A Sharp Inequality on the Exponentiation of Functions on the Sphere

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Abstract

In this paper we show a new inequality which generalizes to the unit sphere the Lebedev-Milin inequality of the exponentiation of functions on the unit circle. It may also be regarded as the counterpart on the sphere of the second inequality in the Szegö limit theorem on the Toeplitz determinants on the circle. On the other hand, this inequality is also a variant of several classical inequalities of Moser-Trudinger type on the sphere. The inequality incorporates the deviation of the center of mass from the origin into the optimal inequality of Aubin for functions with mass centered at the origin, and improves Onofri's inequality with the contribution of the shifting of the mass center explicitly expressed.

1 Introduction

Let $\mathbb{S}^2$ be the unit sphere and for $u \in H^1(\mathbb{S}^2)$ define

$$F_\alpha(u) = \alpha \int_{\mathbb{S}^2} |\nabla u|^2 d\omega + 2 \int_{\mathbb{S}^2} u d\omega - \log \int_{\mathbb{S}^2} e^{2u} d\omega, \tag{1.1}$$ $\boxed{\texttt{MTAO}}$

where the volume form $d\omega$ is normalized so that $\int_{\mathbb{S}^2} d\omega = 1$. The well-known Moser-Trudinger inequality [13] says that F_α is bounded below if and only if $\alpha \geq 1$. Later on Onofri [14] sharpened Moser-Trudinger inequality and showed that for $\alpha \geq 1$ the best lower bound of F_α is equal to zero. Onofri's inequality was based on an inequality established earlier by Aubin [1] who proved that if F_α is restricted to

$$\mathcal{M} := \{u \in H^1(\mathbb{S}^2) : \int_{\mathbb{S}^2} e^{2u} x_i = 0, \quad i = 1, 2, 3\},$$

then for $\alpha > \frac{1}{2}$, F_α is bounded below and the infimum is attained in $\mathcal{M}$. All these inequalities play crucial roles in the "Nirenberg's problem" of prescribing Gaussian curvature, in particular in the work of Chang and Yang ([5] and [4]). In their effort to prescribe Gaussian curvature without additional assumption on the symmetry of the curvature, Chang and Yang have further improved the above Aubin-Onfri inequality by showing that (see Proposition B in [5]) for α sufficient close but less than 1, the lower bound of F_α again is equal to zero for

u in the class $\mathcal{M}$, their work led to the following conjecture:

Conjecture A. For $\alpha \geq \frac{1}{2}$
$$\inf_{u \in \mathcal{M}} F_\alpha(u) = 0.$$

In 1998, Feldman, Froese, Ghoussoub and Gui [7] proved that this conjecture is true for axially symmetric functions when $\alpha > \frac{16}{25} - \epsilon$. Later the second author and Wei [11], and independently Lin [12] proved Conjecture A for axially symmetric functions. In [8] Ghoussoub and Lin showed that Conjecture A holds true for $\alpha \geq \frac{2}{3} - \epsilon$, for some $\epsilon > 0$. Finally Gui and Moradifam proved in [10] that Conjecture A is indeed true. Actually they [10] obtained something stronger than the conjecture, by showing the following uniqueness result for the corresponding Euler-Lagrange equation for the functional F_α.

MAOTheorem **Theorem 1.1** *The following equation*

$$\alpha \Delta u + \frac{e^{2u}}{\int_{\mathbb{S}^2} e^{2u} d\omega} - 1 = 0 \quad on \quad \mathbb{S}^2 \tag{1.2}$$

standardPD

has only constant solutions for $\frac{1}{2} \leq \alpha < 1$.

2 A Refined Aubin-Onofri Type Inequality

The main result in this paper is to establish a variant of Aubin-Onofri inequality. To motivate the study of such type of inequalities, we first recall the classical Lebedev-Milin inequality on the exponentiation of functions defined on the unit circle $\mathbb{S}^1$, which is in spirit similar to that of the Moser-Trudinger inequality for functions defined on $\mathbb{S}^2$.

Assume on $\mathbb{S}^1 \subset \mathbb{R}^2 \sim \mathbb{C}$

$$u(z) = \sum_{k=1}^{\infty} a_k z^k, \quad e^{u(z)} = \sum_{k=0}^{\infty} \beta_k z^k.$$

Then the Lebedev-Milin inequality on the unit circle ([6]) states

$$\log\left(\sum_{k=0}^{\infty} |\beta_k|^2\right) \leq \sum_{k=1}^{\infty} k|a_k|^2 \tag{2.1}$$

LM

if the right hand side is finite , and equality holds if and only if $a_k = \gamma^k/k$ for some $\gamma \in \mathbb{C}$ with $|\gamma| < 1$. This is well known in the community of univalent functions, in particular in connection with Bieberbach conjecture.

Denote D the unit disc on $\mathbb{R}^2$. For any real function u defined on the unit circle, we recall that the right hand side of (2.1) is indeed $H^{\frac{1}{2}}(\mathbb{S}^1)$ norm of u, which can also be identified as the $H^1(D)$ norm of the harmonic extension, which we denote again by u, on the disc D. Then the classical Lebedev-Milin inequality may be written as

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right) - \frac{1}{2\pi}\int_{\mathbb{S}^1} u d\theta \leq \frac{1}{4\pi}||\nabla u||^2_{L^2(D)}. \tag{2.2}$$

ML

It turns out Lebedev-Milin inequality is the "first step" of a string of monotonically increasing inequalities in the Szegö Limit Theorem ([9], 5.5a) on Toeplitz determinants. Here we will just quote the second inequality in the Szegö limit theorem:

$$\log(|\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta|^2 - |\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u e^{i\theta} d\theta|^2) - \frac{1}{\pi}\int_{\mathbb{S}^1} u d\theta \leq \frac{1}{4\pi}||\nabla u||^2_{L^2(D)}. \qquad (2.3) \quad \boxed{\texttt{GS2}}$$

One notes that in the special case when $\int_{\mathbb{S}^1} e^u e^{i\theta} d\theta = 0$, as a direct consequence of the above inequality we have

$$\log(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta) - \frac{1}{2\pi}\int_{\mathbb{S}^1} u d\theta \leq \frac{1}{8\pi}||\nabla u||^2_{L^2(D)}. \qquad (2.4) \quad \boxed{\texttt{GS2cor}}$$

Indeed this special form of the inequality was independently verified by Osgood, Phillips, Sarnak [15] and was used in their study of isospectral compactness for metrics defined on compact surfaces. It was later pointed out by H. Widom ([16]) that it is a direct consequence of the Szegö Limit Theorem. We remark that actually Widom has also pointed out that for all integer k, there is a string of such inequalities for functions u with $\int_{\mathbb{S}^1} e^u e^{ij\theta} d\theta = 0$ for all $1 \leq j \leq k$. In a recent work([3]), Chang and Hang have further explored this angle and established a weaker form of such inequalities for functions defined on the 2-sphere with vanishing higher order of moments.

The relevance to us is the apparent comparison of the inequality of (2.4) on $\mathbb{S}^1$ as compared to Conjecture A in the introduction for functions defined on $\mathbb{S}^2$. This leads us to ask the question if there a corresponding inequality on $\mathbb{S}^2$ similar to that of (2.3), which in the special case when u is in $\mathcal{M}$ reduces to the statement in Conjecture A.

Motivated by this, we consider the following family of functionals in $H^1(\mathbb{S}^2)$:

$$I_\alpha(u) = \alpha\int_{\mathbb{S}^2} |\nabla u|^2 d\omega + 2\int_{\mathbb{S}^2} u d\omega - \frac{1}{2}\log[(\int_{\mathbb{S}^2} e^{2u} d\omega)^2 - \sum_{i=1}^{3}(\int_{\mathbb{S}^2} e^{2u} x_i d\omega)^2] \qquad (2.5) \quad \boxed{\texttt{I\_alpha}}$$

where $\alpha > 0$.

The question we are asking is what is the minimum value of α for which the functional $F_\alpha(u)$ stays non-negative for all functions $u \in H^1(\mathbb{S}^2)$. One notices that if such a minimum value α is $\frac{1}{2}$, then we would recover the statement in Conjecture A. But to our surprise, the answer of the question is actually no, and the minimum value of such α is actually $\frac{2}{3}$. We will present here our analysis, and state the following result as our main theorem.

$\boxed{\texttt{main}}$ **Theorem 2.1** *For any $\alpha > 0$, we have*

$$I_\alpha(u) \geq (\alpha - \frac{2}{3})\int_{\mathbb{S}^2} |\nabla u|^2 d\omega, \quad \forall u \in H^1(\mathbb{S}^2). \qquad (2.6) \quad \boxed{\texttt{ineq}}$$

In particular, when $\alpha \geq \frac{2}{3}$ we have

$$I_\alpha(u) \geq 0, \quad \forall u \in H^1(\mathbb{S}^2). \qquad (2.7) \quad \boxed{\texttt{ineq1}}$$

Furthermore, for $0 < \alpha < \frac{2}{3}$, $\inf_{H^1(\mathbb{S}^2)} I_\alpha(u) = -\infty$.

