Calc. Var. P.D.E. manuscript No.
(will be inserted by the editor)

On the prescribing σ_2 curvature equation on $\mathbb{S}^4$

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Received: date / Accepted: date

Abstract Prescribing σ_k curvature equations are fully nonlinear generalizations of the prescribing Gaussian or scalar curvature equations. For a given a positive function K to be prescribed on the 4-dimensional round sphere, we obtain asymptotic profile analysis for potentially blowing up solutions to the σ_2 curvature equation with the given K; and rule out the possibility of blowing up solutions when K satisfies a non-degeneracy condition. Under the same non-degeneracy condition on K, we also prove uniform a priori estimates for solutions to a family of σ_2 curvature equations deforming K to a positive constant; and under an additional, natural degree condition on a finite dimensional map associated with K, we prove the existence of a solution to the σ_2 curvature equation with the given K using a degree argument involving fully nonlinear elliptic operators to the above deformation.

Keywords σ_k curvature · prescribing curvature · fully nonlinear curvature equation

Mathematics Subject Classification (2000) MSC 58J05 · MSC 53A30 · MSC 35J60 · MSC 35B45

The research of the first and third author is partially supported by NSF through grant DMS-0758601; the first author also gratefully acknowledges partial support from the Minerva Research Foundation and The Charles Simonyi Endowment fund during the academic year 08-09 while visiting Institute of Advanced Study, Princeton. The research of the second author is partially supported by NSF through grant DMS-0103888 and by a Rutgers University Research Council Grant(202132).

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1 Description of main results

Our main results in this paper are (potential) blow up profile analysis, a priori estimates and existence of admissible solutions w to the σ_2 curvature equation

$$\sigma_2(g^{-1} \circ A_g) = K(x), \tag{1}$$

on $\mathbb{S}^4$, where $g = e^{2w(x)} g_c$ is a metric conformal to g_c, with g_c being the canonical background metric on the round sphere $\mathbb{S}^4$, A_g is the the Weyl-Schouten tensor of the metric g,

$$\begin{aligned} A_g &= \frac{1}{n-2}\{Ric - \frac{R}{2(n-1)} g\} \\ &= A_{g_c} - \left[\nabla^2 w - dw \otimes dw + \frac{1}{2}|\nabla w|^2 g_c\right], \end{aligned} \tag{2}$$

and $\sigma_k(\Lambda)$, for any $(1,1)$ tensor Λ on an $n-$dimensional vector space and $k \in \mathbb{N}$, $0 \le k \le n$, is the k-th elementary symmetric function of the eigenvalues of Λ; $K(x)$ is a given function on $\mathbb{S}^4$ with some appropriate assumptions, and an *admissible* solution is defined to be a $C^2(M)$ solution w to (1) such that for all $x \in \mathbb{S}^4$, $A_g(x) \in \Gamma_k^+$, namely, $\sigma_j(g^{-1} \circ A_g) > 0$ for $1 \le j \le k$. Note that $\sigma_1(A_g)$ is simply a positive constant multiple of the scalar curvature of g, so A_g in the Γ_k^+ class is a generalization of the notion that the scalar curvature R_g of g having a fixed $+$ sign.

Note that, since
$$\sigma_2(g^{-1} \circ A_g) = e^{-4w} \sigma_2(g_c^{-1} \circ A_g),$$
so (1) is equivalent to

$$\sigma_2(g_c^{-1} \circ \left[A_{g_c} - \nabla^2 w + dw \otimes dw - \frac{1}{2}|\nabla w|^2 g_c\right]) = K(x) e^{4w(x)}. \tag{3}$$

It is well known that (3) is *elliptic* at an admissible solution; and in fact, any solution w to (3) on $\mathbb{S}^4$ is admissible. There have been a large number of papers on problems related to the σ_k curvature since the work [36] of Viaclovsky a decade ago. It is inadequate to do even a short survey of recent work in the introductory remarks here. We will instead refer the reader to recent surveys [39] by Viaclovsky and [4] by Chang and Chen.

As alluded to above, a similar problem to (3) for the σ_1 curvature was a predecessor to (3). More specifically, if we prescribe a function $K(x)$ on the round n-dimensional sphere $(\mathbb{S}^n, g_c)$ to be the scalar curvature of a metric $g = e^{2w} g_c$ pointwise conformal to g_c, then w satisfies

$$2(n-1)\Delta_{g_c} w + (n-1)(n-2)|\nabla w|^2 = R_{g_c} - K(x) e^{2w}. \tag{4}$$

Similar equation can be formulated for a general manifold. One difference between (4) and (3) is that (4) is semilinear in w, while (3) is fully nonlinear in w. (4) takes on the familiar form
$$2\Delta_{g_c} w = R_{g_c} - K(x) e^{2w}, \tag{5}$$

when $n = 2$, which is the Nirenberg problem. The $n \geq 3$ case of (4) is often written in terms of a different variable $u = e^{(n-2)w/2}$, which would render the equation in the familiar form

$$-4\frac{n-1}{n-2}\Delta_{g_c}u + R_{g_c}u = K(x)u^{\frac{n+2}{n-2}}. \tag{6}$$

The $K(x) \equiv$ const. case of (6) on a general compact manifold is the famous Yamabe problem. (5) and (6) have attracted enormous attention in the last several decades. A large collection of phenomena on the possible behavior of solutions to these equations, and methods and techniques of attacking these problems have been accumulated, which have tremendously enriched our understanding in solving a large class of nonlinear (elliptic) PDEs, and provided guidance in attacking seemingly unrelated problems. It is impossible in the space here to provide even a partial list of references. Please see [23], [25], [33], [31], [8], [24], and the references therein to get a glimpse of the results and techniques in this area.

Directly related to our current work are some work on the (potential) blow up analysis, a priori estimates, and existence of solutions to (5) or (6). It is proved in [19] and [12] that when K is a positive function on $\mathbb{S}^2$, a sequence of blowing up solutions to (5) has only one point blow up and has a well-defined blow up profile, and that when K is a positive C^2 function on $\mathbb{S}^2$ such that $\Delta_{g_c}K(x) \neq 0$ at any of its critical points, no blow up can happen, more precisely, there is an a priori bound on the set of solutions to (5) which depends on the C^2 norm of K, the positive lower bound of K and $|\Delta_{g_c}K(x)|$ near the critical points of K, and the modulus of continuity of the second derivatives of K. Similar results for (6) in the case $n = 3$ were proved in [40] and [12], and for (6) in the case $n \geq 4$ in [29] under a flatness condition of K at its critical points. When K is a Morse function, say, this flatness condition fails when $n \geq 4$. In fact it is proved in [30] that in such cases on $\mathbb{S}^4$, there can be a sequence of solutions to (6) blowing up at more than one points. Later on [15] constructed solutions blowing up on $\mathbb{S}^n$, $n \geq 7$, with unbounded layers of "energy concentration" for certain non-degenerate K.

A natural question concerning the σ_k curvature equations such as (3) is: of the behaviors for solutions to (5) and (6), which do solutions to (3) exhibit?

In the following we will often transform (3) through a conformal automorphism φ of $\mathbb{S}^4$ as follows. Let $|d\varphi(P)|$ denote the factor such that $|d\varphi(P)[X]| = |d\varphi(P)||X|$ for any tangent vector $X \in T_P(\mathbb{S}^4)$, and

$$w_\varphi(P) = w \circ \varphi(P) + \ln|d\varphi(P)|. \tag{7}$$

Then w is a solution to (3) iff w_φ is a solution to

$$\sigma_2(g_c^{-1} \circ \left[A_{g_c} - \nabla^2 w_\varphi + dw_\varphi \otimes dw_\varphi - \frac{1}{2}|\nabla w_\varphi|^2 g_c\right]) = K \circ \varphi(x)e^{4w_\varphi(x)}. \tag{8}$$

Our first results show that solutions to (3) on $\mathbb{S}^4$ exhibits similar behavior as those to (5) on $\mathbb{S}^2$ or (6) on $\mathbb{S}^3$.

Theorem 1 *Consider a family of admissible conformal metrics $g_j = e^{2w_j}g_c$ on $\mathbb{S}^4$ with $\sigma_2(g_j^{-1} \circ A_{g_j}) = K(x)$, where g_c denotes the canonical round metric on $\mathbb{S}^4$ and*

$K(x)$ denotes a C^2 positive function on $\mathbb{S}^4$. Then there exists at most one isolated simple blow up point in the sense that, if $\max w_j = w_j(P_j) \to \infty$, then there exists conformal automorphism φ_j of $\mathbb{S}^4$ such that, if we define $v_j(P) = w_j \circ \varphi_j(P) + \ln |d\varphi_j(P)|$, we have

$$v_j(P) - \frac{1}{4} \ln \frac{6}{K(P_j)} \to 0 \quad in \quad L^\infty(\mathbb{S}^4), \tag{9}$$

and

$$\int_{\mathbb{S}^4} |\nabla v_j|^4 \to 0. \tag{10}$$

In fact, we have the stronger conclusion that the $W^{2,6}$ norm of v_j stays bounded and $v_j - \frac{1}{4} \ln \frac{6}{K(P_j)} \to 0$ in $C^{1,\alpha}(\mathbb{S}^4)$ for any $0 < \alpha < 1/3$.

We also have

Theorem 2 *Let $K(x)$ be a C^2 positive function on $\mathbb{S}^4$ satisfying a non-degeneracy condition*

$$\Delta K(P) \neq 0 \quad whenever \quad \nabla K(P) = 0, \tag{11}$$

and we consider solutions $w(x)$ to (3), with $K(x)$ replaced by

$$K^{[s]}(x) := (1-s)6 + sK(x),$$

for $0 < s \leq 1$, namely,

$$\sigma_2(g_c^{-1} \circ \left[A_{g_c} - \nabla^2 w + dw \otimes dw - \frac{1}{2} |\nabla w|^2 g_c\right]) = K^{[s]}(x) e^{4w(x)}. \tag{3'}$$

Then there exist a priori $C^{2,\alpha}$ estimates on w, uniform in $0 < s \leq 1$, which depend on the C^2 norm of K, the modulus of continuity of $\nabla^2 K$, positive lower bound of $\min K$ and positive lower bound of $|\Delta K(x)|$ in a neighborhood of the critical points of K.

Remark 1 There are several recent papers on the study of the behavior of the singular blow up of solutions to the $\sigma_k(A_g)$ curvature equations on a Riemannian manifold (M, g_0), which are analogues of (1) on (M, g_0) and for general k. More specifically, the equation takes the form

$$\sigma_k(g_0^{-1} \circ \left[A_{g_0} - \nabla^2 w + dw \otimes dw - \frac{1}{2} |\nabla w|^2 g_0\right]) = K(x) e^{2kw(x)}. \tag{12}$$

These papers mostly deal with the case when (M, g_0) is assumed to be *not* conformally equivalent to the round sphere. Compactness and existence of solutions to (12) is proved in [10] for the case $k = 2$ and (M, g_0) a four-dimensional manifold not conformally equivalent to the round sphere; in [26] for the case of a general k, $K \equiv 1$ and (M, g_0) locally conformally flat, and not conformally equivalent to the round sphere; in [18] for the case $k > n/2$ and (M, g_0) not conformally equivalent to the round sphere; and in [35] for the case $k = n/2$ and (M, g_0) not conformally equivalent to the round sphere.

