A NOTE ON RENORMALIZED VOLUME FUNCTIONALS

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Dedicated to Michael Eastwood on his 60th birthday

1. Introduction

The asymptotic expansion of the volume of an asymptotically hyperbolic Einstein (AHE) metric defines invariants of the AHE metric and of a metric in the induced conformal class at infinity. These have been of recent interest, motivated in part by the AdS/CFT correspondence in physics. In this paper we derive some new properties of these invariants.

Let (X^{n+1}, g_+) be AHE with smooth conformal infinity $(M, [g])$, $M = \partial X$. We always assume that X is connected although ∂X need not be. If r is a geodesic defining function associated to a metric g in the conformal class at infinity (see §2 for more details), we have the following volume expansion ([G1]):

For n even,

$$\mathrm{Vol}_{g_+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V_{g_+} + o(1)$$

For n odd,

$$\mathrm{Vol}_{g_+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V_{g_+} + o(1).$$

If n is even, L is independent of the choice of g; if n is odd, V_{g_+} is independent of g. These are invariants of the conformal infinity $(M, [g])$ and the AHE manifold (X, g_+), resp. The constants c_{2k} and the renormalized volume V_{g_+} for n even depend on the choice of representative metric g in the conformal infinity.

The coefficients c_{2k} and L can be written as integrals over M of local expressions in the curvature of g, the so-called renormalized volume coefficients $v^{(2k)}(g)$. (The notation $v^{(2k)}$ is the same as in [G1, CF]. In §3 we also use the notation $v_k = (-2)^k v^{(2k)}$ of [G2].) Changing perspective slightly, one realizes a given conformal manifold $(M^n, [g])$ as the conformal boundary of an AHE manifold (X, g_+) in an

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asymptotic sense (see [FG2]), and the $v^{(2k)}(g)$ are the coefficients in the asymptotic expansion of the volume form of g_+. They have recently been studied in [CF, G2]. They are well-defined for general metrics for all $k \geq 0$ when n is odd but only for $0 \leq k \leq n/2$ when n is even. However, for $n \geq 4$ even they are also defined for all $k \geq 0$ if g is locally conformally flat or conformally Einstein. Directly from the definition of $v^{(2k)}$ (see §2), we have

$$c_{2k} = \frac{1}{n-2k}\int_M v^{(2k)}(g)dv_g, \qquad 0 \leq k \leq \lfloor (n-1)/2 \rfloor,$$

$$L = \int_M v^{(n)}(g)dv_g, \qquad n \text{ even.}$$

When n is even, $V_{g_+} = V_{g_+}(g)$ is a global quantity depending on the choice of g. Nonetheless, its change under conformal rescaling of g can be expressed by an integral of a local expression over the boundary. If $\widehat{g} = e^{2\omega}g$ is a conformally related metric, then

$$V_{g_+}(\widehat{g}) - V_{g_+}(g) = \int_M \mathcal{P}_g(\omega)dv_g,$$

where $\mathcal{P}_g$ is a polynomial nonlinear differential operator whose coefficients depend polynomially on g, g^{-1} and the curvature of g and its covariant derivatives, and whose linear part in ω and its derivatives is $v^{(n)}(g)\omega$ (see [G1]). In particular,

$$\partial_t V_{g_+}(e^{2t\omega}g)|_{t=0} = \int_M v^{(n)}(g)\omega\, dv_g,$$

i.e. V_{g_+} is a conformal primitive of $v^{(n)}$.

Our first result is a formula for V_{g_+} for n odd in terms of a compactification of the AHE metric g_+. If s is any defining function for ∂X, the metric $\overline{g} = s^2 g_+$ is called a compactification of g_+. For any such compactification, ∂X is an umbilic hypersurface relative to $\overline{g}$, i.e. its second fundamental form is a smooth multiple of the induced metric. We will say that $\overline{g}$ is a totally geodesic compactification if the second fundamental form of $M = \partial X$ relative to $\overline{g}$ vanishes identically. If $\overline{g}$ is any compactification, then $e^{2\omega}\overline{g}$ is a totally geodesic compactification for many choices of $\omega \in C^\infty(\overline{X})$; the totally geodesic condition on $e^{2\omega}\overline{g}$ is equivalent to the condition that the normal derivative of ω at ∂X be a specific function determined by $\overline{g}$.