In the rest of the section, we will present the proof of the above theorem. Due to the invariance of $I_\alpha(u)$ by a constant addition, we may confine our discussion in the normalized space

$$\mathcal{H} = \{u \in H^1(\mathbb{S}^2) : \int_{\mathbb{S}^2} e^{2u} d\omega = 1\}. \tag{2.8}$$

normalized

The strategy is to first study the Euler-Lagrange equation of the functional I_α, assuming the critical point is obtained. It turns out for the special value $\alpha = \frac{2}{3}$, for each point $\vec{a}$ in the unit ball $B_1 \subset \mathbb{R}^3$, there is a unique solution $u \in \mathcal{H}$, which we can write down explicitly, of the Euler-Lagrange equation, with the center of mass of e^{2u} being at $\vec{a}$. Based on this analysis, we then study the minimum of $I_\alpha(u)$ over the class of u with a fixed center of mass and verify that it is achieved for each $\alpha > \frac{1}{2}$. Although the infimum of $I_\alpha(u)$ tends to negative infinity as $\vec{a}$ goes to the unit sphere $\mathbb{S}^2 = \partial B_1$ when $\frac{1}{2} < \alpha < \frac{2}{3}$, it turns out that $I_\alpha(u) \geq 0$ for all $u \in H^1(\mathbb{S}^2)$ when $\alpha \geq \frac{2}{3}$, due to the complete understanding of the critical points of $I_{2/3}(u)$ in $\mathcal{H}$ with the center constrained.

We now begin the analysis. For each $u \in \mathcal{H}$, denote

$$a_i = \int_{\mathbb{S}^2} e^{2u} x_i d\omega, \quad i = 1, 2, 3. \tag{2.9}$$

a_i

euler **Proposition 2.1** *The Euler Lagrange equation for the functional I_α in $\mathcal{H}$ is*

$$\alpha \Delta u + \frac{1 - \sum_{i=1}^3 a_i x_i}{1 - \sum_{i=1}^3 a_i^2} e^{2u} - 1 = 0 \quad on \quad \mathbb{S}^2. \tag{2.10}$$

simple

We now study the solution of equation (2.10).

eq-main **Proposition 2.2** *i) When $\alpha \in (0,1)$ and $\alpha \neq \frac{2}{3}$, equation* (2.10) *has only zero solution in $\mathcal{H}$;*

ii) When $\alpha = \frac{2}{3}$, for any $\vec{a} = (a_1, a_2, a_3) \in B_1$, there is a unique solution u to equation (2.10) *in $\mathcal{H}$ such that* (2.9) *holds. In particular, u is axially symmetric about $\vec{a}$ if $\vec{a} \neq (0,0,0)$. After a proper rotation, the solution u is explicitly given by the formula in* (2.17) *below.*

Proof :

To investigate (2.10), recall the Kazdan-Warner condition for the Gaussian curvature equation:

$$\Delta u + K(x) e^{2u} = 1 \text{ on } \mathbb{S}^2, \tag{2.11}$$

ageneral

then

$$\int_{\mathbb{S}^2} (\nabla K(x) \cdot \nabla x_j) e^{2u} d\omega = 0 \text{ for each j= 1,2, 3.} \tag{2.12}$$

KW

If u satisfies the (2.10), then

$$K(x) = \frac{1}{\alpha} \frac{(1 - \sum_{i=1}^3 a_i x_i)}{(1 - \sum_{i=1}^3 a_i^2)} + (1 - \frac{1}{\alpha}) e^{-2u}. \tag{2.13}$$

gaussian

Substituting (2.13) into (2.12), we obtain for each $j = 1, 2, 3$,

$$\frac{1}{\alpha}\frac{1}{(1-\sum_{i=1}^{3} a_i^2)}\int_{\mathbb{S}^2}(\sum_i a_i(\nabla x_i\cdot\nabla x_j))e^{2u}d\omega \;=\; (-2)(1-\frac{1}{\cdot}\alpha)\int_{\mathbb{S}^2}(\nabla u\cdot\nabla x_j)d\omega.$$

Integrate by part the last term, and substitute the term Δu in equation (2.10) and simplify. We then get

$$\sum_i a_i\int_{\mathbb{S}^2}\nabla x_i\cdot\nabla x_j e^{2u}d\omega \;=\; 2(1-\frac{1}{\alpha})\sum_i a_i\int_{\mathbb{S}^2}x_ix_je^{2u}d\omega - 2(1-\frac{1}{\alpha})a_j.$$

We now notice that $\nabla x_i\cdot\nabla x_j = -x_ix_j$ when $i \neq j$, $|\nabla x_j|^2 = 1 - x_j^2$, thus we get

$$(3-\frac{2}{\alpha})a_j\int_{\mathbb{S}^2}e^{2u}d\omega = (3-\frac{2}{\alpha})\sum_i a_i\int_{\mathbb{S}^2}x_ix_je^{2u}\omega.$$

Multiply the above formula by a_j and sum over j, we get

$$(3-\frac{2}{\alpha})\int_{\mathbb{S}^2}\Big(\sum_j a_j^2 - |\sum_i a_ix_i|^2\Big)e^{2u}d\omega = 0.$$

This implies that if $\alpha \neq \frac{2}{3}$, there holds $a_i = 0, i = 1, 2, 3$, since

$$(\sum_{i=1}^{3} a_ix_i)^2 \leq \sum_{i=1}^{3} a_i^2 \;\text{ on }\; \mathbb{S}^2$$

and the equality only holds when $\vec{a} = (a_1, a_1, a_3)$ is the zero vector or x is parallel to the vector (a_1, a_1, a_3) if it is not the zero vector. Therefore, we conclude that when $\alpha \in (0, \frac{2}{3}) \cup (\frac{2}{3}, 1)$, the equation (2.10) have only zero solution, in view of Theorem 1.1.

For $\alpha = \frac{2}{3}$, we assume that u is a solution to the coupled equations (2.9) and (2.10). Without of loss of generality, we may assume that $(a_1, a_2, a_3) = (0, 0, a)$ with $a \in (0, 1)$ and consider

$$\frac{2}{3}\Delta u + \frac{1-ax_3}{1-a^2}e^{2u} - 1 = 0 \;\text{ on }\; \mathbb{S}^2. \tag{2.14}$$

eq-a

We shall use the stereographic projection to transform the equation to be on $\mathbb{R}^2$. Let Π be the stereographic projection $\mathbb{S}^2 \to \mathbb{R}^2$ with respect to the north pole $N = (0, 0, 1)$:

$$y = \Pi(x) := \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

Note that

$$x_3 = \frac{|y|^2-1}{|y|^2+1}, \quad d\omega = \frac{dy}{\pi(|y|^2+1)^2}$$

Suppose u is a solution of (2.14), and let

$$w(y) := u(\Pi^{-1}(y)) - \frac{3}{2}\ln(1+|y|^2) \;\text{ for }\; y \in \mathbb{R}^2.$$

Then w satisfies

$$\Delta w + \frac{6}{1+a}(\mu^2 + |y|^2)e^{2w} = 0 \ \text{ in } \ \mathbb{R}^2 \tag{2.15}$$ $\boxed{\texttt{plane}}$

where $\mu^2 = \frac{1+a}{1-a} > 1, b > 0$ and

$$\int_{\mathbb{R}^2} (\mu^2 + |y|^2)e^{2w} dy = (1+a)\pi. \tag{2.16}$$ $\boxed{\texttt{total}}$

Now it is easy to verify directly that

$$w(y) = -\frac{3}{2}\ln(\mu^2 + |y|^2) + 2\ln\mu + \frac{1}{2}\ln\frac{2}{1+\mu^2}$$

is a solution to (2.15) and (2.16), and hence $u(x)$ defined by

$$u(x) = u(\Pi^{-1}(y)) := \frac{3}{2}\ln\frac{1+|y|^2}{\mu^2+|y|^2} + 2\ln\mu + \frac{1}{2}\ln\frac{2}{1+\mu^2} \tag{2.17}$$ $\boxed{\texttt{solution}}$

is a solution to (2.14). It is also easy to compute that $\int_{\mathbb{S}^2} e^{2u} d\omega = 1$ and

$$\int_{\mathbb{S}^2} e^{2u} x_3 d\omega = \int_{\mathbb{R}^2} e^{2u(\Pi^{-1}(y))} (\frac{|y|^2-1}{|y|^2+1}) \frac{dy}{\pi(|y|^2+1)^2} = a,$$

and therefore u is a solution to (2.9) and (2.10) with $(a_1, a_2, a_3) = (0, 0, a)$.

To show the uniqueness of the solution to (2.14), we will recall a general result regarding the radial symmetry of solutions. Assume $u \in C^2(\mathbb{R}^2)$ satisfies

$$\Delta u + \mathcal{K}(|y|)e^{2u} = 0 \ \text{ in } \ \mathbb{R}^2, \tag{2.18}$$ $\boxed{\texttt{general}}$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathcal{K}(|y|)e^{2u} dy = \beta < \infty, \tag{2.19}$$ $\boxed{\texttt{16Pi}}$

where $\mathcal{K}(y) := \mathcal{K}(|y|) \in C^2(\mathbb{R}^2)$ is a non constant positive function satisfying

$$\begin{array}{ll} (K1) & \Delta \ln(\mathcal{K}(|y|)) \geq 0, \quad y \in \mathbb{R}^2 \\ (K2) & \lim\limits_{|y|\to\infty} \dfrac{|y|\mathcal{K}'(|y|)}{\mathcal{K}(|y|)} = 2l > 0, \quad y \in \mathbb{R}^2. \end{array}$$

The following general symmetry result is proven in [10].