Remark 2 Theorems 1 and **2**, with the uniform estimates in Theorem 2 for solutions to ($3'$) only for $0 < s_0 \le s \le 1$ and the estimates possibly depending on $0 < s_0 < 1$, were obtained several years ago and were announced in [20]. The details were written up in [13] and presented by the second author on several occasions, including at the 2006 Banff workshop "Geometric and Nonlinear Analysis". The current work can be considered as a completion of [13]. As mentioned above similar statements for (5) and (6) were obtained earlier in [19], [12], [29], [40], among others. When applying these estimates to the corresponding equation such as (5) and (6) with $K(x)$ replaced by $K^{[s]}(x)$, all previous work stated and proved that the a priori estimates on the solutions remain uniform as long as $0 < s_0 \le s \le 1$, for any fixed s_0. This stems from the dependence of the a priori estimates on a positive lower bound of $|\Delta K(x)|$ near the critical points of K, among other things. Since $\Delta K^{[s]} = s\Delta K(x)$ becomes small when $s > 0$ is small, previous work in this area assumed that the a priori estimates could deteriorate as $s > 0$ becomes small. In these previous work, one has to devise a way to study the problem when $s > 0$ becomes small. [12] and [29] used some kind of "center of mass" analysis via conformal transformations of the round sphere. Technically [12] and [29] used a constrained variational problem to study the "centered problem". In essence the success of these methods was due to the semilinear nature of the relevant equations, so one could still have control on the "centered solution" in some norm weaker than $C^{2,\alpha}$ norm, say, $W^{2,p}$ norm, when $s > 0$ is small, and used these estimates to prove existence of solutions under natural geometric/topological assumptions on K. This approach was problematic for our fully nonlinear equation (3). Due to this difficulty, until recently we have not been successful in using our preliminary version of Theorem 2 (for estimates in the range $0 < s_0 \le s \le s$ which may depend on s_0) and the deformation $K^{[s]}$ above to the equation to establish solutions to (3), under natural geometric/topological assumptions on K. It was our recent realization that in our setting, as well as in those of [12] and [29], the a priori estimates of solutions, under conditions like those in Theorem 2, remain uniform for all $1 \ge s > 0$! A similar observation was made by M. Ji in her work on (5) in [22]. This uniform a prioir estimate for all $1 \ge s > 0$ leads to our next Theorem.

Theorem 3 *Suppose $K(x)$ is a C^2 positive function on $\mathbb{S}^4$ satisfying* (11). *Then the map*

$$G(P,t) = |\mathbb{S}^4|^{-1} \int_{\mathbb{S}^4} K \circ \varphi_{P,t}(x) x \, dvol_{g_c} \in \mathbb{R}^5$$

does not have a zero for $(P,t) \in \mathbb{S}^4 \times [t_1, \infty)$, for t_1 large.

Furthermore, consider G as a map defined on $(t-1)P/t \in B_r(O)$ for $r > (t-1)/t$ and if

$$\deg(G, B_r(O), O) \ne 0, \quad \textit{for } r \ge r_1 = (t_1 - 1)/t_1, \tag{13}$$

then (3) *has a solution.*

In particular, if K has only isolated critical points in the region $\{x \in \mathbb{S}^4 : \Delta K(x) < 0\}$ and

$$\sum_{x \in \mathbb{S}^4 : \Delta K(x) < 0, \nabla K(x) = 0} ind(\nabla K(x)) \ne 1,$$

where $ind(\nabla K(x))$ stands for the index of the vector field $\nabla K(x)$ at its isolated zero x, then (13) *holds, therefore,* (3) *has a solution.*

A corollary of the proof for the $W^{2,p}$ estimate in Theorem 1 is a bound on a functional determinant whose critical points are solutions to (3). We recall that the relevant functional determinant is defined, similar to [9], through

$$II[w] = \fint_{\mathbb{S}^4} \left((\Delta_0 w)^2 + 2|\nabla_0 w|^2 + 12w\right) dvol_{g_c},$$

$$C_K[w] = 3\log\left(\fint_{\mathbb{S}^4} Ke^{4w}\, dvol_{g_c}\right),$$

and

$$\begin{aligned} Y[w] &= \frac{1}{36}\left(\fint_{\mathbb{S}^4} R^2\, dvol_{g_c} - \fint_{\mathbb{S}^4} R_0^2\, dvol_{g_c}\right) \\ &= \fint_{\mathbb{S}^4} \left(\Delta_0 w + |\nabla_0 w|^2\right)^2 dvol_{g_c} - 4\fint_{\mathbb{S}^4} |\nabla_0 w|^2\, dvol_{g_c}. \end{aligned}$$

$F[w] = Y[w] - II[w] + C_K[w]$ is the relevant functional determinant and a critical point of $F[w]$ is a solution of (3). It is known that, for any conformal transformation φ of $(\mathbb{S}^4, g_c)$, $Y[w_\varphi] = Y[w]$, and $II[w_\varphi] = II[w]$.

There is a similar functional determinant and a variational characterization for solutions to the prescribing Gaussian curvature problem on $\mathbb{S}^2$. Chang, Gursky and Yang proved in [12] that this functional is bounded on the set of solutions to the prescribing Gaussian curvature problem on $\mathbb{S}^2$ for any positive function K on $\mathbb{S}^2$ to be prescribed. Our corollary is in the same spirit.

Corollary 1 *Let $K(x)$ be a given positive C^2 function on $\mathbb{S}^4$. Then there is a bound C depending on K only through the C^2 norm of K, a positive upper and lower bound of K on $\mathbb{S}^4$, such that*

$$|F[w]| \le C$$

for all admissible solutions w to (3)*.*

Theorem 1 will be established using blow up analysis, Liouville type classification results of entire solutions, and integral type estimates for such fully nonlinear equations from [9] and [20]. Theorem 2 will be established using a weaker version of Theorem 1 and a Kazdan-Warner type identity satisfied by the solutions. The weaker version of Theorem 1 only needs to establish

$$v_j(P) - \frac{1}{4}\frac{6}{K(P_j)} \to 0 \quad \text{pointwise on } \mathbb{S}^4 \setminus \{-P_j\}, \text{ and bounded in } L^\infty(\mathbb{S}^4), \qquad (9')$$

instead of (9), (10) and the $W^{2,6}$ norm estimates. A degree argument for a fully nonlinear operator associated with (3) and Theorem 2 will be used to establish Theorem 3. To streamline our presentation, we will first outline the main steps for proving Theorem 3, assuming Theorem 2 and all the other needed ingredients. In the remaining sections, we will first provide a proof for (9′) in Theorem 1 and for Theorem 2, before finally providing a proof for the $W^{2,6}$ norm estimates in Theorem 1 and for Corollary 1.

2 Proof of Theorem 3

The first and third parts of Theorem 3 is contained in [6] and [12]. We will establish the second part of Theorem 3 by formulating the existence of a solution to (3) as a degree problem for a nonlinear map and linking the degree of this map to that of G.

By a fibration result from [5], [6], [3] and [29], see also [1], [32] for early genesis of these ideas, if we define

$$\mathscr{S}_0 = \{v \in C^{2,\alpha}(\mathbb{S}^4) : \int_{\mathbb{S}^4} e^{4v(x)} x \, dvol_{g_c} = 0\},$$

then the map $\pi : (v, \xi) \in \mathscr{S}_0 \times B \mapsto C^{2,\alpha}(\mathbb{S}^4)$ defined by

$$\pi(v, \xi) = v \circ \varphi_{P,t}^{-1} + \ln |d\varphi_{P,t}^{-1}|,$$

with B denoting the open unit ball in $\mathbb{R}^5$ and $\xi = rP$, $P \in \mathbb{S}^4$, $r = (t-1)/t$, $t \geq 1$, is a C^2 diffeomorphism from $\mathscr{S}_0 \times B$ onto $C^{2,\alpha}(\mathbb{S}^4)$. Thus $(v, P, t) \in \mathscr{S}_0 \times \mathbb{S}^4 \times [1, \infty)$ provide global coordinates for $C^{2,\alpha}(\mathbb{S}^4)$ (with a coordinate singularity at $t = 1$, similar to the coordinate singularity of polar coordinates at $r = 0$) through

$$w = v \circ \varphi_{P,t}^{-1} + \ln |d\varphi_{P,t}^{-1}|.$$

w solves (3) with K replaced by $K^{[s]}$ iff v solves

$$\sigma_2(A_v) = K^{[s]} \circ \varphi_{P,t} e^{4v}. \tag{14}$$

Then the estimates for w in Theorem 2 turn into the following estimates for v and t.

Proposition 1 *Assume that K is a positive C^2 function on $\mathbb{S}^4$ satisfying the non-degeneracy condition* (11)*, and let w be a solution to* (3) *with K replaced by $K^{[s]}$ and $(v, P, t) \in \mathscr{S}_0 \times \mathbb{S}^4 \times [1, \infty)$ be the coordinates of w defined in the paragraph above. Then there exist t_0 and $\varepsilon(s) > 0$ with $\lim_{s \to 0} \varepsilon(s) = 0$, such that*

$$t \leq t_0 \quad \textit{and} \quad ||v||_{C^{2,\alpha}(\mathbb{S}^4)} < \varepsilon(s). \tag{15}$$

A proof for Proposition 1 will be postponed to the end of the next section.

We treat (14) as a nonlinear map

$$F^{[s]}[v, \xi] := e^{-4v(x)} \sigma_2(g_c^{-1} \circ \left[A_{g_c} - \nabla^2 v + dv \otimes dv - \frac{1}{2} |\nabla v|^2 g_c\right]) - K^{[s]} \circ \varphi_{P,t},$$

from $\mathscr{S}_0 \times B$ into $C^{\alpha}(\mathbb{S}^4)$, for $0 < s \leq 1$, where $\xi = (t-1)P/t \in B$. Proposition 1 implies that there is a neighborhood $\mathscr{N} \subset \mathscr{S}_0$ of $0 \in \mathscr{S}_0$ and $0 < r_0 = (t_0 - 1)/t_0 < 1$ such that $F^{[s]}$ does not have a zero on $\partial(\mathscr{N} \times B_r)$ for all $r_0 \leq r < 1$ and $0 < s \leq 1$. According to [28], there is a well defined degree for $F^{[s]}$ on $\mathscr{N} \times B_{r_0}$ and it is independent of $0 < s \leq 1$. We will compute this degree of $F^{[s]}$, for $s > 0$ small, through the degree of a finite dimensional map.