Theorem 1.1. *If $n \geq 3$ is odd, g_+ is AHE, and $\overline{g}$ is a totally geodesic compactification of g_+, then*

$$V_{g_+} = C_{n+1}\int_X v^{(n+1)}(\overline{g})\, dv_{\overline{g}}, \qquad C_{n+1} = \frac{2^{n-1}(n+1)(\frac{n-1}{2})!^{\,2}}{n!}. \tag{1.1}$$

In the special case $n = 3$, Theorem 1.1 follows from a result of Anderson [An] (see also [CQY] for a different proof) expressing the Gauss-Bonnet formula in terms of V_{g_+}. Anderson showed that

$$8\pi^2\chi(X) = \frac{1}{4}\int_X |W|^2_{g_+} dv_{g_+} + 6V_{g_+}, \tag{1.2}$$

where W is the Weyl tensor and $|W|^2 = W^{ijkl}W_{ijkl}$. On the other hand, for a totally geodesic compactification, the boundary term vanishes in the Gauss-Bonnet formula for the compact manifold-with-boundary $(X, \overline{g})$. It was observed in [CGY] that in dimension 4 the Pfaffian is a multiple of $\frac{1}{4}|W|^2 + 4\sigma_2(g^{-1}P)$, where P denotes the Schouten tensor and $\sigma_k(g^{-1}P)$ the k-th elementary symmetric function of the eigenvalues of the endomorphism $g^{-1}P$. Since $v^{(4)}(g) = \frac{1}{4}\sigma_2(g^{-1}P)$, the Gauss-Bonnet formula for $(X, \overline{g})$ can be written

$$8\pi^2\chi(X) = \int_X \left[\tfrac{1}{4}|W|^2_{\overline{g}} + 16v^{(4)}(\overline{g})\right] dv_{\overline{g}}. \tag{1.3}$$

Comparing (1.2) and (1.3) and recalling that $\int |W|^2$ is conformally invariant gives (1.1).

For $n > 3$ odd, a generalization of Anderson's formula expressing the renormalized volume as a linear combination of the Euler characteristic and the integral of a pointwise conformal invariant has been established in [CQY]. We do not use this identity, but instead use the idea of its proof to directly relate the renormalized volume to the integral of $v^{(n+1)}(\overline{g})$ for a particular totally geodesic compactification. The fact that (1.1) then holds for any totally geodesic compactification follows using the result of [G2] that under conformal change, the $v^{(2k)}$ depend on at most two derivatives of the conformal factor.

Our second result concerns the renormalized volume functionals $\mathcal{F}_k(g)$ defined by

$$\mathcal{F}_k(g) = (-2)^k \int_M v^{(2k)}(g)\, dv_g$$

on the space of metrics on a connected compact manifold M. This normalization is chosen so that $\mathcal{F}_k(g) = \int_M \sigma_k(g^{-1}P)\, dv_g$ if g is locally conformally flat; the conformal properties of the functionals $\int_M \sigma_k(g^{-1}P)\, dv_g$ have been intensively studied during the last decade. In [CF] it was shown that if $2k \neq n$, then the Euler-Lagrange equation for $\mathcal{F}_k$ under conformal change subject to the constraint that the volume is constant is $v^{(2k)}(g) = c$. In [G2], it was shown that if a background metric g_0 in the conformal class is fixed and one writes $g = e^{2\omega}g_0$, then the Euler-Lagrange equation $v^{(2k)}(e^{2\omega}g_0) = c$ is second order in ω even though for $k \geq 2$, $v^{(2k)}(g)$ depends on $2k - 2$ derivatives of g.

Any Einstein metric satisfies $v^{(2k)}(g) = c$, so is a critical point of $\mathcal{F}_k$. In this paper, we identify the second variation at a general critical point of $\mathcal{F}_k$ under conformal change subject to the constant volume constraint and use this to show that Einstein metrics of nonzero scalar curvature are local extrema. Let $\mathcal{C}$ denote a conformal class of metrics on M and let $\mathcal{C}_1$ denote the subset of metrics of unit volume.