$\boxed{\texttt{th-general}}$ **Proposition 2.3** *Assume that* $\mathcal{K}(y) = \mathcal{K}(|y|) > 0$ *satisfies* $(K1)-(K2)$, *and* u *is a solution to* (2.18)-(2.19) *with* $l + 1 < \beta \leq 4$. *Then* u *must be radially symmetric.*

Applying Proposition 2.3 to (2.15) and (2.16) with $l = 1, \beta = 3$, we conclude that the solution to (2.15) and (2.16) must be radially symmetric. Furthermore, such a radial solution must be unique by Theorem 1.5 of [12]. Therefore we have finished the proof of the Proposition (2.2). ■

Proof of Theorem 2.1

For any $\vec{a} = (a_1, a_2, a_3) \in B_1 := \{|a| < 1\} \subset \mathbb{R}^2$, let us define

$$\mathcal{M}_{\vec{a}} := \{u \in \mathcal{H} \subset H^1(\mathbb{S}^2) : \int_{\mathbb{S}^2} e^{2u} x_i = a_i, \quad i = 1, 2, 3\}. \tag{2.20}$$

minimizing

First we consider a constrained minimizing problem on $\mathcal{M}_{\vec{a}}$:

$$m(\alpha, \vec{a}) := \min_{u \in \mathcal{M}_{\vec{a}}} I_\alpha(u)$$

and recall the following compactness result:

compact **Proposition 2.4** *For any $\alpha > \frac{1}{2}, \vec{a} = (a_1, a_2, a_3) \in B_1$, there exists $C_{\alpha,\vec{a}} \in \mathbb{R}$ such that*

$$I_\alpha(u) \geq C_{\alpha,\vec{a}}, \quad \forall u \in \mathcal{M}_{\vec{a}}. \tag{2.21}$$

lowerbound

Furthermore, there is a positive constant $M_{\alpha,|\vec{a}|,C} > 0$ depending only on $\alpha, |\vec{a}| < 1$ and C such that $\|u\|_{H^1(\mathbb{S}^2)} \leq M_{\alpha,|\vec{a}|,C}$ in the sub level set $I^C_{\alpha,\vec{a}} := \{u \in \mathcal{M}_{\vec{a}}, I_\alpha(u) \leq C\}$.

Proof:

This result may be known to researchers in the area, although it seems not stated or proven explicitly in the literature. Here we will give a sketch of proof following Proposition 2.1 of [3].

Assume that for some $\alpha > \frac{1}{2}, \vec{a} = (a_1, a_2, a_3) \in B_1$, there is a sequence $u_k \in \mathcal{M}_{\vec{a}}, k = 1, 2, \cdots$ such that $I_\alpha(u_k) \to -\infty$ as $k \to \infty$. Then

$$\bar{u}_k := \int_{\mathbb{S}^2} u_k d\omega \to -\infty, \quad k \to \infty.$$

By the classical Moser-Trudinger inequality, we have

$$\int_{\mathbb{S}^2} |\nabla u_k|^2 d\omega \geq -2\bar{u}_k \to \infty, \quad k \to \infty.$$

Let $m_k = (\int_{\mathbb{S}^2} |\nabla u_k|^2 d\omega)^{\frac{1}{2}}$ and

$$v_k = \frac{u_k - \bar{u}_k}{m_k}.$$

Then, when k is sufficiently large, by the assumption v_k satisfies

$$\ln(\int_{\mathbb{S}^2} e^{2m_k v_k} d\omega) = -2\bar{u}_k \geq \alpha \int_{\mathbb{S}^2} |\nabla u_k|^2 d\omega = \alpha m_k^2.$$

Assume that v_k converges weakly to v in $H^1(\mathbb{S}^2)$ as $k \to \infty$ and

$$|\nabla v_k|^2 d\omega \to |\nabla v|^2 d\omega + \sigma \quad \text{and} \quad \frac{e^{2m_k v_k} d\omega}{\int_{\mathbb{S}^2} e^{2m_k v_k} d\omega} \to \nu, \quad k \to \infty.$$

in measure, where $\sigma(\mathbb{S}^2) = \nu(\mathbb{S}^2) = 1$. Then, by Proposition 2.1 in [3] we have,

$$\{x \in \mathbb{S}^2 : \sigma(x) \geq \alpha\} = \{P\} \quad \text{and} \quad \nu = \delta_P$$

for some $P \in \mathbb{S}^2$ since $\alpha > \frac{1}{2}$. Note that here we have a normalized area of the unit sphere being 1 with the measure ω while in [3] the area of the unit sphere is 4π.

This leads to a contradiction that

$$\vec{a} = \int_{\mathbb{S}^2} e^{2u_k} x d\omega = \frac{\int_{\mathbb{S}^2} e^{2m_k v_k} x d\omega}{\int_{\mathbb{S}^2} e^{2m_k v_k} d\omega} \to P, \quad k \to \infty.$$

The argument also shows that $\|\nabla u\|_{L^2(\mathbb{S}^2)}$ is bounded in the sub level set $I^C_{\alpha,\vec{a}} := \{u \in \mathcal{M}_{\vec{a}}, I_\alpha(u) \le C\}$ for any fixed $C \in \mathbb{R}$. The Jensen inequality and the convexity of the exponential function as well as (4.2) lead to the boundedness of $\bar{u} = \int_{\mathbb{S}^2} u d\omega$ in the set $I^C_{\alpha,\vec{a}}$.

Therefore, Proposition 2.4 holds. ∎

From here it is standard to show that there exists a minimizer $u_{\alpha,\vec{a}} \in \mathcal{M}_{\vec{a}}$ of (2.20) satisfying

$$\alpha \Delta u + e^{2u}(\rho - \sum_{i=1}^{3} \beta_i x_i) = 1, \quad x \in \mathbb{S}^2 \tag{2.22}$$

minimizer

for some $\rho \in \mathbb{R}$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$.

To be more precise, for a fixed $\alpha > \frac{1}{2}$ there is a minimizing sequence of $u_k \in \mathcal{M}_{\vec{a}}, k = 1, 2, \cdots$ of I_α such that u_k is bounded in $H^1(\mathbb{S}^2)$ and u_k converges weakly to $u_{\alpha,\vec{a}}$ in $H^1(\mathbb{S}^2)$ and $u_{\alpha,\vec{a}} \in \mathcal{M}_{\vec{a}}$. (See, e.g., the proof of Theorem 5.1 of [3].) Hence $u_{\alpha,\vec{a}}$ is a minimizer of $\min_{u \in \mathcal{M}_{\vec{a}}} I_\alpha(u)$. It is easy to see that

$$\rho = 1 + \sum_{i=1}^{3} \beta_i a_i.$$

Using Kazdan-Warner condition (2.12), we obtain

$$2(\frac{1}{\alpha} - \frac{3}{2}) \sum_{i=1}^{3} \beta_i \int_{\mathbb{S}^2} x_i x_j e^{2u} d\omega = 2(\frac{1}{\alpha} - 1)\rho a_j - \beta_j, \quad j = 1, 2, 3. \tag{2.23}$$

condition

In particular, when $\alpha = \frac{2}{3}$, we have

$$\beta_j = \frac{a_j}{1 - |\vec{a}|^2}, \quad j = 1, 2, 3.$$

Then equation (2.22) is equivalent to (2.10) when $\alpha = \frac{2}{3}$.

After a proper rotation so that $\vec{a}$ points to the north pole and using the stereographic project $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$, the solution is uniquely determined by

$$u_{\frac{2}{3},|\vec{a}|}(x) := \frac{3}{2} \ln \frac{1 + |y|^2}{\mu^2 + |y|^2} + 2 \ln \mu + \frac{1}{2} \ln \frac{2}{1 + \mu^2}$$

where $\mu^2 = \frac{1+|\vec{a}|}{1-|\vec{a}|} > 1$.

Hence, by direct computations we have

$$\begin{aligned}\int_{\mathbb{S}^2} |\nabla u_{\frac{2}{3},|\vec{a}|}|^2 d\omega &= \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{9(\mu^2-1)^2}{4}\frac{|y|^2}{(|y|^2+1)^2(|y|^2+\mu^2)^2}dy \\ &= \frac{9}{4}\frac{(\mu^2+1)\ln(\mu^2)-2(\mu^2-1)}{\mu^2-1},\end{aligned} \tag{2.24}$$

gradient

and

$$\begin{aligned}\int_{\mathbb{S}^2} u_{\frac{2}{3},|\vec{a}|} d\omega &= \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{3}{2}\ln(\frac{1+|y|^2}{\mu^2+|y|^2})\frac{4}{(1+|y|^2)^2}dy + \ln(\mu^2) + \frac{1}{2}\ln(\frac{2}{1+\mu^2}) \\ &= -\frac{3}{2}\frac{\mu^2\ln(\mu^2)+\mu^2-1}{\mu^2-1} + \ln(\mu^2) + \frac{1}{2}\ln(\frac{2}{1+\mu^2}).\end{aligned}$$

Then we can calculate

$$\min_{u\in\mathcal{M}_{\vec{a}}} I_{\frac{2}{3}}(u) = I_{\frac{2}{3}}(u_{\frac{2}{3},|\vec{a}|}) = 0.$$

Furthermore,

$$\begin{aligned}\min_{u\in\mathcal{M}_{\vec{a}}} I_\alpha(u) \le I_\alpha(u_{\frac{2}{3},|\vec{a}|}) &= (\alpha-\frac{2}{3})\int_{\mathbb{S}^2} |\nabla u_{\frac{2}{3},|\vec{a}|}|^2 d\omega \\ &= (\alpha-\frac{2}{3})\frac{9}{4|\vec{a}|}\big(-2|\vec{a}| + \ln\frac{1+|\vec{a}|}{1-|\vec{a}|}\big).\end{aligned} \tag{2.25}$$

upper

In particular, if $\alpha < \frac{2}{3}$ we have that $\min_{u\in\mathcal{M}_{\vec{a}}} I_\alpha(u) \to -\infty$ as $|\vec{a}| \to 1$. This establishes the proof of Theorem 2.1. ■

3 Uniqueness and symmetry

For a better understanding of $I_\alpha(u)$, particularly for $\alpha \ne \frac{2}{3}$, we need to consider the minimizer $u_{\alpha,\vec{a}}$ in (2.22) more closely.