We first use the implicit function theorem to define this map and link the solutions to (14) to the zeros of this map. Note that $F^{[0]}[0,(t-1)P/t]=0$ and $D_vF^{[0]}[0,(t-1)P/t](\eta)=-6\Delta\eta-24\eta$. If Π denotes the projection from $C^{\alpha}(\mathbb{S}^4)$ into

$$Y:=\{f\in C^{\alpha}(\mathbb{S}^4):\int_{\mathbb{S}^4} fx_j\,dvol_{g_c}=0\quad\text{for } j=1,\cdots,5\}$$

defined by $\Pi(f)=f-5|\mathbb{S}^4|^{-1}\sum_{j=1}^{5}(\int_{\mathbb{S}^4} fx_j\,dvol_{g_c})x_j$, then we can apply the implicit function theorem to $\Pi\circ F^{[s]}$ at $v=0$ to conclude

Proposition 2 *There exist some neighborhood $\mathscr{N}_\varepsilon\subset\mathscr{N}$ of $0\in\mathscr{S}_0$ and $s_0>0$, such that for all $0<s<s_0$, $(P,t)\in\mathbb{S}^4\times[1,t_0]$, there exists a* unique $v=v(x;P,t,s)\in\mathscr{N}_\varepsilon$, *depending differentiably on (P,t,s) such that*

$$\Pi\circ F^{[s]}[v(x;P,t,s),(t-1)P/t]=0. \tag{16}$$

Furthermore, there exists some $C>0$ such that, for $0<s\leq s_0$, $1\leq t\leq t_0$,

$$||v(x;P,t,s)||_{C^{2,\alpha}(\mathbb{S}^4)}\leq C||K^{[s]}\circ\varphi_{P,t}-6||_{C^{\alpha}(\mathbb{S}^4)}=Cs||K\circ\varphi_{P,t}-6||_{C^{\alpha}(\mathbb{S}^4)}. \tag{17}$$

(16) implies that

$$F^{[s]}[v(x;P,t,s),(t-1)P/t]=\sum_{j=1}^{5}\Lambda_j(P,t,s)x_j,$$

for some Lagrange multipliers $\Lambda_j(P,t,s)$, which depend differentiably on (P,t,s). Or, equivalently,

$$\sigma_2(A_{v(x;P,t,s)})=\left(K^{[s]}\circ\varphi_{P,t}+\sum_{j=1}^{5}\Lambda_j(P,t,s)x_j\right)e^{4v(x;P,t,s)}. \tag{18}$$

A zero of the map $\Lambda^{[s]}(P,t):=(\Lambda_1(P,t,s),\cdots,\Lambda_5(P,t,s))$ corresponds to a solution to (14). Propositions 1 and 2 say that, for $s_0>0$ small, all solutions $v\in\mathscr{S}_0$ to (14), for $0<s\leq s_0$, are in $\mathscr{N}_\varepsilon$, thus correspond to the zeros of the map $\Lambda^{[s]}(P,t)$.

Remark 3 $||K\circ\varphi_{P,t}-6||_{C^{\alpha}(\mathbb{S}^4)}$ could become unbounded when $t\to\infty$; yet thanks to the bound $1\leq t\leq t_0$ from Proposition 1, it remains bounded in terms of $||K||_{C^{\alpha}(\mathbb{S}^4)}$ in the range $1\leq t\leq t_0$. Note also that $||K\circ\varphi_{P,t}-6||_{L^p(\mathbb{S}^4)}$ remains bounded in terms of $||K||_{L^\infty(\mathbb{S}^4)}$ even in the range $1\leq t<\infty$. It is essentially this bound and the applicability of $W^{2,p}$ estimates in the semilinear setting of [12] and [29] which allowed them to handle their cases without using the bound $1\leq t\leq t_0$.

Remark 4 The implicit function theorem procedure here works also in the setting of [6], [12] and [29] using $W^{2,p}$ space, as does Proposition 1 in the setting of [12] and [29], and can be used to simplify the arguments there.

At this point, we need the following Kazdan-Warner type identity for solutions to (3).

Proposition 3 *Let w be a solution to* (3). *Then, for* $1 \le j \le 5$,

$$\int_{\mathbb{S}^4} \langle \nabla K(x), \nabla x_j \rangle e^{4w(x)}\, dvol_{g_c} = 0. \tag{19}$$

Proposition 3 is a special case of the results in [37] and [21]. But in the special case of $\mathbb{S}^4$, it is a direct consequence of the variational characterization of the solution to (3), as given after the statement of Theorem 1. A solution w to (3) is a critical point of $F[w] = Y[w] - I\!I[w] + C_K[w]$ there, thus satisfies, for any one-parameter family of conformal diffeomorphisms φ_s of $\mathbb{S}^4$ with $\phi_0 = $Id,

$$\frac{d}{ds}\Big|_{s=0} F[w_{\varphi_s}] = 0,$$

with $w_{\varphi_s} = w \circ \varphi_s + \log|d\varphi_s|$. Since $Y[w_{\varphi_s}] = Y[w]$ and $I\!I[w_{\varphi_s}] = I\!I[w]$, a solution w of (3) thus satisfies

$$\frac{d}{ds}\Big|_{s=0} C_K[w_{\varphi_s}] = \frac{d}{ds}\Big|_{s=0} \left(⨍_{\mathbb{S}^4} K \circ \varphi_s^{-1} e^{4w}\, dvol_{g_c} \right) = 0,$$

which is (19).

Applying (19) to $v(x;P,t,s)$, a solution to (18), we obtain,

$$\int_{\mathbb{S}^4} \langle \nabla \left(K^{[s]} \circ \varphi_{P,t}(x) + \sum_{j=1}^{5} \Lambda_j(P,t,s) x_j \right), \nabla x_k \rangle e^{4v(x;P,t,s)}\, dvol_{g_c} = 0, \quad \text{for } 1 \le k \le 5,$$

from which we obtain, for $1 \le k \le 5$,

$$\begin{aligned} &- \sum_{j=1}^{5} \Lambda_j(P,t,s) \int_{\mathbb{S}^4} \langle \nabla x_j, \nabla x_k \rangle e^{4v(x;P,t,s)}\, dvol_{g_c} \\ =& \int_{\mathbb{S}^4} \langle \nabla \left(K^{[s]} \circ \varphi_{P,t}(x) \right), \nabla x_k \rangle e^{4v(x;P,t,s)}\, dvol_{g_c} \\ =& s \int_{\mathbb{S}^4} \langle \nabla \left(K \circ \varphi_{P,t}(x) \right), \nabla x_k \rangle e^{4v(x;P,t,s)}\, dvol_{g_c}. \end{aligned}$$

As in [6], [12] and [29], we define

$$A^{[s]}(P,t) = (4|\mathbb{S}^4|)^{-1} \int_{\mathbb{S}^4} \langle \nabla \left(K \circ \varphi_{P,t}(x) \right), \nabla x \rangle e^{4v(x;P,t,s)}\, dvol_{g_c} \in \mathbb{R}^5.$$

Since

$$\left(\int_{\mathbb{S}^4} \langle \nabla x_j, \nabla x_k \rangle e^{4v(x;P,t,s)}\, dvol_{g_c} \right)$$

is positive definite, we conclude that

$$\deg(A^{[s]}, B_{r_0}, O) = -\deg(\Lambda^{[s]}, B_{r_0}, O),$$

for $s_0 > s > 0$ provided that one of them is well defined.

Using $v(x;P,t,s) \in \mathscr{S}_0$, and $\Delta x = -4x$ on $\mathbb{S}^4$, we have, as in [6], [12] and [29],

$$A^{[s]}(P,t) = G(P,t) + I + \Pi,$$

where
$$I = |\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4}(K\circ\varphi_{P,t}(x)-K(P))\,x\left(e^{4v(x;P,t,s)}-1\right)dvol_{g_c},$$
and
$$\Pi = -(4|\mathbb{S}^4|)^{-1}\int_{\mathbb{S}^4}(K\circ\varphi_{P,t}(x)-K(P))\,\langle\nabla x,\nabla e^{4v(x;P,t,s)}\rangle\,dvol_{g_c}.$$
We could have fixed $t_0 \geq t_1$ such that $G(P,t) \neq 0$ for $t = t_0$, and there will be a $\delta > 0$ such that $|G(P,t)| \geq \delta$ for $t = t_0$. Since (17) implies that
$$||v(x;P,t,s)||_{C^{2,\alpha}(\mathbb{S}^4)} = O(s), \quad \text{uniformly for } (P,t) \in \mathbb{S}^4\times[1,t_0],$$
we find that, by fixing $s_0 > 0$ small if necessary,
$$|I| + |\Pi| \leq \frac{1}{2}|G(P,t)|, \quad \text{for } 0 < s \leq s_0 \text{ and } t = t_0.$$
This implies that $A^{[s]}(P,t)\cdot G(P,t) > 0$ for $0 < s \leq s_0$ and $t = t_0$. Therefore
$$-\deg(\Lambda^{[s]},B_{r_0},O) = \deg(A^{[s]},B_{r_0},O) = \deg(G,B_{r_0},O) \neq 0,$$
for $0 < s \leq s_0$. Finally, we now prove
$$\deg(F^{[s]},\mathscr{N}\times B_{r_0},O) = -\deg(\Lambda^{[s]},B_{r_0},O), \tag{20}$$
for $0 < s \leq s_0$, from which follows the existence of a solution to (3).

The verification of (20) is routine, but requires several steps. First, we may perturb K, if necessary, within the class of functions satisfying the conditions in Theorem 3 such that the corresponding $G(P,t)$ has only isolated and non-degenerate zeros in $B_{r_0}(O)$. We will prove momentarily that for $s > 0$ small, the zeros of $\Lambda^{[s]}$ for $s > 0$ small will be close to the zeros of $G(P,t)$ and are isolated, non-degenerate. Therefore the zeros of $F^{[s]}$ in $\mathscr{N}\times B_{r_0}$ are isolated and non-degenerate. This can be argued as follows. First, it follows from (18) that
$$\Lambda^{[s]}(\xi)\cdot x = (\mathrm{Id}-\Pi)\left(e^{-4v(x;\xi,s)}\sigma_2(A_{v(x;\xi,s)}) - K^{[s]}\circ\varphi_{P,t}\right).$$
Using (17), we can then write
$$e^{-4v(x;\xi,s)}\sigma_2(A_{v(x;\xi,s)}) = 6 - 6\Delta v(x;\xi,s) - 24v(x;\xi,s) + Q(v(x;\xi,s)),$$
with $||Q(v(x;\xi,s))||_Y \lesssim ||v(x;\xi,s)||_X^2 \lesssim s^2$. Therefore, using
$$(\mathrm{Id}-\Pi)(1) = (\mathrm{Id}-\Pi)\,(6\Delta v(x;\xi,s) + 24v(x;\xi,s)) = 0,$$
and
$$\Lambda^{[s]}(\xi) = 5|\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4}\left(\Lambda^{[s]}(\xi)\cdot x\right)x\,dvol_{g_c},$$
we have
$$\Lambda^{[s]}(\xi) = -5sG(P,t) + 5|\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4}[(\mathrm{Id}-\Pi)\,(Q(v(x;\xi,s)))]\,x\,dvol_{g_c},$$
with $|(\mathrm{Id}-\Pi)\,(Q(v(x;\xi,s)))| \lesssim s^2$, so the zeros of $\Lambda^{[s]}(\xi)$ for $s > 0$ small are close to the zeros of $G(P,t)$. We can further use the implicit function theorem to prove that for $s > 0$ small there is a (unique) non-degenerate zero of $\Lambda^{[s]}(\xi)$ near each zero of $G(P,t)$.

Remark 5 This argument shows that, for each non-degenerate zero of $G(P,t)$, if we associate (P,t) with the center of mass of $\varphi_{P,t}$,

$$C.M(\varphi_{P,t}) := |\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4}\varphi_{P,t}(x)\,dvol_{g_c} \in B_1(O)$$

as a geometric representation of (P,t), then for $s>0$ small, there is a unique solution w to (3) whose center of mass approaches $C.M(\varphi_{P,t})$, for our argument gives rise to a solution

$$w(x) = v(\cdot;P',t')\circ\varphi^{-1}_{P',t'}(x) + \ln|d\varphi^{-1}_{P',t'}(x)|$$

with (P',t') approaching (P,t), and $v(x;P',t')$ approaching 0 as $s\to 0$, thus the center of mass of w is

$$|\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4} e^{4w(x)}x\,dvol_{g_c} = |\mathbb{S}^4|^{-1}\int_{\mathbb{S}^4} e^{4v(y)}\varphi_{P',t'}(y)\,dvol_{g_c} \to C.M(\varphi_{P,t})$$

as $s\to 0$.