Theorem 1.2. *Let (M, g) be a unit volume connected compact Einstein manifold of dimension $n \geq 3$ with nonzero scalar curvature which is not isometric to S^n with the standard metric (normalized to have unit volume). Suppose $1 \leq k \leq n$ and if n is even assume that $k \neq n/2$. Then the second variation of $(\mathcal{F}_k|_{\mathcal{C}_1})$, $(\mathcal{F}_k|_{\mathcal{C}_1})''$, is a definite quadratic form on $T_g\mathcal{C}_1$ whose sign is as follows:*

(1) *Let $k < n/2$.*
  - *If $R > 0$, then $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is positive definite.*
  - *If $R < 0$, then $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is positive definite for k odd and negative definite for k even.*

(2) *Let $k > n/2$. Then all signs are reversed:*
  - *If $R > 0$, then $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is negative definite.*
  - *If $R < 0$, then $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is negative definite for k odd and positive definite for k even.*

For S^n with the (normalized) standard metric, the only change is that $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is semi-definite with the indicated sign and with $n+1$-dimensional nullspace.

Of course, one concludes from Theorem 1.2 that $\mathcal{F}_k|_{\mathcal{C}_1}$ has a local maximum or minimum at an Einstein metric, with sign determined as in the statement of the theorem. The max-min conclusion follows also for S^n since the null directions for the Hessian arise from conformal diffeomorphisms.

If g is locally conformally flat or if $k = 1$ or 2, then $(-2)^k v^{(2k)}(g) = \sigma_k(g^{-1}P)$. In these cases Theorem 1.2 follows from Theorem 2 of [V], which is concerned with the second variation of the functionals $\int_M \sigma_k(g^{-1}P)\, dv_g$. (Theorem 1.2 corrects a sign error in the statement of Theorem 2 of [V] for $k > n/2$.) Theorem 1.2 indicates that for $k > 2$ and g not locally conformally flat, $(-2)^k v^{(2k)}(g)$ is the natural replacement for $\sigma_k(g^{-1}P)$. The special case $k = 3$, $n > 6$, of Theorem 1.2 was first proved by Guo-Li [GL] by direct computation of $(\mathcal{F}_3|_{\mathcal{C}_1})''$ from the explicit formula for $v^{(6)}(g)$.

As mentioned above, Theorem 1.2 follows from a formula which we derive for the second variation of $\mathcal{F}_k$ at a general critical point (Theorem 3.1). This second variation formula is an immediate consequence of a formula derived in [ISTY] and rederived in [G2] for the first conformal variation of $v^{(2k)}(g)$: by the result of [CF], the first conformal variation of $\mathcal{F}_k$ is integration against a multiple of $v^{(2k)}(g)$, so the second variation of $\mathcal{F}_k$ is integration against the first conformal variation of $v^{(2k)}(g)$. The principal part of these variations is a symmetric contravariant 2-tensor $L^{ij}_{(k)}(g)$ defined by (3.6) which was derived in [ISTY] and analyzed in some detail in [G2]. We also state a general condition in terms of $L^{ij}_{(k)}(g)$ which is sufficient for definiteness of the second variation of $\mathcal{F}_k$ for non-Einstein critical points and which generalizes a criterion of Viaclovsky in the cases $k = 1$, 2 or g locally conformally flat when g has (possibly nonconstant) negative scalar curvature.

If n is even, $\mathcal{F}_{n/2}(g)$ is conformally invariant as noted above, so conformal variations of $\mathcal{F}_{n/2}(g)$ are trivial. A natural substitute for $\mathcal{F}_{n/2}(g)$ as far as conformal variations is concerned is the renormalized volume $V_{g_+}(g)$ of an AHE metric g_+ with conformal infinity $(M, [g])$. V_{g_+} is a conformal primitive of $v^{(n)}$ as noted above, just as $(-2)^{-k}\frac{1}{n-2k}\mathcal{F}_k$ is a conformal primitive of $v^{(2k)}$ if $2k \neq n$. So critical points of $V_{g_+}|_{\mathcal{C}_1}$ are precisely solutions of $v^{(n)}(g) = c$. The identification of the first variation of $v^{(2k)}(g)$ from [ISTY, G2] holds just as well for $2k = n$, so this gives immediately the second variation of V_{g_+} in terms of the tensor $L^{ij}_{(n/2)}(g)$ (Theorem 3.2). Einstein metrics are critical points for $V_{g_+}|_{\mathcal{C}_1}$, and upon evaluating the second variation at an Einstein metric, we deduce the following analogue of Theorem 1.2. We take M to be connected and formulate the result for a general AHE manifold (X, g_+) such that $\mathcal{C} = (M, [g])$ is one of the connected components of its conformal infinity. We fix arbitrarily a representative of the conformal infinity on each of the other connected components and view V_{g_+} as a function of the metric in the conformal class on M.