First, we can rotate the coordinates properly so that $\beta_1 = \beta_2 = 0$. From (2.23) we see that $\beta_3 \ne 0$ if $\vec{a} \ne (0,0,0)$. Without loss of generality, we assume that $a_3 \ge 0$. In view of (2.23), we have in particular $\rho = 1 + \beta_3 a_3$ and (2.22) becomes

$$\alpha\Delta u + e^{2u}(1+\beta_3(a_3-x_3)) = 1, \quad x\in\mathbb{S}^2. \tag{3.1}$$

reduced

Also (2.23) is reduced to

$$2(\frac{1}{\alpha}-\frac{3}{2})\beta_3\int_{\mathbb{S}^2}(x_3)^2 e^{2u_{\alpha,\vec{a}}}d\omega = 2(\frac{1}{\alpha}-1)\rho a_3 - \beta_3. \tag{3.2}$$

beta_3

Using (3.2) and the fact that

$$(a_3)^2 = (\int_{\mathbb{S}^2} x_3 e^{2u_{\alpha,\vec{a}}})^2 d\omega \le \int_{\mathbb{S}^2}(x_3)^2 e^{2u_{\alpha,\vec{a}}}d\omega \le 1,$$

we can obtain

$$\frac{a_3}{1-a_3^2} \le \beta_3 \le \frac{2(\frac{1}{\alpha}-1)a_3}{1-a_3^2}, \qquad \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}] \tag{3.3}$$

beta-small

and

$$\frac{a_3}{1-a_3^2} \ge \beta_3 \ge \frac{2(\frac{1}{\alpha}-1)a_3}{1-a_3^2}, \qquad \text{if } \alpha \in [\frac{2}{3}, 1]. \tag{3.4}$$

beta-large

We first show that when $\alpha \in (1/2, 1)$ is fixed, the solution set $\{u_{\alpha,\vec{a}}\}$ with parameters $\vec{a}$ has a simple structure near the trivial solution $u = 0$ with $(\alpha, \vec{a}) = (\alpha, (0, 0, 0))$.

iqueness_a **Proposition 3.1** *Fix $\alpha \in (1/2, 1)$. There is a constant $\delta(\alpha) > 0$ sufficiently small such that when $0 < |\vec{a}| < \delta(\alpha)$, (3.1) has a unique solution $u_{\alpha,\vec{a}}$ in $\mathcal{M}_a := \mathcal{M}_{\vec{a}}$, which is therefore axially symmetric around $\vec{a}$.*

Proof. We only need to consider the special case $\vec{a} = (0, 0, a_3)$ after a proper rotation. From (3.3) and (3.4), we know that $\beta \to 0$, as $a_3 \to 0$ and hence $u_{\alpha,\vec{a}}$ converges to the trivial solution $u = 0$ as a_3 goes to zero. Furthermore, from (3.2) we obtain

$$\lim_{a_3 \to 0} \frac{\beta}{a_3} = 3(1-\alpha).$$

Suppose there is a sequence of $\{a^{(k)} := a_3^{(k)}, k = 1, 2, \cdots\}$ with $|a^{(k)}| \to 0$ as $k \to \infty$ such that $\min_{u \in \mathcal{M}_{\vec{a}}} I_\alpha(u)$ has two distinct solutions $u_1^{(k)}, u_2^{(k)}$ which satisfies (3.1) with $\beta_1^{(k)}, \beta_2^{(k)}$ respectively.

From (3.3) and (3.4), we know that $\beta_1^{(k)}, \beta_2^{(k)} \to 0$, as $k \to \infty$ and hence $u_1^{(k)}, u_2^{(k)}$ converge in $C^2(\mathbb{S}^2)$ to the trivial solution $u = 0$ as k goes to infinity, due to the uniqueness result Theorem 1.1. Furthermore, from (3.2) we obtain

$$\lim_{k \to \infty} \frac{\beta_i^{(k)}}{a^{(k)}} = 3(1-\alpha), \quad i = 1, 2.$$

Let $w_k := u_1^{(k)} - u_2^{(k)}$, and $m_k := \|u_1^{(k)} - u_2^{(k)}\|_{L^\infty(\mathbb{S}^2)}$. Note that w_k satisfies

$$\alpha \Delta w_k + (e^{2u_1^{(k)}} - e^{2u_2^{(k)}})\big(1 + \beta_1^{(k)}(a^{(k)} - x_3)\big) = e^{2u_2^{(k)}}(\beta_2^{(k)} - \beta_1^{(k)})(a^{(k)} - x_3).$$

Multiplying the above equation by $a^{(k)} - x_3$ and integrating on $\mathbb{S}^2$, we can obtain

$$|\beta_1^{(k)} - \beta_2^{(k)}| \le C\|u_1^{(k)} - u_2^{(k)}\|_{L^\infty(\mathbb{S}^2)}$$

for some positive constant C.

Hence, after taking a proper subsequence, we know that w_k/m_k converges in $C^2(\mathbb{S}^2)$ to some nontrivial function $\phi \in C^2(\mathbb{S}^2)$, and $(\beta_1^{(k)} - \beta_2^{(k)})/m_k$ converges to a constant $\mu_0 \in \mathbb{R}$. Furthermore, we have

$$\int_{\mathbb{S}^2} \phi d\omega = \int_{\mathbb{S}^2} \phi x_3 d\omega = 0$$

and

$$\alpha \Delta \phi + 2\phi + \mu_0 x_3 = 0, \quad x \in \mathbb{S}^2.$$

Multiplying the above equation by x_3 and integrating on $\mathbb{S}^2$, we obtain $\mu_0 = 0$. Since $\alpha \in (1/2, 1)$ and the first and second eigenvalue of the Laplacian on $\mathbb{S}^2$ are $\lambda_1 = 2, \lambda_2 = 6$ respectively, this leads to a contradiction. The proposition is proven. ■

Next we shall show a uniqueness result for α close to $\frac{2}{3}$ when $\vec{a}$ is fixed.

uniqueness **Proposition 3.2** *Fix $\vec{a} \in B_1$. There is a constant $\delta(\vec{a}) > 0$ sufficiently small such that when $|\alpha - \frac{2}{3}| < \delta$, (3.1) has a unique solution in $\mathcal{M}_a := \mathcal{M}_{\vec{a}}$, which is therefore axially symmetric.*

Proof. We only need to consider solutions to (3.1) in $\mathcal{M}_{\vec{a}}$ with possible different β_3. Assume the contrary that there is a sequence of $\alpha_k, k = 1, 2, \cdots$ such that $\alpha_k \to \frac{2}{3}$ as $k \to \infty$, and (3.1) has distinct solutions $u_{1,k}, u_{2,k}$ corresponding to possibly distinct values of $\beta_3 = \beta_k^1, \beta_k^2$ and distinct rotations $\vec{a}_k^1, \vec{a}_k^2$ of $\vec{a}$ respectively. It is well-known that these solutions are smooth and uniformly bounded. By the uniqueness of solution to (2.14), it is easy to see that $\vec{a}_k^1, \vec{a}_k^2$ converge to $\vec{a} = (0, 0, a_3)$, and β_k^1, β_k^2 converge to $\beta_3 = \frac{|\vec{a}|}{1-|\vec{a}|^2} = \frac{a_3}{1-a_3^2}$.

In view of (2.23), it is also easy to see that

$$|\beta_k^1 - \beta_k^2| \le C(|\vec{a}|)|\alpha - \frac{2}{3}| \times \|u_{1,k} - u_{2,k}\|_{L^\infty(\mathbb{S}^2)}$$

for some positive constant C depending only on $|\vec{a}| \in (0, 1)$.