Now $\deg(F^{[s]},\mathscr{N}\times B_{r_0},O)$ is well defined in the manner of [28], and according to Propositions 2.1–2.4 of [28],

$$\deg(F^{[s]},\mathscr{N}\times B_{r_0},O) = \sum_{\xi\in B_{r_0}(O):\Lambda^{[s]}(\xi)=0} \mathrm{ind}(DF^{[s]}[v(x;\xi,s),\xi]),$$

where $\mathrm{ind}(DF^{[s]}[v(x;\xi,s),\xi])$ refers to the index of the linear operator $DF^{[s]}[v(x;\xi,s),\xi]$, and is computed as $(-1)^{\beta}$, with β denoting the number of negative eigenvalues of $DF^{[s]}[v(x;\xi,s),\xi]$. We also have

$$\deg(\Lambda^{[s]},B_{r_0},O) = \sum_{\xi\in B_{r_0}(O):\Lambda^{[s]}(\xi)=0} \mathrm{ind}(D\Lambda^{[s]}(\xi)).$$

To compute $DF^{[s]}[v(x;\xi,s),\xi]$, we identify $\xi\in\mathbb{R}^5$ with $\xi\cdot x\in\mathrm{span}\{x_1,\cdots,x_5\}$, and write the differential of $F^{[s]}$ in the direction of $\dot v$ as $D_vF^{[s]}[v(x;\xi,s),\xi](\dot v)$, or simply $D_vF^{[s]}(\dot v)$, and the differential of $F^{[s]}$ in the direction of $\dot\xi\cdot x$ as $D_\xi F^{[s]}[v(x;\xi,s),\xi](\dot\xi)$. Then

$$D_\xi F^{[s]}[v(x;\xi,s),\xi](\dot\xi) = -s\dot\xi\cdot\nabla_\xi\left(K\circ\varphi_{P,t}\right),$$

and

$$D_vF^{[s]}[v(x;\xi,s),\xi](\dot v) = M^{ij}[v(x;\xi,s)]\nabla^{v(x;\xi,s)}_{ij}\dot v - 4K^{[s]}\circ\varphi_{P,t}\dot v,$$

where $M^{ij}[v(x;\xi,s)]$ stands for the Newton tensor associated with $\sigma_2(e^{-2v(x;\xi,s)}A_{v(x;\xi,s)})$, and $\nabla^{v(x;\xi,s)}_{ij}$ stands for the covariant differentiation in the metric $e^{2v(x;\xi,s)}g_c$. Thus,

$$DF^{[s]}[v(x;\xi,s),\xi](\dot v+\dot\xi\cdot x) = M^{ij}[v(x;\xi,s)]\nabla^{v(x;\xi,s)}_{ij}\dot v - 4K^{[s]}\circ\varphi_{P,t}\dot v - s\dot\xi\cdot\nabla_\xi\left(K\circ\varphi_{P,t}\right).$$

At a fixed zero ξ of $\Lambda^{[s]}(\xi)=0$, we define a family of deformed linear operators $L_{\tau,s}$ for $0\le\tau\le 1$ by

$$L_{\tau,s}(\dot v+\dot\xi\cdot x) = M^{ij}[v^{[\tau]}]\nabla^{v^{[\tau]}}_{ij}\dot v - 4K^{[s\tau]}\circ\varphi_{P,t}\dot v - s\dot\xi\cdot\nabla_\xi\left(K\circ\varphi_{P,t}\right),$$

where $v^{[\tau]} = \tau v(x;\xi,s)$, and $\dot v\in X := \{\dot v\in C^{2,\alpha}(\mathbb{S}^4) : \int_{\mathbb{S}^4}\dot v(x)x_j = 0, j=1,\cdots,5\}$. Then $L_{\tau,s}$ defines self-adjoint operators with respect to the metric $e^{2v^{[\tau]}}g_c$, thus its eigenvalues are all real. We first assume the

Claim For $s > 0$ small and $0 \leq \tau \leq 1$, the spectrum of $L_{\tau,s}$ does not contain zero.

Thus $\mathrm{ind}(L_{0,s}) = \mathrm{ind}(L_{1,s}) = \mathrm{ind}(DF^{[s]}[v(x;\xi,s),\xi])$. We will next establish

$$\mathrm{ind}(L_{0,s}) = (-1)^{1+\gamma}, \quad \text{for } s > 0 \text{ small}, \tag{21}$$

where γ is the number of positive eigenvalues of $\nabla G(P,t)$ at ξ, and

$$\gamma = \text{ the number of negative eigenvalues of } D_\xi \Lambda^{[s]}(\xi) \text{ for } s > 0 \text{ small}. \tag{22}$$

First note that

$$L_{0,s}(\dot{v} + \dot{\xi} \cdot x) = -6\Delta \dot{v} - 24\dot{v} - s\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right),$$

so if $\dot{v} + \dot{\xi} \cdot x$ is an eigenfunction corresponding to a negative eigenvalue $-\lambda$, with $\dot{v} \in X$, then

$$-6\Delta \dot{v} - 24\dot{v} - s\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right) = -\lambda(\dot{v} + \dot{\xi} \cdot x).$$

Taking projection in $\mathrm{span}\{x_1, \cdots, x_5\}$, we find

$$-s\nabla G(P,t)\dot{\xi} = -\frac{\lambda}{5}\dot{\xi},$$

and taking projection in X, we find

$$-6\Delta \dot{v} - 24\dot{v} - s\Pi\left(\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right)\right) = -\lambda \dot{v}. \tag{23}$$

If $\dot{\xi} \neq 0$, then $\dot{\xi}$ is an eigenvector of $\nabla G(P,t)$ with eigenvalue $\frac{\lambda}{5s} > 0$; and if $\dot{\xi} = 0$, then $\dot{v} \neq 0$ solves $-6\Delta \dot{v} - 24\dot{v} = -\lambda \dot{v}$, which is possible for some $\lambda > 0$ iff $-\lambda = -24$ and $\dot{v} = \text{constant}$. Conversely, for any eigenvector $\dot{\xi} \neq 0$ of $\nabla G(P,t)$ with eigenvalue $\mu > 0$, the operator $-6\Delta - 24 + 5s\mu$ is an isomorphism from X to Y for $s > 0$ small, so we can solve (23) as $(-6\Delta - 24 + 5s\mu)\dot{v} = s\Pi\left(\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right)\right)$ for $\dot{v} \in X$ and $\dot{v} + \dot{\xi} \cdot x$ becomes an eigenfunction of $L_{0,s}$ with eigenvalue $-5s\mu$. Therefore we conclude (21).

We now establish (22) to prove $\mathrm{ind}(DF^{[s]}[v(x;\xi,s),\xi]) = -\mathrm{ind}(D_\xi \Lambda^{[s]}(\xi))$ for $s > 0$ small. From (18), which can be written as $\Lambda^{[s]}(\xi) \cdot x = F^{[s]}[v(x;\xi,s),\xi])$, we obtain

$$D_v F^{[s]}(D_\xi v(x;\xi,s)(\dot{\xi})) + D_\xi F^{[s]}(\dot{\xi}) = D_\xi \Lambda^{[s]}(\xi)(\dot{\xi}) \cdot x.$$

Taking projections in X and $\mathrm{span}\{x_1, \cdots, x_5\}$, respectively, and using $D_\xi F^{[s]}(\dot{\xi}) = -s\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right)$, we obtain

$$\Pi\left(D_v F^{[s]}(D_\xi v(x;\xi,s)(\dot{\xi}))\right) - s\Pi\left(\dot{\xi} \cdot \nabla_\xi \left(K \circ \varphi_{P,t}\right)\right) = 0, \tag{24}$$

and

$$\fint_{\mathbb{S}^4} (\mathrm{Id} - \Pi)\left(D_v F^{[s]}(D_\xi v(x;\xi,s)(\dot{\xi}))\right) x \, dvol_{g_c} - s\nabla_\xi G(P,t)\dot{\xi} = \frac{1}{5} D_\xi \Lambda^{[s]}(\xi)(\dot{\xi}). \tag{25}$$

Writing

$$D_v F^{[s]}(D_\xi v(x;\xi,s)(\dot\xi)) = (-6\Delta - 24)(D_\xi v(x;\xi,s)(\dot\xi)) + \Theta(D_\xi v(x;\xi,s)(\dot\xi)),$$

we find, using (17), that $||\Theta(D_\xi v(x;\xi,s)(\dot\xi))||_Y \lesssim s||D_\xi v(x;\xi,s)(\dot\xi)||_X$. Thus

$$\Pi\left(D_v F^{[s]}(\cdot)\right) : X \mapsto Y$$

is an isomorphism for $s > 0$ small and has an inverse Ψ, and we can solve $D_\xi v(x;\xi,s)(\dot\xi)$ in terms of $\dot\xi$ from (24):

$$D_\xi v(x;\xi,s)(\dot\xi) = \Psi\left(s\Pi\left(\dot\xi \cdot \nabla_\xi (K \circ \varphi_{P,t})\right)\right) := s\Upsilon(\dot\xi).$$

Using this in (25), we find

$$\begin{aligned}\frac{1}{5} D_\xi \Lambda^{[s]}(\xi) &= -s\nabla_\xi G(P,t) + s\fint_{\mathbb{S}^4} [(\mathrm{Id} - \Pi)\circ\Theta\circ\Upsilon]\, x\, dvol_{g_c} \\ &= -s\left(\nabla_\xi G(P,t) + O(s)\right).\end{aligned}$$

Thus for $s > 0$ small, γ matches the number of negative eigenvalues of $D_\xi \Lambda^{[s]}(\xi)$, and we can conclude that $\mathrm{ind}(DF^{[s]}(v(x;\xi,s),\xi)) = -\mathrm{ind}(D_\xi \Lambda^{[s]}(\xi))$.

In the remainder of this section, we provide proof for our **Claim** above, leaving the proof for Proposition 1 to the end of the next section.

Proof (of **Claim***)* Suppose that for (a sequence of) $s > 0$ small and some $0 \le \tau \le 1$, $L_{\tau,s}$ has $\dot v + \dot\xi \cdot x$, with $\dot v \in X$, $\dot\xi \in \mathbb{R}^5$, as eigenfunction with zero eigenvalue. Then, taking projections in $\mathrm{span}\{x_1,\cdots,x_5\}$ and X, respectively, we obtain

$$\Pi\left[M^{ij}[v^{[\tau]}]\nabla_{ij}^{v^{[\tau]}}\dot v - 4K^{[s\tau]}\circ\varphi_{P,t}\dot v\right] - s\Pi\left[\dot\xi\cdot\nabla_\xi (K\circ\varphi_{P,t})\right] = 0, \tag{26}$$

and

$$\fint_{\mathbb{S}^4} (\mathrm{Id} - \Pi)\left[M^{ij}[v^{[\tau]}]\nabla_{ij}^{v^{[\tau]}}\dot v - 4K^{[s\tau]}\circ\varphi_{P,t}\dot v\right] x\, dvol_{g_c} - s\nabla_\xi G(P,t)\dot\xi = 0. \tag{27}$$

Using (17) again, we find

$$M^{ij}[v^{[\tau]}]\nabla_{ij}^{v^{[\tau]}}\dot v - 4K^{[s\tau]}\circ\varphi_{P,t}\dot v = -6\Delta\dot v - 24\dot v + \Theta^{\tau,s}(\dot v),$$

with $||\Theta^{\tau,s}(\dot v)||_Y \lesssim s\tau||\dot v||_X$. Thus, for $s > 0$ small, we can solve $\dot v$ from (26) to obtain

$$\dot v = \Psi\left(s\Pi\left[\dot\xi\cdot\nabla_\xi (K\circ\varphi_{P,t})\right]\right) = s\Upsilon(\dot\xi).$$

Thus $\dot\xi \ne 0$ and we can normalize it so that $|\dot\xi| = 1$. Using this in (27), we find

$$s(\mathrm{Id} - \Pi)\circ\Theta^{\tau,s}\circ\Upsilon(\dot\xi) - s\nabla_\xi G(P,t)\dot\xi = 0.$$

Using $||\Theta^{\tau,s}|| \lesssim s\tau$, we find this impossible for $s > 0$ small under our non-degeneracy assumption on the zeros of $G(P,t)$.

