let u be a minimizer for (6.13), then u is smooth and for some real numbers β_k and γ_k,

$$\begin{aligned} -\Delta u &= 0 \text{ in } D; \\ 2a \frac{\partial u}{\partial \nu} - \frac{e^u}{\int_{\mathbb{S}^1} e^u d\theta} &= -\frac{1}{2\pi} + \sum_{k=1}^m (\beta_k \cos k\theta + \gamma_k \sin k\theta) e^u. \end{aligned}$$

Here ν is the unit outer normal direction of S^1. Let

$$v = u - \log\left(2a \int_{\mathbb{S}^1} e^u d\theta\right), \tag{6.14}$$

then v is smooth and

$$\begin{aligned} -\Delta v &= 0 \text{ in } D; \\ \frac{\partial v}{\partial \nu} + \frac{1}{4\pi a} &= e^v + \sum_{k=1}^{m} \left(c_k e^{ik\theta} + \overline{c_k} e^{-ik\theta} \right) e^v; \\ \int_{\mathbb{S}^1} e^v e^{ik\theta} d\theta &= 0 \text{ for } k = 1, \cdots, m. \end{aligned}$$

Here $c_1, \cdots, c_m$ are complex constants. Next we claim $c_k = 0$ for all k. For the case $m = 1$, this is proved in [OsPS, lemma 2.6].

Lemma 6.2. *Let $m \in N$, $\alpha > 0$, $v \in C^\infty \left(\overline{D} \right)$ such that $\int_{\mathbb{S}^1} e^v e^{ik\theta} d\theta = 0$ for $k = 1, \cdots, m$ and*

$$\begin{aligned} -\Delta v &= 0 \textit{ in } D; && (6.15) \\ \frac{\partial v}{\partial \nu} + \alpha &= e^v + \sum_{k=1}^{m} \left(c_k e^{ik\theta} + \overline{c_k} e^{-ik\theta} \right) e^v; && (6.16) \end{aligned}$$

here ν is the unit outer normal direction of S^1 and $c_1, \cdots, c_m$ are complex constants, then $c_k = 0$ for $1 \le k \le m$.

Proof. We write

$$\begin{aligned} v|_{\mathbb{S}^1} &= \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad a_k \in C, \overline{a_k} = a_{-k}; \\ e^v|_{\mathbb{S}^1} &= \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}, \quad b_k \in C, \overline{b_k} = b_{-k}. \end{aligned}$$

It follows from the assumption that

$$b_k = 0 \text{ for } 1 \le |k| \le m. \tag{6.17}$$

Using (6.15) and (6.16) we see

$$\sum_{k=-\infty}^{\infty} |k| \, a_k e^{ik\theta} + \alpha = \left(1 + \sum_{j=1}^{m} c_j e^{ij\theta} + \sum_{j=1}^{m} \overline{c_j} e^{-ij\theta} \right) \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}.$$

Compare the constant term on both sides and using (6.17) we get $b_0 = \alpha$. On the other hand, for $k \neq 0$, we have

$$|k| \, a_k = b_k + \sum_{j=1}^{m} c_j b_{k-j} + \sum_{j=1}^{m} \overline{c_j} b_{k+j}. \tag{6.18}$$

Next we observe that

$$\partial_\theta \left(e^v \right) = e^v \partial_\theta v,$$

hence

$$\sum_{k=-\infty}^{\infty} k b_k e^{ik\theta} = \left(\sum_{j=-\infty}^{\infty} j a_j e^{ij\theta} \right) \left(\sum_{k=-\infty}^{\infty} b_k e^{ik\theta} \right).$$

It follows that

$$k b_k = \sum_{j=-\infty}^{\infty} j a_j b_{k-j}. \tag{6.19}$$

Plug (6.18) into (6.19), we get

$$kb_k = \sum_{j=-\infty}^{\infty} \operatorname{sgn}(j) \left[b_j + \sum_{s=1}^{m} c_s b_{j-s} + \sum_{s=1}^{m} \overline{c_s} b_{j+s} \right] b_{k-j}.$$

In particular, for $1 \leq k \leq m$, it becomes

$$\begin{aligned} kb_k &= \sum_{j=1}^{k} b_j b_{k-j} + \sum_{s=1}^{m} c_s \sum_{j=1}^{k+s} b_{j-s} b_{k-j} + \sum_{s=1}^{k} \overline{c_s} \sum_{j=1}^{k-s} b_{j+s} b_{k-j} \\ &\quad + \sum_{s=k+1}^{m} \overline{c_s} \sum_{j=k-s+1}^{0} b_{j+s} b_{k-j}. \end{aligned}$$

Using (6.17) we get $\alpha^2 c_k = 0$, hence $c_k = 0$.

It follows from Lemma 6.2 that the function v defined in (6.14) satisfies

$$\begin{aligned} -\Delta v &= 0 \text{ in } D; \\ \frac{\partial v}{\partial \nu} + \frac{1}{4\pi a} &= e^v \text{ on } S^1. \end{aligned}$$

Since $\frac{1}{4\pi a} \in (m, m+1)$, it follows from [OsPS, lemma 2.3] that v is a constant function. Hence any minimizer of (6.13) must be 0. In another word, for any $u \in \mathcal{S}_m$,

$$\log \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u d\theta \right) \leq a \|\nabla u\|_{L^2(D)}^2 .$$

Let $a \to \frac{1}{4\pi(m+1)}$, we get (6.1).

If $u \in \mathcal{S}_m$ such that

$$\log \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u d\theta \right) = \frac{\|\nabla u\|_{L^2(D)}^2}{4\pi (m+1)},$$

then u is smooth and for some real numbers β_k and γ_k,

$$\begin{aligned} -\Delta u &= 0 \text{ in } D; \\ \frac{1}{2\pi (m+1)} \frac{\partial u}{\partial \nu} - \frac{e^u}{\int_{\mathbb{S}^1} e^u d\theta} &= -\frac{1}{2\pi} + \sum_{k=1}^{m} (\beta_k \cos k\theta + \gamma_k \sin k\theta) e^u. \end{aligned}$$

Let

$$v = u - \log \frac{\int_{\mathbb{S}^1} e^u d\theta}{2\pi (m+1)},$$

it follows from Lemma 6.2 that

$$\begin{aligned} -\Delta v &= 0 \text{ in } D; \\ \frac{\partial v}{\partial \nu} + m + 1 &= e^v \text{ on } S^1. \end{aligned}$$

By [Wa, theorem 7], we can find $\xi \in C$ with $|\xi| < 1$ such that

$$v(z) = \log \frac{(m+1)\left(1 - |\xi|^2\right)}{|1 - \xi z^{m+1}|^2}.$$

Using the fact $\int_{\mathbb{S}^1} u d\theta = 0$, we see $u(z) = \log \frac{1}{|1-\xi z^{m+1}|^2}$.

At last calculation shows for any $\xi \in C$ with $|\xi| < 1$, if we write $u_\xi(z) = \log \frac{1}{|1-\xi z^{m+1}|^2}$, then $u_\xi \in \mathcal{S}_m$ and

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^{u_\xi} d\theta\right) = \log \frac{1}{1-|\xi|^2} = \frac{1}{4\pi(m+1)} \|\nabla u_\xi\|^2_{L^2(D)}.$$

Theorem 6.1 follows.

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