Let

$$\phi_k = \frac{u_{1,k} - u_{2,k}}{\|u_{1,k} - u_{2,k}\|_{L^\infty(\mathbb{S}^2)}}.$$

It is standard to verify that $u_{1,k}, u_{2,k}$ converges to $u_{\frac{2}{3},\vec{a}}$ in $C^2(\mathbb{S}^2)$ with $\vec{a} = (0, 0, a_3)$, as $k \to \infty$, and ϕ_k converges, after passing to a subsequence, in $C^2(\mathbb{S}^2)$ to ϕ with $||\phi||_{L^\infty(\mathbb{S}^2)} = 1$. Furthermore, ϕ satisfies (5.2) and the linearized equation

$$\frac{2}{3}\Delta\phi + \frac{2(1 - a_3^2)}{(1 - a_3x_3)^2}\phi = 0, \quad x \in \mathbb{S}^2. \tag{3.5}$$ linear3.5

Now consider the eigenvalue problem

$$\Delta\phi + \frac{\lambda(1 - a^2)}{(1 - ax_3)^2}\phi = 0, \quad x \in \mathbb{S}^2. \tag{3.6}$$ eigenvalue

for a fixed $a \in (0, 1)$. Note that for any $a \in (0, 1)$, the transformation $T_a : \mathbb{S}^2 \to \mathbb{S}^2$ given by

$$T_a(x) = (\frac{\sqrt{1 - a^2}x_1}{1 - ax_3}, \frac{\sqrt{1 - a^2}x_1}{1 - ax_3}, \frac{x_3 - a}{1 - ax_3})$$

is a conformal transformation. Indeed,

$$T_a = \Pi^{-1}(\sqrt{\frac{1 - a}{1 + a}}\Pi) : \mathbb{S}^2 \to \mathbb{S}^2$$

and

$$det(dT_a) = \frac{(1 - a^2)}{(1 - ax_3)^2}.$$

Then we observe that $P(T_a(x))$ is an eigenfunction to (3.6) if and only if $P(x)$ is a spherical harmonics. Therefore, (3.6) has only eigenvalues $\lambda = m(m+1)$ for a nonnegative integer m. This leads to a contradiction to (3.5) since $\lambda = 3$ is not an eigenvalue of (3.6). The proof is complete. ■

We can show the axial symmetry of a minimizer to $\min_{u\in\mathcal{M}_{\vec{a}}} I_\alpha(u)$ for most cases, though it is still not completely resolved whether a given minimizer is always axially symmetric.

symmetry **Proposition 3.3** *Fix $\vec{a} \in B_1$, assume that for $\alpha > 1/2$, $u_{\alpha,\vec{a}}$ is a solution to (3.1) in $\mathcal{M}_a := \mathcal{M}_{\vec{a}}$. Then $u_{\alpha,\vec{a}}$ must be axially symmetric when either i) $\alpha \in (1/2, 2/3]$ or ii) $\alpha \geq 1$ or iii) $\alpha \in (2/3, 1)$ and $|\vec{a}| \leq \frac{1-\alpha}{2\alpha-1}$.*

Proof. For this purpose, we choose the stereographic project $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$ from the north pole $N = (0, 0, 1)$. By (3.3) and (3.4), we have

$$0 \leq \beta_3 < \frac{1}{1-a_3}, \quad \rho - \beta_3 > 0.$$

Set

$$w_{\alpha,\vec{a}}(y) := u_{\alpha,\vec{a}}(\Pi^{-1}(y)) - \frac{1}{\alpha}\ln(1+|y|^2) + \frac{1}{2}\ln(\frac{4(\rho-\beta_3)}{\alpha}) \ \text{ for } \ y \in \mathbb{R}^2.$$

Let μ be a positive constant with $\mu^2 = \frac{\rho+\beta_3}{\rho-\beta_3} > 1$. Then $w_{\alpha,\vec{a}}$ satisfies

$$\Delta w + \mathcal{K}(|y|)e^{2w} = 0 \ \text{ in } \ \mathbb{R}^2 \tag{3.7}$$ general-pl

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathcal{K}(|y|)e^{2w}dy = \frac{2}{\alpha} \tag{3.8}$$ general-to

where

$$\mathcal{K}(|y|) := (\mu^2 + |y|^2)(1+|y|^2)^{\frac{2}{\alpha}-3}.$$

i) When $\frac{1}{2} < \alpha < \frac{2}{3}$, it is easy to see that $\mathcal{K}(|y|)$ satisfies $(K1) - (K2)$ with $l = \frac{2}{\alpha} - 2$. By Proposition 2.3, we know that $w_{\alpha,\vec{a}}(y)$ is radially symmetric and hence $u_{\alpha,\vec{a}}(y)$ must be axially symmetric and $a_1 = a_2 = 0$.

ii) When $\alpha > 1$, then $\mathcal{K} > 0$ is not constant and decreasing in $r = |y|$. The standard moving plane method can lead to the radial symmetry of $w_{\alpha,\vec{a}}(y)$. Indeed, the radially symmetric solution is also unique (see Theorem 1.4 of [12]).

When $\alpha = 1$, by (3.2), we know that $\beta_3 = 0$ and hence (3.1) becomes (1.2) with $\alpha = 1$. It is well known that there is a unique solution to $u_{1,\vec{a}} \in \mathcal{M}_{\vec{a}}$ which is axially symmetric about $\vec{a}$.

iii) When $\frac{2}{3} < \alpha < 1$, if $\mu^2(3 - \frac{2}{\alpha}) \leq 1$ we have

$$\Delta \ln \mathcal{K}(|y|) = \frac{4[\mu(r^2+1) + \sqrt{3-\frac{2}{\alpha}}(r^2+\mu^2)]}{(r^2+1)^2(r^2+\mu^2)^2}\Big(\mu(r^2+1) - \sqrt{3-\frac{2}{\alpha}}(r^2+\mu^2)\Big) \geq 0$$

and hence $(K1) - (K2)$ are satisfied. In particular, in view of (3.4), $(K1) - (K2)$ hold when

$$a_3 = |\vec{a}| \leq \frac{1-\alpha}{2\alpha-1}. \tag{3.9}$$ a_3

By Proposition 2.3 with $l = \frac{2}{\alpha} - 2$, under the condition (3.9), $w_{\alpha,\vec{a}}(y)$ must be radially symmetric and hence $u_{\alpha,\vec{a}}(x)$ must be axially symmetric, $a_1 = a_2 = 0$. ■

4 Estimates of the minimum of I_α on $\mathcal{M}_{\vec{a}}$

In this section, we shall estimate for $\alpha \in (1/2, 1)$

$$m(\alpha, a) := \inf_{u \in \mathcal{M}_{\vec{a}}, |\vec{a}|=a} I_\alpha(u), \quad \forall a \in [0, 1). \tag{4.1}$$

minimizer_a

In view of Proposition 2.4, we know that $m(\alpha, a)$ is a continuous function of $a \in [0, 1)$ for any fixed $\alpha \in (1/2, 1)$.

We have the following estimates.

energy **Theorem 4.1** *There hold pointwise in* $a \in [0, 1)$

$$m(\alpha, a) \geq \begin{cases} (\frac{2}{\alpha} - 3)\ln(1 - a^2), & \alpha \in (1/2, 2/3), \\ \alpha(\frac{1}{\alpha} - \frac{3}{2})\ln(1 - a^2), & \alpha \in (2/3, 1). \end{cases} \tag{4.2}$$

lowerbound

and

$$m(\alpha, a) \leq \begin{cases} (\frac{2}{\alpha} - 3)\ln(1 - a^2), & \alpha \in (2/3, 1), \\ \frac{3\alpha}{2a}(\frac{1}{\alpha} - \frac{3}{2})\big(\ln(1 - a^2) - 2(\ln(1 + a) - a)\big), & \forall \alpha \in (1/2, 1). \end{cases} \tag{4.3}$$

upperbound

There also holds asymptotically as $a \to 1$

$$m(\alpha, a) \leq (\frac{1}{\alpha} - \frac{3}{2})\ln(1 - a^2)\big(1 + o(1)\big), \quad \alpha \in (1/2, 1). \tag{4.4}$$

upper-appr

Proof.

We first recall Onofri's inequality

$$F_1(u) = \int_{\mathbb{S}^2} |\nabla u|^2 d\omega + 2\int_{\mathbb{S}^2} u d\omega - \log \int_{\mathbb{S}^2} e^{2u} d\omega \geq 0, \quad u \in H^1(\mathbb{S}^2). \tag{4.5}$$

onofri

In view of (2.6) and (4.5), it is easy to see by interpolation that for $\alpha \in (2/3, 1]$

$$I_\alpha(u) \geq \alpha(\frac{1}{\alpha} - \frac{3}{2})\ln(1 - a^2), \quad \forall u \in \mathcal{M}. \tag{4.6}$$

lower-larg

As we know from previous discussion, there is a minimizer $u_{\alpha,a}$ to the minimization problem (4.1), which is a solution to (3.1) with $a_3 = a, \beta_3 = \beta_3(a)$ satisfying (3.3) and (3.4). Also from Proposition 3.1, the solution $u_{\alpha,a}$ forms a curve smooth curve parametrized by $a \in (0, \delta(\alpha))$. Furthermore, the linearized operator of (3.1) is a Fredholm operator on the tangent space of $\mathcal{M}_{\vec{a}}$ at any solution u of (3.1) on $\mathcal{M}_{\vec{a}}$. By the compactness of solutions of (3.1) for $a_3 = a \in [0, 1 - \epsilon]$ for any fix $\epsilon \in (0, 1)$ and the analyticity of equation (3.1) in term of u, it can be shown by the global bifurcation theory (see, e.g, Theorem 9.1.1 in [2]) that any solution set of (3.1) can be extended globally with either $a \to 0, 1$ or being a closed loop. In particular, by Proposition 3.1 there exists a branch of solution set which extends to $a = 0$ in one direction and to $a = 1$ in the other direction. Note that we do not know in general the uniqueness of the solution for a fixed $a \in (0,1)$, there might be more branches, and each branch of solutions might contain portions which are not minimizers of (4.1).