3 Proof of $(9')$, (10), Theorem 2 and Proposition 1

Proof (of $(9')$ *and* (10)*)* The full strength of (9) is established as soon as the $W^{2,3}$ estimates are established — the latter is a step in proving the $W^{2,6}$ estimates.

If there is a sequence of solutions w_j to (3) such that $\max w_j = w_j(P_j) \to \infty$, then we choose conformal automorphism $\phi_j = \phi_{P_j,t_j}$ of $\mathbb{S}^4$, such that the rescaled function

$$v_j(P) = w_j \circ \phi_j(P) + \ln|d\phi_j(P)|, \tag{28}$$

satisfies the normalization condition

$$v_j(P_j) = \frac{1}{4}\ln\frac{6}{K(P_j)}. \tag{29}$$

If we use stereographic coordinates for $\mathbb{S}^4$, with P_j as the north pole, then

$$y\left(\phi_j(P)\right) = t_j y(P), \text{ for } P \in \mathbb{S}^4,$$

and

$$v_j(P) = w_j \circ \phi_j(P) + \ln\frac{t_j\left(1+|y(P)|^2\right)}{1+t_j^2|y(P)|^2}.$$

v_j would satisfy

$$\sigma_2(A_{v_j}) = K \circ \phi_j e^{4v_j}. \tag{30}$$

The normalization in (29) amounts to choosing t_j such that

$$w_j(P_j) - \ln t_j = \frac{1}{4}\ln\frac{6}{K(P_j)}.$$

Thus, $t_j \to \infty$, and for any $P \in \mathbb{S}^4$,

$$v_j(P) \le \frac{1}{4}\ln\frac{6}{K(P_j)} + \ln\frac{t_j^2\left(1+|y(P)|^2\right)}{1+t_j^2|y(P)|^2}. \tag{31}$$

(31) implies that, away from $-P_j$, v_j has an upper bound independent of j. Together with (30), the local gradient and higher derivative estimates of [17], there exists a subsequence, still denoted as $\{v_j\}$, such that, $P_j \to P_*$, and for any $\delta > 0$,

$$v_j \to v_\infty \text{ in } C^{2,\alpha}\left(\mathbb{S}^4 \setminus B_\delta(-P_*)\right), \quad \text{for some limit} \quad v_\infty. \tag{32}$$

We also have

$$\sigma_2(A_{v_\infty}) = K(P_*)e^{4v_\infty} \qquad \text{on} \quad \mathbb{S}^4 \setminus \{-P_*\}, \tag{33}$$

$$\int_{\mathbb{S}^4} K(P_*)e^{4v_\infty}\, dvol_{g_c} \le \liminf_{j\to\infty}\int_{\mathbb{S}^4} K\circ\phi_j e^{4v_j}\, dvol_{g_c} = 16\pi^2, \tag{34}$$

$$v_\infty(P_*) = \frac{1}{4}\ln\frac{6}{K(P_*)}, \quad \nabla v_\infty(P_*) = 0, \tag{35}$$

$$v_\infty(P) \le \frac{1}{4}\ln\frac{6}{K(P_*)} + \ln\frac{1+|y(P)|^2}{|y(P)|^2}. \tag{36}$$

A Liouville type classification result in [10] and [26] says that

$$v_\infty - \frac{1}{4}\ln\frac{6}{K(P_*)} = \ln|d\phi|$$

for some conformal automorphism ϕ of $\mathbb{S}^4$, which together with (35) implies that

$$v_\infty \equiv \frac{1}{4}\ln\frac{6}{K(P_*)}. \tag{37}$$

Thus for any $\delta > 0$,

$$\lim_{j\to\infty}\int_{\mathbb{S}^4\setminus B_\delta(-P_*)} K\circ\phi_j e^{4v_j}\,dvol_{g_c} = 6\left|\mathbb{S}^4\setminus B_\delta(-P_*)\right|.$$

Together with the Gauss-Bonnet formula

$$\int_{\mathbb{S}^4} K\circ\phi_j e^{4v_j}\,dvol_{g_c} = 6\left|\mathbb{S}^4\right|,$$

we have

$$\lim_{j\to\infty}\int_{B_\delta(-P_*)} K\circ\phi_j e^{4v_j}\,dvol_{g_c} = 6\left|B_\delta(-P_*)\right|.$$

This allows us to apply our Theorem in [20] on $B_\delta(-P_*)$ for small $\delta > 0$ to conclude that $\exists C > 0$, such that

$$\max_{\mathbb{S}^4} v_j \le C. \tag{38}$$

Next we declare the

Claim There exists $C' > 0$ such that

$$\min_{\mathbb{S}^4} v_j \ge -C'. \tag{39}$$

The Claim can be proved making use of the information that $R_{v_j} = R_{w_j}\circ\phi_j \ge 0$, which implies

$$2 - \Delta v_j - |\nabla v_j|^2 \ge 0. \tag{40}$$

Thus

$$\begin{aligned} v_j(P) - \bar{v_j} &= \int_{\mathbb{S}^4} (-\Delta v_j(Q))\, G(P,Q)\, dvol_{g_c}(Q) \\ &\ge -2\int_{\mathbb{S}^4} G(P,Q)\, dvol_{g_c}(Q), \end{aligned} \tag{41}$$

where $G(P,Q)$ is the Green's function of $-\Delta$ on $\mathbb{S}^4$. Integrating (40) over $\mathbb{S}^4$ implies that

$$2 \ge \fint_{\mathbb{S}^4} |\nabla v_j|^2 \ge \text{const.} \left(\fint_{\mathbb{S}^4} |v_j(P) - \bar{v_j}|^4\right)^{\frac{1}{2}}. \tag{42}$$

(32), (37), (41), and (42) conclude the Claim and (9′).

Next we prove the integral estimate (10). This can be seen by looking at the integral version of the equation

$$\int_{\mathbb{S}^4} \left[2|\nabla v_j|^2 + 2\Delta v_j - 6\right] \langle \nabla v_j, \nabla \eta \rangle + \Delta \eta |\nabla v_j|^2 + \left(K \circ \phi_j e^{4v_j} - 6\right) \eta = 0.$$

If we plug in $\eta = v_j$, we obtain

$$\int_{\mathbb{S}^4} \left(6 - 2|\nabla v_j|^2 - 3\Delta v_j\right) |\nabla v_j|^2 = \int_{\mathbb{S}^4} \left(K \circ \phi_j e^{4v_j} - 6\right) v_j.$$

Using $\Delta v_j \leq 2 - |\nabla v_j|^2$, we have

$$\int_{\mathbb{S}^4} |\nabla v_j|^4 \leq \int_{\mathbb{S}^4} \left(K \circ \phi_j e^{4v_j} - 6\right) v_j, \tag{43}$$

which converges to 0 by (32), (37), (38), (39) and the Dominated Convergence Theorem.

Next, we prove Theorem 2. We will first prove that, under our non-degeneracy conditions on K, there is a bound $C > 0$ depending on the quantities as in the statements of Theorem 2, but uniform in $0 < s \leq 1$, such that any solution w of (3) with $K^{[s]}$ satisfies $\max_{\mathbb{S}^4} w_j \leq C$. Once we have the bound $\max_{\mathbb{S}^4} w_j \leq C$, the $C^{2,\alpha}$ estimates follow from known theory of fully nonlinear elliptic equations.

Proof (of Theorem 2) Suppose, on the contrary, that $\max_{\mathbb{S}^4} w_j \to \infty$ (for a sequence of K's, which we write as a single K for simplicity, satisfying the bounds in Theorem 2). Then, as proved above, (9′) holds. Let P_j, t_j be as defined in the earlier part of the proof. We will then prove the following estimates:

$$|\nabla K(P_j)| = \frac{o(1)}{t_j}, \quad \text{as } j \to \infty, \tag{44}$$

and

$$\Delta K(P_j) \to 0, \quad \text{as } j \to \infty. \tag{45}$$

(44) and (45) would contradict our hypotheses on K.

The main idea is to examine the Kazdan-Warner identity in the light of the asymptotic profile of w_j as given in Theorem 1.

For each w_j, we choose stereographic coordinates with P_j as the north pole. For $P = (x_1, \cdots, x_5) \in \mathbb{S}^4$, let its stereographic coordinates be $y = (y_1, \cdots, y_4)$. Also set $x' = (x_1, \cdots, x_4)$. Then

$$\begin{cases} x_i = \dfrac{2y_i}{1+|y|^2}, & i = 1,2,3,4, \\ x_5 = \dfrac{|y|^2 - 1}{|y|^2 + 1}. \end{cases} \tag{46}$$

For any $\varepsilon > 0$, there exists $M > 0$ such that for any P with $|y(P)| > M$, we have

$$K(P) = K(P_j) + \sum_{i=1}^{4} a_i x_i + \sum_{k,h=1}^{4} b_{hk} x_h x_k + r(P), \tag{47}$$

with

$$|r(P)| \le \varepsilon |x'|^2, \quad |x'||\nabla r(P)| \le \varepsilon |x'|^2, \quad |x'|^2 |\nabla^2 r(Q)| \le \varepsilon |x'|^2. \tag{48}$$

We can identify $a_i = \nabla_i K(P_j)$, $b_{hk} = \nabla_{hk} K(P_j)$, and we may assume that b_{hk} is diagonalized: $b_{hk} = \delta_{hk} b_h$. Then, using

$$\nabla x_1 = (1 - x_1^2, -x_1 x_2, \cdots, -x_1 x_5), \cdots, \nabla x_5 = (-x_5 x_1, \cdots, -x_5 x_4, 1 - x_5^2),$$

and

$$\nabla x_i \cdot \nabla x_h = \delta_{ih} - x_i x_h,$$

we have, for $1 \le h \le 4$,

$$\langle \nabla K, \nabla x_h \rangle = a_h - \sum_{i=1}^{4} a_i x_i x_h + 2 b_h x_h - 2 \sum_{i=1}^{4} b_i x_i^2 x_h + \nabla r \cdot \nabla x_h.$$

So we can fix M large such that, when $|y| > M$,

$$\begin{cases} \langle \nabla K, \nabla x_h \rangle = a_h + 2 b_h x_h + r_1(P), \\ \quad |r_1(P)| \le \varepsilon |x'|. \end{cases} \tag{49}$$

From the Kazdan-Warner identity, we have

$$\begin{aligned} 0 &= \int_{\mathbb{S}^4} \langle \nabla K^{[s]}, \nabla x_h \rangle e^{4w_j} \, dvol_{g_c} \\ &= s \int_{\mathbb{S}^4} \langle \nabla K, \nabla x_h \rangle e^{4w_j} \, dvol_{g_c} \end{aligned}$$

Thus, the deformation parameter s is divided out from the Kazdan-Warner identity to give

$$\begin{aligned} 0 &= \int_{\mathbb{S}^4} \langle \nabla K, \nabla x_h \rangle e^{4w_j} \, dvol_{g_c} \\ &= \int_{|y| \le M} \langle \nabla K, \nabla x_h \rangle e^{4w_j} \, dvol_{g_c} + \int_{|y| > M} \langle \nabla K, \nabla x_h \rangle e^{4w_j} \, dvol_{g_c}. \end{aligned}$$

Remark 6 It is this property that K, not $K^{[s]}$, can be used in the Kazdan-Warner identity that allows us to obtain bounds on w uniform in $0 < s \le 1$. This also applies to the settings in [12] and [29] to make the estimates there uniform in $0 < s \le 1$ in the respective deformations.