Theorem 1.3. *Let $n \geq 2$ be even. Let (M, g) be a unit volume connected compact Riemannian manifold with constant nonzero scalar curvature which is not isometric to S^n with the standard metric (normalized to have unit volume). If $n \geq 4$, assume that g is Einstein. Let (X, g_+) be AHE and suppose that $(M, [g])$ is one of the connected components of its conformal infinity. The second variation of $V_{g_+}|_{\mathcal{C}_1}$ is a definite quadratic form on $T_g\mathcal{C}_1$ whose sign is as follows:*

- *If $R < 0$, then $\left(V_{g_+}|_{\mathcal{C}_1}\right)''$ is negative definite.*
- *If $R > 0$, then $\left(V_{g_+}|_{\mathcal{C}_1}\right)''$ is positive definite if $n \equiv 0 \mod 4$ and negative definite if $n \equiv 2 \mod 4$.*

For S^n with the (normalized) standard metric, $\left(V_{g_+}|_{\mathcal{C}_1}\right)''$ is semi-definite with the indicated sign and with $n+1$-dimensional nullspace.

We also state a sufficient condition for definiteness of the second variation of V_{g_+} for non-Einstein critical points which is analogous to the condition mentioned above for the $\mathcal{F}_k$.

Colin Guillarmou has informed us that he has proved Theorem 1.3 in joint work with S. Moroianu and J.-M. Schlenker.

The results of [CF, G2] and Theorem 1.2 indicate the importance of the renormalized volume functionals in conformal geometry, which we will hopefully continue to explore in future works.

2. Renormalized Volume

Let g_+ be an asymptotically hyperbolic Einstein (AHE) metric on X^{n+1} with smooth conformal infinity $(M,[g])$, where $M = \partial X$. Let g be a metric in the conformal class on M. One can uniquely identify a neighborhood of ∂X with $[0,\epsilon) \times \partial X$ so that g_+ takes the normal form

$$g_+ = r^{-2}\left(dr^2 + g_r\right) \tag{2.1}$$

for a 1-parameter family g_r of metrics on M with $g_0 = g$. The defining function r is called the geodesic defining function associated to g. A boundary regularity result ([CDLS, H, BH]) shows that g_r is smooth up to $r = 0$ if n is odd, and has a polyhomogeneous expansion as $r \to 0$ if n is even. The family g_r is even to order n; in particular $\partial_r g_r|_{r=0} = 0$. Thus the geodesic compactification $\overline{g}_{geod} = r^2 g_+$ is totally geodesic. Any other compactification which induces the same boundary metric can be written as $\overline{g} = e^{2\omega}\overline{g}_{geod}$ for some $\omega \in C^\infty(\overline{X})$ satisfying $\omega = 0$ on ∂X. Such a compactification $\overline{g}$ is totally geodesic if and only if $\omega = O(r^2)$.

The renormalized volume coefficients $v^{(2k)}(g)$ are defined for $0 \le k \le \lfloor n/2 \rfloor$ by the asymptotic expansion

$$\left(\frac{\det g_r}{\det g}\right)^{1/2} = \sum_{k=0}^{\lfloor n/2\rfloor} v^{(2k)}(g) r^{2k} + o(r^n). \tag{2.2}$$

If n is odd, the definition can be extended to all $k \ge 0$ by considering metrics g_+ of the form (2.1) for which g_r is even to infinite order and for which $\mathrm{Ric}(g_+) + n g_+$ vanishes to infinite order. The $v^{(2k)}(g)$ are local curvature invariants of g which are determined by an inductive algorithm; see [G1, G2]. Clearly $v^{(0)}(g) = 1$. The next three are given by:

$$\begin{aligned}
v^{(2)}(g) &= -\frac{R}{4(n-1)} \\
v^{(4)}(g) &= \frac{1}{4}\sigma_2(g^{-1}P) = \frac{1}{8}\left[(P^j{}_j)^2 - P^{ij}P_{ij}\right] \\
v^{(6)}(g) &= -\frac{1}{8}\left[\sigma_3(g^{-1}P) + \frac{1}{3(n-4)}P^{ij}B_{ij}\right],
\end{aligned}$$

where $P_{ij} := \frac{1}{n-2}\left[R_{ij} - Rg_{ij}/2(n-1)\right]$ and $B_{ij} := \nabla^k\nabla_k P_{ij} - \nabla^k\nabla_j P_{ik} - P^{kl}W_{kijl}$ are the Schouten and Bach tensors of g, and $\sigma_k(g^{-1}P)$ is the k-th elementary symmetric function of the eigenvalues of the endomorphism $g^{-1}P$.