Nevertheless, by the compactness result Proposition 2.4 again, there are only finite numbers of smooth branches of solutions to (3.1) for $a \in [0, 1-\epsilon]$ with $\epsilon > 0$, and we can find a piecewise smooth solution curve $u_{\alpha,a(\tau)}, \tau \in (0,\infty)$ to (3.1) in $\mathcal{M}_a := \mathcal{M}_{\vec{a}}$ with $\vec{a} = (0,0,a)$ such that $a(\tau) \to 1$ as $\tau \to \infty$. Furthermore, the singular set

$$\mathcal{S} := \{\tau : a'(\tau) \text{ does not exists ; or } a'(\tau) = 0\,; \text{ or } \frac{\partial u_{\alpha,a(\tau)}}{\partial a} \text{ does not exists in } C^2(\mathbb{S}^2)\}$$

does not have accumulative point. For a fixed $a \in (0,1)$, there is some $\mathcal{T} > 0$ depending on α such that $a(\mathcal{T}) = a$. We have

$$a'(\tau) \int_{\mathbb{S}^2} e^{2u_{\alpha,a(\tau)}} \frac{\partial u_{\alpha,a(\tau)}}{\partial a} d\omega = 0, \quad \tau \in [0, \mathcal{T}] \setminus \mathcal{S}$$

and

$$a'(\tau) \int_{\mathbb{S}^2} e^{2u_{\alpha,a(\tau)}} x_3 \frac{\partial u_{\alpha,a(\tau)}}{\partial a} d\omega = a'(\tau), \quad \tau \in [0, \mathcal{T}] \setminus \mathcal{S}.$$

Now using (3.1) and the above equalities we obtain

$$\frac{\partial I_\alpha(u_{\alpha,a(\tau)})}{\partial \tau} = -2(\beta(\tau) - \frac{a(\tau)}{1 - a^2(\tau)})a'(\tau), \quad \tau \in [0, \mathcal{T}] \setminus \mathcal{S}.$$

Using similar arguments, we can find a solution curve of (4.1), still denoted by $u_{\alpha,a(\tau)}, \tau \in [0, \mathcal{T}]$, which is piecewise smooth, but may have finite discontinuous points $\tau_i, i = 1, 2, \cdots, N$ with $\tau_0 = 0, \tau_N = a$. Moreover, it can be chosen that $a(\tau)$ is continuous, and $u_{\alpha,a(\tau)}$ has both left limit and right limit at $\tau_i, i = 0, 1, 2, \cdots, N$ in $C^2(\mathbb{S}^2)$ and $I_\alpha(u_{\alpha,a(\tau)}) = m(\alpha, a(\tau))$ is continuous. Furthermore, $a'(\tau) > 0, \tau \in (\tau_i, \tau_{i+1}), i = 0, 1, 2, \cdots, N$.

Hence, using (3.3) and (3.4) with $a_3 = a(\tau)$ and $\beta_3 = \beta_3(\tau)$, in view of $I_\alpha(u_{\alpha,0}) = 0$ we have for $\alpha \in (1/2, 2/3]$

$$\begin{aligned} 0 \geq I_\alpha(u_{\alpha,a}) &= \int_0^{\mathcal{T}} \frac{\partial I_\alpha(u_{\alpha,a(\tau)})}{\partial \tau} d\tau \\ &= \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} \frac{\partial I_\alpha(u_{\alpha,a(\tau)})}{\partial \tau} d\tau \geq \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} (\frac{2}{\alpha} - 3) \frac{2a(\tau)a'(\tau)}{1 - a^2(\tau)} d\tau \end{aligned}$$

and for $\alpha \in (2/3, 1]$

$$\begin{aligned} 0 \leq I_\alpha(u_{\alpha,a}) &= \int_0^{\mathcal{T}} \frac{\partial I_\alpha(u_{\alpha,a(\tau)})}{\partial \tau} d\tau \\ &= \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} \frac{\partial I_\alpha(u_{\alpha,a(\tau)})}{\partial \tau} d\tau \leq \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} (\frac{2}{\alpha} - 3) \frac{2a(\tau)a'(\tau)}{1 - a^2(\tau)} d\tau. \end{aligned}$$

Hence the first inequalities in both (4.2) and (4.3) are proven.

Next, we will estimate $m(\alpha, a(\tau))$ from above by using suitable auxiliary functions. Define

$$\tilde{u}_{\alpha,\mu}(x) = \tilde{u}_{\alpha,\mu}(\Pi^{-1}(y)) := \frac{1}{\alpha} \ln \frac{1 + |y|^2}{\mu^2 + |y|^2}, \quad y \in \mathbb{R}^2. \tag{4.7}$$ test

Direct computations show that

$$\int_{\mathbb{S}^2} |\nabla \tilde{u}_{\alpha,\mu}|^2 d\omega = \frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla \tilde{u}_{\alpha,\mu}(\Pi^{-1}(y))|^2 dy = \frac{1}{\alpha^2(\mu^2-1)}\big(2(1-\mu^2)+(\mu^2+1)\ln(\mu^2)\big),$$

$$\int_{\mathbb{S}^2} \tilde{u}_{\alpha,\mu} d\omega = \frac{1}{4\pi}\int_{\mathbb{R}^2} \tilde{u}_{\alpha,\mu}(\Pi^{-1}(y))\frac{4}{(1+|y|^2)^2}dy = -\frac{1}{\alpha(\mu^2-1)}\big((1-\mu^2)+\mu^2\ln(\mu^2)\big),$$

and

$$\int_{\mathbb{S}^2} e^{2\tilde{u}_{\alpha,\mu}} d\omega = \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{2\tilde{u}_{\alpha,\mu}(\Pi^{-1}(y))}\frac{4}{(1+|y|^2)^2}dy = \frac{1-\mu^{2-4/\alpha}}{(\frac{2}{\alpha}-1)(\mu^2-1)}$$

$$\begin{aligned}\int_{\mathbb{S}^2} e^{2\tilde{u}_{\alpha,\mu}} x_3 d\omega &= \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{2\tilde{u}_{\alpha,\mu}(\Pi^{-1}(y))}\frac{4(|y|^2-1)}{(1+|y|^2)^3}dy \\ &= \frac{(\frac{2}{\alpha})\mu^{4-4/\alpha}+(\frac{2}{\alpha}-2)(\mu^2-\mu^{2-4/\alpha})-\frac{2}{\alpha}}{(\frac{2}{\alpha}-1)(\frac{2}{\alpha}-2)(\mu^2-1)^2}\end{aligned}$$

Hence, we have

$$I_\alpha(\tilde{u}_{\alpha,\mu}) = (\frac{3}{2}-\frac{1}{\alpha})\ln(\mu^2)\big(1+o(1)\big) \tag{4.8}$$

as $a \to 1$ and $\mu \to \infty$.

Now we compute the center of mass of $\tilde{u}_{\alpha,\mu}$

$$a_{\alpha,\mu} := \frac{\int_{\mathbb{S}^2} e^{2\tilde{u}_{\alpha,\mu}} x_3 d\omega}{\int_{\mathbb{S}^2} e^{2\tilde{u}_{\alpha,\mu}} d\omega} = 1 - \frac{2(\frac{2}{\alpha}-1)(1-\mu^2)+2(\mu^{\frac{4}{\alpha}-2}-1)}{(\frac{2}{\alpha}-2)(\mu^2-1)(\mu^{\frac{4}{\alpha}-2}-1)}.$$

It is easy to see that for $\alpha \in (1/2,1)$

$$a_{\alpha,\mu} \to 1 \text{ as } \ \mu \to \infty; \quad a_{\alpha,\mu} \to 0 \text{ as } \ \mu \to 1.$$

Hence for any fixed $a \in (0,1)$ there is at least a positive number $\mu(a)$ such that $a_{\alpha,\mu(a)} = a$ and

$$\mu^2(a) = \frac{\alpha}{(1-\alpha)(1-a)}\big(1+o(1)\big), \text{ as } \ a \to 1.$$

Letting $\mu = \mu(a)$, we can choose a constant $c = c(a)$ such that $v_{\alpha,a} := u_{\alpha,\mu(a)} + c(a) \in \mathcal{M}_a$ and

$$I_\alpha(v_{\alpha,a}) = (\frac{1}{\alpha}-\frac{3}{2})\ln(1-a^2)\big(1+o(1)\big) \text{ as } \ a \to 1.$$

Hence (4.4) holds.

Therefore, the proof is complete. ■

Remark 4.2 *In view of* (4.5)*, we can then derive a lower bound of the gradient L^2 norm of a minimizer $u_{\alpha,a}$ of* (4.1) *asymptotically*

$$\int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \geq -\frac{\ln(1-a^2)}{\alpha}\big(1+o(1)\big) \ \textit{as} \ \ a \to 1;$$

While for $\alpha \in (1/2, 2/3)$*, using* (2.7) *a better lower bound can also be obtained*

$$\int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \geq -\frac{3\ln(1-a^2)}{2\alpha}\big(1+o(1)\big) \ \text{as} \ \ a \to 1.$$

By (2.6) *and the above gradient estimates, we can obtain for* $\alpha \in (2/3, 1)$ *an asymptotical lower bound*

$$m(\alpha, a) \geq (\alpha - \frac{2}{3}) \int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \geq \frac{2}{3}(\frac{1}{\alpha} - \frac{3}{2}) \ln(1-a^2)\big(1+o(1)\big) \ \text{as} \ \ a \to 1.$$

However, this energy lower bound is not as good as (4.6)*. When* $\alpha \in (1/2, 2/3)$*, similar energy lower bound does not follow immediately from the gradient lower bound. If it did, it would be a sharp one as it would coincide with the upper bound. Nevertheless, it is expected that*

$$m(\alpha, a) = (\frac{1}{\alpha} - \frac{3}{2}) \ln(1-a^2)\big(1+o(1)\big) \ \text{as} \ \ a \to 1.$$