We estimate

$$\begin{aligned}\int_{|y|\le M}\langle\nabla K,\nabla x_h\rangle e^{4w_j}\,dvol_{g_c} &\le C\int_{|y|\le M} e^{4w_j}\left(\frac{2}{1+|y|^2}\right)^4 dy\\ &= C\int_{|z|\le\frac{M}{t_j}} e^{4v_j}\left(\frac{2}{1+|z|^2}\right)^4 dz\\ &\le C\left(\frac{M}{t_j}\right)^4,\quad \text{using (38) and (39).}\end{aligned}$$

$$\begin{aligned}&\int_{|y|>M}\langle\nabla K,\nabla x_h\rangle e^{4w_j}\,dvol_{g_c}\\ =&a_h\int_{|y|>M} e^{4w_j}\,dvol_{g_c}+2b_h\int_{|y|>M} e^{4w_j}x_h\,dvol_{g_c}+\int_{|y|>M} e^{4w_j}r_1(P)\,dvol_{g_c}.\end{aligned}$$

The following estimates will complete the proof of (44).

$$\lim_{j\to\infty}\int_{|y|>M} e^{4w_j}\,dvol_{g_c}=\frac{6}{K(P_*)}\left|\mathbb{S}^4\right|. \tag{50}$$

$$\int_{|y|>M} e^{4w_j}x_h\,dvol_{g_c}=\frac{o(1)}{t_j},\quad \text{as } j\to\infty\ (1\le h\le 4). \tag{51}$$

$$\int_{|y|>M} e^{4w_j}r_1(P)\,dvol_{g_c}=\frac{o(1)}{t_j},\quad \text{as } j\to\infty. \tag{52}$$

Here are the verifications of the above estimates.

$$\int_{|y|>M} e^{4w_j}\,dvol_{g_c}=\int_{|z|>\frac{M}{t_j}} e^{4v_j}\left(\frac{2}{1+|z|^2}\right)^4 dz\to\frac{6}{K(P_*)}\left|\mathbb{S}^4\right|,$$

by (32), (38) and (39).

$$\begin{aligned}\int_{|y|>M} e^{4w_j}x_h\,dvol_{g_c} &=\int_{|z|>\frac{M}{t_j}}\frac{2t_jz_h}{1+t_j^2|z|^2}e^{4v_j}\left(\frac{2}{1+|z|^2}\right)^4 dz\\ &=\int_{|z|>\frac{M}{t_j}}\frac{2t_jz_h}{1+t_j^2|z|^2}\left(e^{4v_j}-\frac{6}{K(P_*)}\right)\left(\frac{2}{1+|z|^2}\right)^4 dz\\ &=\int_{|z|>\delta}+\int_{\delta>|z|>\frac{M}{t_j}},\end{aligned}$$

with

$$\left|\int_{\delta>|z|>\frac{M}{t_j}}\right|\le C\int_{\delta>|z|>\frac{M}{t_j}}\frac{1}{t_j|z|}dz\le\frac{C\delta^3}{t_j}.$$

For any given $\varepsilon>0$, we can first fix $\delta>0$ such that $C\delta^3<\varepsilon$. Then using the convergence of v_j to $\frac{1}{4}\ln\frac{6}{K(P_*)}$ on $|z|>\delta$, we can fix J such that when $j\ge J$, we have $\left|e^{4v_j}-\frac{6}{K(P_*)}\right|<\varepsilon$. Then

$$\left|\int_{|z|>\delta}\right|\le\varepsilon\int_{|z|>\delta}\frac{1}{t_j|z|}\left(\frac{2}{1+|z|^2}\right)^4 dz\le\frac{C\varepsilon}{t_j}.$$

These together prove the second estimate above. (52) follows similarly.

Finally

$$\langle \nabla K, \nabla x_5 \rangle = -\sum_{i=1}^{4} a_i x_i x_5 - 2\sum_{i=1}^{4} b_i x_i^2 x_5 + \nabla r \cdot \nabla x_5.$$

We may fix M large so that $|\nabla r \cdot \nabla x_5| \leq \varepsilon |x'|^3$ when $|y| > M$. In

$$\begin{aligned} 0 &= \int_{\mathbb{S}^4} \langle \nabla K, \nabla x_5 \rangle e^{4w_j}\, dvol_{g_c} \\ &= \int_{|y|\leq M} \langle \nabla K, \nabla x_5 \rangle e^{4w_j}\, dvol_{g_c} + \int_{|y|>M} \langle \nabla K, \nabla x_5 \rangle e^{4w_j}\, dvol_{g_c}, \end{aligned} \tag{53}$$

$$\left| \int_{|y|\leq M} \langle \nabla K, \nabla x_5 \rangle e^{4w_j}\, dvol_{g_c} \right| \leq C \left(\frac{M}{t_j} \right)^4, \tag{54}$$

as before.

$$\left| \int_{|y|>M} x_i x_5 e^{4w_j}\, dvol_{g_c} \right| = \frac{o(1)}{t_j}$$

as in the proof of (51). Thus

$$\left| \int_{|y|>M} a_i x_i x_5 e^{4w_j}\, dvol_{g_c} \right| = \frac{o(1)}{t_j^2}. \tag{55}$$

$$\begin{aligned} &\int_{|y|>M} x_i^2 x_5 e^{4w_j}\, dvol_{g_c} \\ =&\int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j z_i}{1+t_j|z|^2} \right)^2 \frac{t_j|z|^2-1}{t_j|z|^2+1} e^{4v_j} \left(\frac{2}{1+|z|^2} \right)^4 dz \\ =&\int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j z_i}{1+t_j|z|^2} \right)^2 \frac{t_j|z|^2-1}{t_j|z|^2+1} \left(e^{4v_j} - \frac{6}{K(P_*)} \right) \left(\frac{2}{1+|z|^2} \right)^4 dz \\ &+ \frac{6}{K(P_*)} \int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j z_i}{1+t_j|z|^2} \right)^2 \frac{t_j|z|^2-1}{t_j|z|^2+1} \left(\frac{2}{1+|z|^2} \right)^4 dz \end{aligned}$$

Note that

$$\begin{aligned} &\int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j z_i}{1+t_j|z|^2} \right)^2 \frac{t_j|z|^2-1}{t_j|z|^2+1} \left(\frac{2}{1+|z|^2} \right)^4 dz \\ \asymp& \frac{4}{t_j^2} \int_{|z|>\frac{M}{t_j}} \frac{z_i^2}{|z|^4} \left(\frac{2}{1+|z|^2} \right)^4 dz \\ \asymp& \frac{1}{t_j^2} \int_{|z|>\frac{M}{t_j}} \frac{1}{|z|^2} \left(\frac{2}{1+|z|^2} \right)^4 dz \\ \asymp& \frac{1}{t_j^2} \left(\int_0^\infty (\frac{2}{1+r^2})^4 r\, dr \right) |\mathbb{S}^3| \end{aligned} \tag{56}$$

Similarly, we can prove

$$\left| \int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j z_i}{1+t_j|z|^2} \right)^2 \frac{t_j|z|^2-1}{t_j|z|^2+1} \left(e^{4v_j} - \frac{6}{K(P_*)} \right) \left(\frac{2}{1+|z|^2} \right)^4 dz \right| = \frac{o(1)}{t_j^2}, \quad (57)$$

and

$$\begin{aligned} &\int_{|y|>M} |x'|^3 e^{4w_j} \\ =&\int_{|z|>\frac{M}{t_j}} \left(\frac{2t_j|z|}{1+t_j^2|z|^2} \right)^3 e^{4v_j} \left(\frac{2}{1+|z|^2} \right)^4 dz \\ =&\frac{O(1)}{t_j^3}. \end{aligned} \quad (58)$$

To put things together, we multiply (53) by t_j^2 and use (54), (55), (56), and (58) to see that

$$0 = o(1) - 2\Delta K(P_j) \left(|\mathbb{S}^3| \int_0^\infty (\frac{2}{1+r^2})^4 r dr + o(1) \right) + \frac{o(1)}{t_j},$$

which shows (45).

Proof (of Proposition 1) First, by Theorem 2, there is a $C > 0$ depending on K and $0 < \alpha < 1$ such that any solution w to (3) with K substituted by $K^{[s]}$ and $0 < s \leq 1$ satisfies

$$||w||_{C^{2,\alpha}(\mathbb{S}^4)} < C. \quad (59)$$

Since $v = w \circ \varphi_{P,t} + \ln|d\varphi_{P,t}|$ is chosen such that

$$⨍_{\mathbb{S}^4} e^{4v(y)} y \, dvol_{g_c} = 0,$$

we obtain, in terms of w and (P,t),

$$0 = ⨍_{\mathbb{S}^4} e^{4w(x)} \varphi_{P,t}^{-1}(x) \, dvol_{g_c} = ⨍_{\mathbb{S}^4} e^{4w(x)} \varphi_{P,t^{-1}}(x) \, dvol_{g_c}, \quad (60)$$

Due to (59), there is a $\delta > 0$ such that

$$⨍_{\mathbb{S}^4} e^{4w(x)} \geq \delta.$$

If there existed a sequence of solutions w_j for which $t_j \to \infty$, we would have, computing in stereographic coordinates in which P_j is placed at the north pole, $\varphi_{P_j, t_j^{-1}}(x) \to (0, \cdots, 0, -1)$ except at $x = P_j$, therefore, in view of (59),

$$⨍_{\mathbb{S}^4} e^{4w(x)} \varphi_{P,t^{-1}}(x) \, dvol_{g_c} \to (0, \cdots, 0, -⨍_{\mathbb{S}^4} e^{4w(x)}) \neq 0,$$

contradicting (60) above. This implies the existence of some t_0 such that $t \le t_0$. Using this and (59) in the relation between w and v, we find an upper bound for $||v||_{C^{2,\alpha}(\mathbb{S}^4)}$. Finally using the equation for v:

$$\sigma_2(A_v) = K^{[s]} \circ \varphi_{P,t} e^{4v},$$

in which the right hand side has an upper bound in $C^{2,\alpha}(\mathbb{S}^4)$ due to $C^{2,\alpha}(\mathbb{S}^4)$ estimates of v and the bound $t \le t_0$, we find higher derivative bounds for v. Then as $s \to 0$, a subsequence of v would converge to a limit v_∞ in $C^{2,\alpha}(\mathbb{S}^4)$, which satisfies

$$\sigma_2(A_{v_\infty}) = 6e^{4v_\infty} \quad \text{and} \quad \fint_{\mathbb{S}^4} e^{4v_\infty(x)} x \, dvol_{g_c} = 0.$$

This implies that $v_\infty \equiv 0$. Since this limit v_∞ is unique, we obtain that $v \to 0$ in $C^{2,\alpha}(\mathbb{S}^4)$ as $s \to 0$, which is the remaining part of (15).