Similarly using (4.5) *and* (4.3)*, we can obtain, when* $\alpha \in (2/3, 4/5)$*, a lower bound of the gradient* L^2 *norm of a minimizer* $u_{\alpha,a}$ *of* (4.1) *point wisely in* a

$$\int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \geq -\frac{4-5\alpha}{2\alpha(1-\alpha)} \ln(1-a^2), \qquad \forall a \in [0, 1],$$

which leads to the following lower bound point wisely in a

$$m(\alpha, a) \geq (\alpha - \frac{2}{3}) \int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \geq -\frac{(4-5\alpha)(2\alpha-3)}{4\alpha(1-\alpha)} \ln(1-a^2).$$

We note that the upper bound in (4.3) *is not optimal which leads to the technical condition* $\alpha < 4/5$ *in the above estimates instead of the natural range up to* $\alpha < 1$*.*

On the other hand, using (2.7) *and* (4.3)*, we can also derive an upper bound of the gradient* L^2 *norm of a minimizer* $u_{\alpha,a}$ *of* (4.1) *point wisely in* a *when* $\alpha \in (2/3, 1)$

$$\int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \leq -\frac{3}{\alpha} \ln(1-a^2), \qquad \forall a \in [0, 1].$$

Similarly, using (2.7) *and* (4.4)*, we can derive an upper bound of the gradient* L^2 *norm of a minimizer* $u_{\alpha,a}$ *of* (4.1) *asymptotically in* a *when* $\alpha \in (2/3, 1)$

$$\int_{\mathbb{S}^2} |\nabla u_{\alpha,a}|^2 d\omega \leq -\frac{3}{2\alpha} \ln(1-a^2)\big(1+o(1)\big) \quad \text{as} \ \ a \to 1.$$

Similar upper bounds seem not follow immediately when $\alpha \in (1/2, 2/3)$*.*

Remark 4.3 *The following technical questions still remain open:*

1) Should $u_{\alpha,\vec{a}}(x)$ *always be axially symmetric for all* $\alpha \in (\frac{2}{3}, 1)$ *and* $\vec{a} \in B_1$*?*

2) Is the minimizer $u_{\alpha,\vec{a}}(x)$ *unique determined? In particular, is* β *uniquely determined? We know that if* β *is uniquely determined by* α *and* $\vec{a}$*, then the axially symmetric solution* $u_{\alpha,\vec{a}}(y)$ *is unique.*

3) Fixed $\alpha \in (\frac{1}{2}, 1), \vec{a} \in B_1$, for any given $\vec{\beta} = \beta_3\vec{a}/|\vec{a}|, 0 < \beta_3 < \frac{1}{1-|\vec{a}|}, \rho = 1 + \beta_3|\vec{a}|$, there is a unique axially symmetric solution u to (2.22) *with the corresponding w solving* (3.7)- (3.8)*, following Theorem 1.5 of [12]. However, it is not clear whether the center of mass $A\vec{a}/|\vec{a}|$ of u is still $\vec{a}$ and the total mass M is still* 1 *or not. Certainly for some such β_3, ρ, we should have $A \neq |\vec{a}|, M \neq 1$ since otherwise* (3.3) *or* (3.4) *should hold. This implies that a solution to* (3.7)-(3.8) *may not a solution to the minimizing problem* $\min_{u \in \mathcal{M}_{\vec{a}}} I_\alpha(u)$. *Nevertheless, we note that $M[1 + \beta_3(|\vec{a}| - |A|)] = 1$ still holds.*

4) Can we compute or estimate more accurately $m(\alpha, a)$. We need to get more information on the minimizer. In particular, what is the asymptotic behavior of the solution $u_{\alpha,\vec{a}}$ as $\vec{a} \in B_1$ goes to the unit sphere? If we have a detailed profile of the solution in different regions, we might get an optimal asymptotic estimate of $m(\alpha, a)$.

We note that the answers to all above questions are known for $\alpha = \frac{2}{3}$ as shown in Theorem 2.1. The technical questions and other related problems will be studied in a forthcoming paper.

5 Second Variation of I_α

Now we consider another technical aspect of I_α: the second variation of I_α in $H^1(\mathbb{S}^2)$, in an effort to understand I_α better.

Fixed a solution $u \in \mathcal{H}$ to (2.10) and (2.9), for any $\phi \in H^1(\mathbb{S}^2)$ we have

$$D^2 I_\alpha(u)(\phi, \phi) = 2\alpha \int_{\mathbb{S}^2} |\nabla\phi|^2 d\omega + \frac{8}{(1-|\vec{a}|^2)^2}\Big(\int_{\mathbb{S}^2} e^{2u}(1 - \vec{a}\cdot x)\phi d\omega\Big)^2$$
$$-\frac{4}{1-|\vec{a}|^2}\Big(\int_{\mathbb{S}^2} e^{2u}(1-\vec{a}\cdot x)\phi^2 d\omega + (\int_{\mathbb{S}^2} e^{2u}\phi d\omega)^2 - \sum_{i=1}^{3}(\int_{\mathbb{S}^2} e^{2u} x_i \phi d\omega)^2\Big).$$

In particular, at $u \equiv 0$ for any $\phi \in H^1(\mathbb{S}^2)$

$$D^2 I_\alpha(0)(\phi, \phi) = 2\alpha \int_{\mathbb{S}^2} |\nabla\phi|^2 d\omega - 4\int_{\mathbb{S}^2} \phi^2 d\omega + 4(\int_{\mathbb{S}^2} \phi d\omega)^2 + 4\sum_{1}^{3}(\int_{\mathbb{S}^2} x_i\phi d\omega)^2.$$

Let $\phi = \sum_{n=0}^{\infty} b_n\phi_n$ where $\{\phi_n\}_0^\infty$ is an orthonormal basis of $H^1(\mathbb{S}^2)$ formed by spherical harmonics in an increasing order of eigenvalues λ_n. Note that $\phi_0 = 1, \phi_i = \sqrt{3}x_i, i = 1, 2, 3$ and $\lambda_i = 0, \lambda_1 = \lambda_2 = \lambda_3 = 2$ and $\lambda_4 = 6$.

Then $\forall \phi \in H^1(\mathbb{S}^2)$

$$D^2 I_\alpha(0)(\phi, \phi) = 2\alpha \sum_{n=0}^{\infty} \lambda_n b_n^2 - 4\sum_{n=0}^{\infty} b_n^2 + 4b_0^2 + \frac{4}{3}\sum_{n=1}^{3} b_n^2$$
$$\geq (4\alpha - \frac{8}{3})\sum_{n=1}^{3} b_n^2 + (12\alpha - 4)\sum_{n=4}^{\infty} b_n^2.$$

In particular, the linearized equation of equation (2.10) at $u \equiv 0$ is

$$\alpha\Delta\phi + 2\phi - 2\int_{\mathbb{S}^2} \phi d\omega - 2\sum_{i=1}^{3} x_i \int_{\mathbb{S}^2} x_i\phi d\omega = 0 \quad \text{on} \quad \mathbb{S}^2. \tag{5.1}$$

linear

It has a kernel $\mathcal{K}_\alpha = \{\phi_0\}$ if $\alpha \neq \frac{2}{3}$ and $\mathcal{K}_{\frac{2}{3}} = \{\phi_i, i = 0, 1, 2, 3\}$ when $\alpha = \frac{2}{3}$.

Hence, it is easy to conclude from the above discussion the following

l variation **Proposition 5.1** *i) when $\alpha \in (\frac{2}{3}, 1)$, $D^2 I_\alpha(0)(\phi, \phi) \geq 0$ and the equality holds only when ϕ is a constant function; In particular, $D^2 I_\alpha(0)$ is positive definite when restricted to $\mathcal{H}$;*

ii) when $\alpha = \frac{2}{3}$, $D^2 I_\alpha(0)(\phi, \phi) \geq 0$ and the equality holds only when ϕ is expanded by $\phi_0 = 1, \phi_i = \sqrt{3}x_i, i = 1, 2, 3$;

iii) when $\alpha < \frac{2}{3}$, $D^2 I_\alpha(0)$ is not non-negative. In particular, $I_\alpha(u) < 0$ for some $u \in H^1(\mathbb{S}^2)$..

This fact gives a simple explanation of the critical value of α being $2/3$, compared to Theorem 2.1.

Finally, we shall look at the second variation of I_α at the nontrivial explicit solution when $\alpha = \frac{2}{3}$. We can rewrite the solution as

$$u_{\frac{2}{3},\vec{a}}(x) = -\frac{3}{2} \ln(1 - \vec{a} \cdot x) + \ln(1 - |\vec{a}|^2), \quad x \in \mathbb{S}^2.$$

Then, for any $\phi \in H^1(\mathbb{S}^2)$, from Theorem 2.1 we have

$$D^2 I_{\frac{2}{3}}(u_{\frac{2}{3},\vec{a}})(\phi, \phi)$$

$$= \frac{4}{3} \int_{\mathbb{S}^2} |\nabla \phi|^2 d\omega + 8(1 - |\vec{a}|^2)^2 \Big(\int_{\mathbb{S}^2} \frac{\phi}{(1 - \vec{a} \cdot x)^2} d\omega \Big)^2 - 4(1 - |\vec{a}|^2) \int_{\mathbb{S}^2} \frac{\phi^2}{(1 - \vec{a} \cdot x)^2} d\omega$$

$$-4(1 - |\vec{a}|^2)^3 \Big(\Big(\int_{\mathbb{S}^2} \frac{\phi}{(1 - \vec{a} \cdot x)^3} d\omega \Big)^2 - \sum_{i=1}^{3} \Big(\int_{\mathbb{S}^2} \frac{x_i \phi}{(1 - \vec{a} \cdot x)^3} d\omega \Big)^2 \Big) \geq 0.$$

In particular, if we consider the second variation of $I_{\frac{2}{3}}$ at $u_{\frac{2}{3},\vec{a}}$ on $\mathcal{M}_{\frac{2}{3},\vec{a}}$, we only need to deal with $\phi \in H^1(\mathbb{S}^2)$ with

$$\int_{\mathbb{S}^2} \frac{\phi}{(1 - \vec{a} \cdot x)^3} d\omega = 0, \qquad \int_{\mathbb{S}^2} \frac{x_i \phi}{(1 - \vec{a} \cdot x)^3} d\omega = 0, \ i = 1, 2, 3. \tag{5.2}$$ constraint

In this setting, it also holds that

$$\int_{\mathbb{S}^2} \frac{\phi}{(1 - \vec{a} \cdot x)^2} d\omega = 0.$$

Hence we have

$$D^2 I_{\frac{2}{3}}(u_{\frac{2}{3},\vec{a}})(\phi, \phi) = \frac{4}{3} \int_{\mathbb{S}^2} |\nabla \phi|^2 d\omega - 4(1 - |\vec{a}|^2) \int_{\mathbb{S}^2} \frac{\phi^2}{(1 - \vec{a} \cdot x)^2} d\omega \geq 0.$$

6 Monotonicity

In this section, we shall discuss and prove the first monotonicity formula of the analogue of the Szegö Limit theorem on $\mathbb{S}^2$. Following [9] (Section 2.1-2.2, Chapter 2), for any given function $f \geq 0, f \not\equiv 0$, we denote a measure $d\nu = f d\omega$ on $\mathbb{S}^2$ and orthogonalize the functions $f_0 = 1, f_1 = \sqrt{3}x_1, f_2 = \sqrt{3}x_2, f_3 = \sqrt{3}x_3$ with respect to this measure. Note that f_0, f_1, f_2, f_3 form an orthonormal basis for spherical harmonics of order less than or equal to 1 with respect to the measure $d\omega$.

Differing from the case on S^1 as discussed in [9], we only construct $\phi_0, \phi_1, \phi_2, \phi_3$ such that ϕ_0, ϕ_1 form an orthonormal basis for functions generated by f_0, f_i respectively for $i = 1, 2, 3$.

Denote the inner product on $L^2(\mathbb{S}^2, d\nu)$ by $<, >$. We define

$$D_0 =< f_0, f_0 >= \int_{S^2} f d\omega$$

and

$$D_{0,i} = det \begin{pmatrix} < f_0, f_0 > & < f_0, f_i > \\ < f_i, f_0 > & < f_i, f_i > \end{pmatrix}, \quad i = 1, 2, 3$$

and

$$D_1 = \frac{1}{3} \sum_{i=1}^{3} D_{0,i}.$$

It is easy to see that $\phi_0 = D_0^{-1/2} f_0$ and

$$\phi_i = (D_0 D_{0,i})^{-1/2} \begin{vmatrix} < f_0, f_0 > & < f_0, f_i > \\ f_0 & f_i \end{vmatrix} = l_{i,0} f_0 + l_{i,1} f_i$$

where

$$l_{i,0} = -(D_0 D_{0,i})^{-1/2} < f_0, f_i >, \quad i = 1, 2, 3,$$
$$l_{i,1} = (D_0 D_{0,i})^{-1/2} < f_0, f_0 >= (\frac{D_0}{D_{0,i}})^{1/2}, \quad i = 1, 2, 3.$$

Now we state the following stage one monotonicity relation on $\mathbb{S}^2$, which may be considered as the counterpart on $\mathbb{S}^2$ of the Szegö monotonicity theorem on S^1.

monotone **Proposition 6.1** *We have*

$$D_1 = (\int_{\mathbb{S}^2} f d\omega)^2 - \sum_{i=1}^{3} (\int_{\mathbb{S}^2} f x_i d\omega)^2 \geq 0$$

and

$$\ln D_0 - \int_{\mathbb{S}^2} f d\omega \leq \ln D_1 - 2 \int_{\mathbb{S}^2} f d\omega. \tag{6.1}$$ monotonici

Proof It is easy to see that

$$\sum_{i=1}^{3} (\int_{\mathbb{S}^2} f x_i d\omega)^2 \leq \sum_{i=1}^{3} \int_{\mathbb{S}^2} f d\omega \cdot \int_{\mathbb{S}^2} f x_i^2 d\omega \leq (\int_{\mathbb{S}^2} f d\omega)^2.$$

To show (6.1), we shall follow the proof of Theorem a of [9] and prove

$$\mu_i := \inf_{a_i \in R} \int_{\mathbb{S}^2} |a_i + f_i|^2 d\nu$$

is attained at $a_i = \frac{l_{i,0}}{l_{i,1}}$ and

$$\mu_i = l_{i,1}^{-2} = \frac{D_{0,i}}{D_0}, \quad i = 1, 2, 3.$$

This can be seen from the folowing two facts. First,

$$\mu_i \leq \int_{\mathbb{S}^2} |\frac{l_{i,0}}{l_{i,1}} + f_i|^2 d\nu \leq l_{i,1}^{-2} < \phi_i, \phi_i >= l_{i,1}^{-2}$$

Second, if we write $a_i + f_i = b_0\phi_0 + b_i\phi_i$ where $b_i l_{i,1} = 1$, hence

$$\int_{\mathbb{S}^2} |a_i + f_i|^2 d\nu = b_0^2 + b_i^2 \geq b_i^2 = l_{i,1}^{-2}.$$

Therefore

$$\begin{aligned}
\frac{D_1}{D_0} =& \frac{1}{3}\sum_{i=1}^{3} \mu_i \\
=& \frac{1}{3} \inf_{a_i \in R} \int_{\mathbb{S}^2} \sum_{i=1}^{3} |a_i + f_i|^2 d\nu \\
=& \inf_{c_i \in R} \int_{\mathbb{S}^2} \sum_{i=1}^{3} |c_i + x_i|^2 d\nu
\end{aligned}$$

For any $(c_1, c_2, c_3) \in \mathbb{R}^3$, let

$$\eta(x) = \sum_{i=1}^{3} |c_i + x_i|^2, \quad x \in \mathbb{S}^2.$$

We know that

$$\eta(x) = 1 + \sum_{i=1}^{3} c_i^2 + \sum_{i=1}^{3} 2c_i x_i \geq 0$$

and

$$\ln(\int_{\mathbb{S}^2} \eta d\nu) = \ln(\int_{\mathbb{S}^2} \eta f d\omega) \geq \int_{\mathbb{S}^2} \ln(\eta f) d\omega \geq \int_{\mathbb{S}^2} \ln(\eta) d\omega + \int_{\mathbb{S}^2} \ln f d\omega.$$

We claim that for any $(c_1, c_2, c_3) \in \mathbb{R}^3$,

$$\int_{\mathbb{S}^2} \ln(\eta) d\omega \geq 0.$$

For this purpose, we assume without loss of generality that $(c_1, c_2, c_3) = (0, 0, t), t \geq 0 \in \mathbb{R}$ after a possible rotation, and define

$$g(t) = \int_{\mathbb{S}^2} \ln(\eta) d\omega = \int_{\mathbb{S}^2} \ln(1 + t^2 + 2tx_3) d\omega = \frac{1}{2} \int_{-1}^{1} \ln(1 + t^2 + 2tx_3) dx_3.$$

Straightforward computations lead to

$$g(t) = \ln(1+t)(1 + \frac{t^2+1}{2t}) + \ln(|t-1|)(\frac{t^2+1}{2t} - 1) - 1, \ t \in (0,1) \cup (1, \infty)$$

and $g(1) = 2\ln 2 - 1$. It is easy to check that $\lim_{t \to 0^+} g(t) = 0$ and $\lim t \to \infty g(t) = \infty$ and $g(t)$ is continuous in $(0, \infty)$ and $g \in C^1(0, \infty)$. Indeed, $g'(1) = 1$ and

$$g'(t) = \frac{1}{4t^2} \big(4t + (t^2 - 1) \ln(\frac{1+t}{1-t})^2\big), \quad t \in (0,1) \cup (1, \infty)$$

is continuous in $(0, \infty)$.

It is easy to check by differentiation again that $g'(t) > 0, t \in (0, \infty)$ and hence $g(t) > 0, t \in (0, \infty)$. This proves the claim.

Hence,

$$\ln(D_1/D_1) \geq \int_{\mathbb{S}^2} \ln f d\omega$$

and (6.1) follows. This completes the proof.

Remark 6.1 *If we choose* $f = e^{2u}$, (2.7) *is equivalent to*

$$\ln D_1 - 2 \int_{\mathbb{S}^2} f d\omega \leq \frac{4}{3} \int_{\mathbb{S}^2} |\nabla u|^2 d\omega.$$

The factor $\frac{4}{3}$ *in the above inequality makes the inequality weaker than the Szegö limit theorem on* S^1*, where the factor is* 1 *for the corresponding term. Given the optimal constant in* (2.7)*, we may not expect that the Szegö limit theorem on* $\mathbb{S}^2$ *holds fully in its original* S^1 *form.*

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