4 Proof of the $W^{2,6}$ estimates of Theorem 1 and of Corollary 1

For the $W^{2,6}$ bound for v_j, we write v for v_j and σ_2 for $\sigma_2(e^{-2v_j} g_c^{-1} \circ A_{v_j}) = K \circ \varphi_j$, and adapt the argument for the $W^{2,p}$ estimates in [9] of Chang-Gursky-Yang and push the argument to $p = 6$. We will first prove a $W^{2,3}$ estimate for v_j, with the bound depending on an upper bound of $\sigma_2 = K \circ \varphi_j$, a positive lower bound for σ_2, and an upper bound for $\int_{\mathbb{S}^4} |\nabla_0 (K \circ \varphi_j)|^2 dvol_{g_c}$. Then we will extend the $W^{2,3}$ estimate to $W^{2,6}$ estimate for the v_j in terms of an upper bound of $\sigma_2 = K \circ \varphi_j$, a positive lower bound for σ_2, and an upper bound for $\int_{\mathbb{S}^4} |\nabla_0 (K \circ \varphi_j)|^4 dvol_{g_c}$. Since $\int_{\mathbb{S}^4} |\nabla_0 (K \circ \varphi_j)|^4 dvol_{g_c} = \int_{\mathbb{S}^4} |\nabla_0 K|^4 dvol_{g_c}$, we see that a bound for the $W^{2,3}$ norm of v_j is given in terms of an upper bound of K, a positive lower bound for K, and an upper bound for $\int_{\mathbb{S}^4} |\nabla_0 K|^4$. This will suffice for proving (9).

Proof (of the $W^{2,6}$ estimates of Theorem 1) First we list a few key ingredients for these $W^{2,p}$ estimates, mostly adapted from [9]. As in [9] we explore two differential identities, which in the case of $\mathbb{S}^4$, are

$$\begin{aligned} S_{ij} \nabla^2_{ij} R &= 6 tr E^3 + R|E|^2 + 3\Delta\sigma_2 + 3(|\nabla E|^2 - \frac{|\nabla R|^2}{12}) \\ &\ge 6 tr E^3 + \frac{R^3}{12} - 2\sigma_2 R + 3\Delta\sigma_2 - \frac{3|\nabla\sigma_2|^2}{2\sigma_2}, \end{aligned} \tag{61}$$

following (5.10) of [9], with

$$S_{ij} = \frac{\partial \sigma_2(A)}{\partial A_{ij}} = -R_{ij} + \frac{1}{2} R g,$$

and

$$\begin{aligned} & S_{ij} \nabla^2_{ij} |\nabla v|^2 \\ = & \frac{R^3}{144} - \frac{tr E^3}{2} - \frac{\sigma_2 R}{12} - \frac{R|\nabla v|^4}{2} - 2 S_{ij} \nabla_i |\nabla v|^2 \nabla_j v + S_{ij} \nabla_l A^\circ_{ij} \nabla_l v \\ & - 2e^{-2v} S_{ij} \nabla_i v \nabla_j v + 2Re^{-2v} |\nabla v|^2 + \frac{Re^{-4v}}{2} - \langle \nabla v, \nabla \sigma_2 \rangle - 2\sigma_2 e^{-2v}, \end{aligned} \tag{62}$$

following (5.44) of [9] and the fact that $A^0_{ij} = g^0_{ij}$ in the case of $\mathbb{S}^4$. Here the differentiations are in the metric g.

In (61) and (62) we used $|E|^2 = \frac{R^2}{12} - 2\sigma_2$ and

$$|\nabla E|^2 - \frac{|\nabla R|^2}{12} \geq -\frac{|\nabla \sigma_2|^2}{2\sigma_2}. \tag{63}$$

(63) can be proven as in (7.26) of [9], but can also be seen to be based on the general fact that $\{\sigma_k\}^{1/k}$ is concave in its argument as follows: set $F(A_{ij}) = \{\sigma_k(A_{ij})\}^{1/k}$, then

$$S_{ij} = \frac{\partial \sigma_k}{\partial A_{ij}} = kF^{k-1}\frac{\partial F}{\partial A_{ij}}, \quad \text{and} \quad \nabla\sigma_k = S_{ij}\nabla A_{ij} = kF^{k-1}\frac{\partial F}{\partial A_{ij}}\nabla A_{ij}. \tag{64}$$

So

$$\nabla_l S_{ij} = kF^{k-1}\frac{\partial^2 F}{\partial A_{ij}\partial A_{IJ}}\nabla_l A_{IJ} + k(k-1)F^{k-2}\frac{\partial F}{\partial A_{ij}}\frac{\partial F}{\partial A_{IJ}}\nabla_l A_{IJ}.$$

Thus

$$\begin{aligned} &\sum_l \nabla_l S_{ij}\nabla_l A_{ij} \\ =&kF^{k-1}\frac{\partial^2 F}{\partial A_{ij}\partial A_{IJ}}\nabla_l A_{IJ}\nabla_l A_{ij} + k(k-1)F^{k-2}\frac{\partial F}{\partial A_{ij}}\frac{\partial F}{\partial A_{IJ}}\nabla_l A_{IJ}\nabla_l A_{ij} \\ \leq&\frac{(k-1)|\nabla\sigma_k|^2}{k\sigma_k} \qquad\qquad \text{using concavity of } F \text{ and (64).} \end{aligned} \tag{65}$$

In the case of $2k = n = 4$, $A_{ij} = E_{ij} + \frac{R}{12}g_{ij}$, and $S_{ij} = \frac{R}{4}g_{ij} - E_{ij}$. So

$$\sum_l \nabla_l S_{ij}\nabla_l A_{ij} = \sum_l \{\frac{\nabla_l R}{4}g_{ij} - \nabla_l E_{ij}\}\{\nabla_l E_{ij} + \frac{\nabla_l R}{12}g_{ij}\} = \frac{|\nabla R|^2}{12} - |\nabla E|^2,$$

and by (65)

$$\frac{|\nabla R|^2}{12} - |\nabla E|^2 \leq \frac{|\nabla\sigma_2|^2}{2\sigma_2}.$$

Because of $S_{ij,j} = 0$, which is a consequence of Bianchi identity, we can use (61) and (62) to obtain

$$\begin{aligned} 0 &= \int_{\mathbb{S}^4} S_{ij}\nabla^2_{ij}(R + 12|\nabla v|^2) \\ &\geq \int_{\mathbb{S}^4} \frac{R^3}{6} - 6R|\nabla v|^4 - 24S_{ij}\nabla_i|\nabla v|^2\nabla_j v \\ &\quad + 12S_{ij}\nabla_l A^\circ_{ij}\nabla_l v + 24Re^{-2v}|\nabla v|^2 - 24e^{-2v}S_{ij}\nabla_i v\nabla_j v \\ &\quad - 12\langle\nabla v, \nabla\sigma_2\rangle + (6e^{-4v} - 2\sigma_2)R - 24e^{-2v}\sigma_2 - \frac{3|\nabla\sigma_2|^2}{2\sigma_2}, \end{aligned}$$

from which we can estimate $\int_{\mathbb{S}^4} R^3$ in terms of the other terms:

$$\begin{aligned}\int_{\mathbb{S}^4} \frac{R^3}{6} \leq \int_{\mathbb{S}^4} & 6R|\nabla v|^4 + 24 S_{ij}\nabla_i |\nabla v|^2 \nabla_j v \\ & - 12 S_{ij}\nabla_l A^{\circ}_{ij}\nabla_l v - 24 R e^{-2v}|\nabla v|^2 + 24 e^{-2v} S_{ij}\nabla_i v \nabla_j v \\ & + 12\langle \nabla v, \nabla \sigma_2\rangle - (6e^{-4v} - 2\sigma_2)R + 24\sigma_2 e^{-2v} + \frac{3|\nabla \sigma_2|^2}{2\sigma_2}.\end{aligned} \tag{66}$$

The integrations are done in the g metric, but due to the L^∞ estimates on v, the integrals in g metric are comparable to those in g_c. The terms that require careful treatments are

$$\int_{\mathbb{S}^4} S_{ij}\nabla_i |\nabla v|^2 \nabla_j v$$

and

$$\int_{\mathbb{S}^4} R|\nabla v|^4 \leq \left[\int_{\mathbb{S}^4} R^3\right]^{1/3} \left[\int_{\mathbb{S}^4} |\nabla v|^6\right]^{2/3} \leq \frac{\varepsilon}{3}\int_{\mathbb{S}^4} R^3 + \frac{2\varepsilon^{-1/2}}{3}\int_{\mathbb{S}^4} |\nabla v|^6. \tag{67}$$

The term $\int_{\mathbb{S}^4} S_{ij}\nabla_i |\nabla v|^2 \nabla_j v$ can be estimated as (5.53) in [9]

$$\begin{aligned}& \int_{\mathbb{S}^4} S_{ij}\nabla_i |\nabla v|^2 \nabla_j v \\ =& -\int_{\mathbb{S}^4} |\nabla v|^2 S_{ij}\nabla^2_{ij} v \\ =& -\int_{\mathbb{S}^4} |\nabla v|^2 S_{ij}\{-\frac{A_{ij}}{2} + \frac{A^0_{ij}}{2} - \nabla_i v \nabla_j v + \frac{|\nabla v|^2}{2} g_{ij}\} \\ =& \int_{\mathbb{S}^4} |\nabla v|^2 \{\sigma_2 + S_{ij}\nabla_i v \nabla_j v - \frac{R|\nabla v|^2}{2} - \frac{S_{ij}A^0_{ij}}{2}\} \\ =& \int_{\mathbb{S}^4} |\nabla v|^2 \{\sigma_2 - R_{ij}\nabla_i v \nabla_j v - \frac{S_{ij}A^0_{ij}}{2}\} \\ \leq& \int_{\mathbb{S}^4} |\nabla v|^2 \sigma_2.\end{aligned} \tag{68}$$

where in the last line we used $(R_{ij}) \geq 0$ when $g \in \Gamma_2^+$ in dimension 4 and $S_{ij}A^0_{ij} \geq 0$ on $\mathbb{S}^4$. The terms in the second line of (66) can be estimated in terms of $\int_{\mathbb{S}^4} Re^{-2v}|\nabla v|^2$, which in turn can be estimated as

$$\int_{\mathbb{S}^4} Re^{-2v}|\nabla v|^2 \lesssim \{\int_{\mathbb{S}^4} R^3\}^{1/3}\{\int_{\mathbb{S}^4} |\nabla v|^3\}^{2/3} \leq \frac{\varepsilon}{3}\int_{\mathbb{S}^4} R^3 + \frac{2\varepsilon^{-1/2}}{3}\int_{\mathbb{S}^4} |\nabla v|^3. \tag{69}$$

The terms in the last line of (66) can be estimated in terms of upper bound of σ_2, a lower bound of σ_2, and $\int_{\mathbb{S}^4} |\nabla \sigma_2|^2$, in a trivial way. The term $\int_{\mathbb{S}^4} |\nabla v|^6$ in (67) can be estimated as

$$\int_{\mathbb{S}^4} |\nabla v|^6 \leq \left[\int_{\mathbb{S}^4} |\nabla v|^4\right]^{3/4} \left[\int_{\mathbb{S}^4} |\nabla v|^{12}\right]^{1/4}, \tag{70}$$

and, as in (5.73) in [9],

$$\begin{aligned}
&\left[\int_{\mathbb{S}^4}|\nabla v|^{12}\right]^{1/4}\\
\lesssim &\int_{\mathbb{S}^4}|\nabla^2 v|^3+|\nabla v|^6+e^{-3v}|\nabla v|^3 \qquad (71)\\
\lesssim &\int_{\mathbb{S}^4}R^3+|\nabla v|^6+1,
\end{aligned}$$

here in the last line we used

$$S_{ij}=S^0_{ij}+2\nabla^2_{ij}v-2(\Delta v)g_{ij}+2\nabla_i v\nabla_j v+|\nabla v|^2 g_{ij}, \qquad (72)$$

$$R=R_0e^{-2v}-6\Delta v+6|\nabla v|^2, \qquad (73)$$

$$\qquad (74)$$

and

$$0\le (S_{ij})\le (Rg_{ij}).$$

Using (71) in (70) and noting that $\int_{\mathbb{S}^4}|\nabla v|^4$ is small, we obtain

$$\int_{\mathbb{S}^4}|\nabla v|^6\lesssim\left[\int_{\mathbb{S}^4}|\nabla v|^4\right]^{3/4}\int_{\mathbb{S}^4}R^3+1, \qquad (75)$$

Using (75), together with (68), (69) and (67) in (66), and noting the smallness of $\int_{\mathbb{S}^4}|\nabla v|^4$, we obtain an upper bound for $\int R^3$ in terms of $\int|\nabla\sigma_2|^2$, upper bound for σ_2 and positive lower bound for σ_2. Note that $\int_{\mathbb{S}^4}|\nabla\sigma_2|^4=\int_{\mathbb{S}^4}|\nabla_0(K\circ\phi_j)|^4=\int_{\mathbb{S}^4}|\nabla_0 K|^4\,dvol_{g_c}$ and using a transformation law like (75), we can estimate

$$\int|\Delta_0 v|^3\,dvol_{g_c}\lesssim\int\left(R^3+|\nabla_0 v|^6\right)dvol_{g_c}\lesssim\int R^3+1,$$

bounded above in terms of $\int|\nabla_0 K|^4\,dvol_{g_c}$, upper bound for K and positive lower bound for K. Then we can use the $W^{2,p}$ theory for the Laplace operator to obtain the full $W^{2,3}$ estimates for v.

Remark 7 In fact, for any solution w to (3), one can obtain an upper bound for the $W^{2,3}$ norm of w in terms of a positive upper and lower bound for K, an upper bound for $\int|\nabla_0 K|^2\,dvol_{g_c}$, and an upper bound for $|w|$ and $\int|\nabla_0 w|^4\,dvol_{g_c}$. A proof would proceed as above, instead of using the smallness of $\int|\nabla_0 w|^4\,dvol_{g_c}$ in proving (75) and the subsequent bound on $\int R^3$ via (66), one uses Proposition 5.20, Proposition 5.22, and Lemma 5.24 in [9] to complete the argument.

To obtain the $W^{2,6}$ estimates of v by iteration, we multiply (61) and (62) by R^p and estimate $\int R^{p+3}$ in terms of the other terms:

$$\int R^pS_{ij}\nabla^2_{ij}R\ge\int\frac{R^{p+3}}{12}+6R^p\,trE^3-2\sigma_2R^{p+1}+3R^p\Delta\sigma_2-\frac{3R^p|\nabla\sigma_2|^2}{2\sigma_2},$$

and

$$
\begin{aligned}
&\int R^p S_{ij}\nabla^2_{ij}|\nabla v|^2\\
=&\int \frac{R^{p+3}}{144}-\frac{R^p trE^3}{2}-\frac{\sigma_2 R^{p+1}}{12}-\frac{R^{p+1}|\nabla v|^4}{2}-2R^pS_{ij}\nabla_i|\nabla v|^2\nabla_j v\\
&+R^pS_{ij}\nabla_l A^{\circ}_{ij}\nabla_l v-2R^pS_{ij}\nabla_i v\nabla_j v+2R^{p+1}|\nabla v|^2-R^p\langle\nabla v,\nabla\sigma_2\rangle-2\sigma_2R^p+\frac{R^{p+1}}{2}.
\end{aligned}
$$

From these we obtain

$$
\begin{aligned}
&\int \frac{R^{p+3}}{6}\\
\le&\int R^pS_{ij}\nabla^2_{ij}\{R+12|\nabla v|^2\}-3R^p\Delta\sigma_2+6R^{p+1}|\nabla v|^4+24R^pS_{ij}\nabla_i|\nabla v|^2\nabla_j v\\
&+3\sigma_2R^{p+1}+\frac{3R^p|\nabla\sigma_2|^2}{2\sigma_2}-12R^pS_{ij}\nabla_l A^{\circ}_{ij}\nabla_l v+24R^pS_{ij}\nabla_i v\nabla_j v-24R^{p+1}|\nabla v|^2\\
&+12R^p\langle\nabla v,\nabla\sigma_2\rangle+24\sigma_2R^p-6R^{p+1}.
\end{aligned}
\tag{76}
$$

The most crucial terms are

$$\int R^pS_{ij}\nabla^2_{ij}R=-p\int R^{p-1}S_{ij}\nabla_iR\nabla_jR, \tag{77}$$

$$
\begin{aligned}
&\int R^pS_{ij}\nabla^2_{ij}|\nabla v|^2\\
=&-p\int R^{p-1}S_{ij}\nabla_iR\nabla_j|\nabla v|^2\\
\le&p\int R^{p-1}\left[S_{ij}\nabla_iR\nabla_jR\right]^{1/2}\left[S_{ij}\nabla_i|\nabla v|^2\nabla_j|\nabla v|^2\right]^{1/2}\\
\le&p\left[\int R^{p-1}S_{ij}\nabla_iR\nabla_jR\right]^{1/2}\left[\int R^{p-1}S_{ij}\nabla_i|\nabla v|^2\nabla_j|\nabla v|^2\right]^{1/2}\\
\le&\frac{p}{2}\int R^{p-1}S_{ij}\nabla_iR\nabla_jR+2p\int R^p|\nabla^2 v|^2|\nabla v|^2\\
\le&\frac{p}{2}\int R^{p-1}S_{ij}\nabla_iR\nabla_jR+Cp\int R^p(R^2+|S^0_{ij}|^2+|\nabla v|^2)|\nabla v|^2.
\end{aligned}
\tag{78}
$$

and

$$
\begin{aligned}
&\int -R^p\Delta\sigma_2\\
=&p\int R^{p-1}\nabla\sigma_2\nabla R\\
\le&p\left[\int|\nabla\sigma_2|^4\right]^{1/4}\left[\int|\nabla R|^2R^{p-2}\right]^{1/2}\left[\int R^{2p}\right]^{1/4}\\
\le&p\left[\int|\nabla\sigma_2|^4\right]^{1/4}\left[\int\frac{R^{p-1}S_{ij}\nabla_iR\nabla_jR}{3\sigma_2}\right]^{1/2}\left[\int R^{2p}\right]^{1/4}\\
\le&\frac{\varepsilon p}{2}\int\frac{R^{p-1}S_{ij}\nabla_iR\nabla_jR}{3\sigma_2}+\frac{p}{2\varepsilon}\left[\int|\nabla\sigma_2|^4\right]^{1/2}\left[\int R^{2p}\right]^{1/2}
\end{aligned}
\tag{79}
$$

Next we claim that the Sobolev inequality in dimension 4 implies

$$\left[\int R^{2p}\right]^{1/2} \lesssim p^2 \int |\nabla R|^2 R^{p-2} + \int R^p |\nabla v|^2 + \int R^p e^{-2v}. \tag{80}$$

Using (80) in (79), we obtain

$$\begin{aligned}&\int -R^p \Delta \sigma_2 \\ \leq &\varepsilon p \int \frac{R^{p-1} S_{ij} \nabla_i R \nabla_j R}{3\sigma_2} + \frac{\varepsilon}{2p} \int R^p |\nabla v|^2 + \frac{p^3}{8\varepsilon^3} \int |\nabla \sigma_2|^4.\end{aligned} \tag{81}$$

Using (77), (78), and (81) in (76) and choosing $\varepsilon > 0$ small, we obtain

$$\begin{aligned}&\int \frac{R^{p+3}}{6} + \frac{p}{4} R^{p-1} S_{ij} \nabla_i R \nabla_j R \\ \lesssim &\int R^{p+1} |\nabla v|^4 + R^{p+2} |\nabla v|^2 + R^p |\nabla \sigma_2|^2 + R^p |\nabla v| |\nabla \sigma_2| + R^{p+1} + 1 \\ \lesssim &\{\int R^{p+3}\}^{\frac{p+1}{p+3}} \{\int |\nabla v|^{2(p+3)}\}^{\frac{2}{p+3}} + \{\int R^{p+3}\}^{\frac{p+2}{p+3}} \{\int |\nabla v|^{2(p+3)}\}^{\frac{1}{p+3}} \\ &+ \{\int R^{2p}\}^{1/2} \{\int |\nabla \sigma_2|^4\}^{1/2} + \{\int R^{2p}\}^{1/2} \{\int |\nabla v|^4\}^{1/4} \{\int |\nabla \sigma_2|^4\}^{1/4} + \int R^{p+1} + 1\end{aligned} \tag{82}$$

Now for $p \leq 3$, we have $2p \leq p+3$ and $2(p+3) \leq 12$. Using the earlier bounds on $\int R^3$ and $\int |\nabla v|^{12}$ from (71), we obtain an upper bound for $\int R^6$ in terms of $\int |\nabla_0 K|^4 \, dvol_{g_c}$, an upper bound and a positive lower bound of K, which again gives a bound for v in $W^{2,6}$.

Proof (of Corollary 1) Let $\delta > 0$ be small such that the argument for (75) and the subsequent $W^{2,3}$ estimate for v via (66) would go through when $\int_{\mathbb{S}^4} |\nabla v|^4 \leq \delta$. For any admissible solution w to (3), Theorem 1 implies that there is a constant $B > 0$ depending on the C^2 norm of K, a positive lower bound of K, and $\delta > 0$, such that if $\max w = w(Q) > B$, then the normalized v defined as in Theorem 1: $v = w \circ \varphi + \ln |d\varphi|$ with $v(Q) = \frac{1}{4} \ln \frac{6}{K(Q)}$, would satisfy

$$\left| v - \frac{1}{4} \ln \frac{6}{K(Q)} \right| \leq \delta, \quad \text{and} \quad \int_{\mathbb{S}^4} |\nabla v|^4 \leq \delta. \tag{83}$$

v also satisfies (8) and then estimate (66), with σ_2 standing for $K \circ \varphi$, is valid for v. Then the $W^{2,3}$ estimate in Theorem 1 would be valid for v, and one obtains a bound for the $W^{2,3}$ norm of v in terms of an upper bound for K, a positive lower bound for K, and an upper bound for $\int_{\mathbb{S}^4} |\nabla_0 K \circ \varphi|^4 \, dvol_{g_c}$, and since $\int_{\mathbb{S}^4} |\nabla_0 K \circ \varphi|^4 \, dvol_{g_c} = \int_{\mathbb{S}^4} |\nabla_0 K|^4 \, dvol_{g_c}$, one can use this estimate to obtain the bound for $F[v]$. Since $I\!I[w] = I\!I[v]$ and $Y[w] = Y[v]$, the bound for $F[w]$ now follows. When the solution w satisfies $w \leq B$, then one can use the Harnack type estimate in [20] to obtain a lower bound for w, and use inequality (43) to obtain an upper bound for $\int_{\mathbb{S}^4} |\nabla w|^4 \, dvol_{g_c}$. Then one can use Remark 7 to obtain the $W^{2,3}$ estimate for w in terms of an upper bound for K, a positive lower bound for K, and an upper bound for $\int_{\mathbb{S}^4} |\nabla_0 K|^2 \, dvol_{g_c}$. Finally these $W^{2,3}$ estimates for w give directly the bound for $F[w]$.

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