The rest of this section is devoted to the proof of Theorem 1.1. The first step is to establish the result for a specific totally geodesic compactification. Let g be a metric in the conformal class at infinity with associated geodesic defining function r and geodesic compactification $r^2 g_+ = dr^2 + g_r$. Theorem 4.1 of [FG1] asserts the existence of a unique $U \in C^\infty(\mathring{X})$ such that $-\Delta_{g_+} U = n$ (our convention is

$\Delta = \nabla^k \nabla_k$) with the asymptotics

$$U = \log r + A + Br^n, \tag{2.3}$$

where $A,\ B \in C^\infty(\overline{X})$ are even functions modulo $O(r^\infty)$ and $A|_{\partial X} = 0$. Then $e^U = re^{A+Br^n}$ is a defining function and

$$\overline{g}_U := e^{2U} g_+ = e^{2(A+Br^n)} \left(dr^2 + g_r\right) \tag{2.4}$$

is a totally geodesic compactification. Theorem 4.3 of [FG1] asserts that

$$V_{g_+} = \int_{\partial X} B|_{\partial X}\, dv_g. \tag{2.5}$$

Proposition 2.1. *Theorem 1.1 holds for $\overline{g} = \overline{g}_U$.*

Proposition 2.1 follows from an argument of [CQY]. The formula of [CQY] mentioned in the introduction for the renormalized volume in terms of the the Euler characteristic and the integral of a pointwise conformal invariant is derived by applying Alexakis' theorem [Al] on the existence of a decomposition of Q-curvature. Our proof of Proposition 2.1 uses an analogous identity expressing the Q-curvature as a multiple of $v^{(n+1)}$ and a divergence. The existence of such a formula can be deduced by general considerations since the integrals of the Q-curvature and $v^{(n+1)}$ agree up to a multiplicative constant on compact Riemannian manifolds. However, an explicit formula of this kind is known: the holographic formula for Q-curvature of [GJ]. So we base our proof on the holographic formula for Q-curvature.

We first recall some properties of the metric $\overline{g}_U$ which were established in [CQY].

Proposition 2.2. *Let $\overline{g}_U$ be given by* (2.4)*, where U is the solution of $-\Delta_{g_+} U = n$ with asymptotics* (2.3) *as above. Then we have*

- *(*[CQY] *Lemma 2.1)*

$$Q(\overline{g}_U) = 0. \tag{2.6}$$

- *(*[CQY] *Lemma 3.1) Let R denote the scalar curvature and Δ the Laplacian for the metric $\overline{g}_U$. Then*

$$\partial_r \Delta^{(n-3)/2} R = -2nn!B \ \text{ on } \partial X. \tag{2.7}$$

- *(*[CQY] *Lemma 3.2) Let $*$ stand for indices in the tangential directions on ∂X. For the covariant derivatives of the curvature tensor R_{ijkl} of $\overline{g}_U$, the following three types of components*

$$\nabla_r^{2k+1} R_{****}, \quad \nabla_r^{2k} R_{r***}, \quad \nabla_r^{2k-1} R_{r*r*},$$

vanish at the boundary for $1 \le 2k+1 \le n-2$.

Proof of Proposition 2.1. The holographic formula for Q-curvature states that for any metric in even dimension $m = n+1$, one has

$$2c_{m/2}Q = v^{(n+1)} + \frac{1}{n+1}\sum_{k=1}^{(n-1)/2}(n+1-2k)p_{2k}^{*}v^{(n+1-2k)}, \tag{2.8}$$

where $c_l^{-1} = (-1)^l 2^{2l} l!(l-1)!$. Here the $v^{(n+1-2k)}$ are the renormalized volume coefficients, p_{2k} is a natural differential operator of order $2k$ with no constant term and with principal part $a_{n+1,k}\Delta^k$, where

$$a_{n+1,k} = \frac{\Gamma\left((n+1-2k)/2\right)}{2^{2k}\, k!\, \Gamma\left((n+1)/2\right)},$$

and p_{2k}^{*} denotes the formal adjoint of p_{2k}. In particular, each term $p_{2k}^{*}v^{(n+1-2k)}$ with $k \geq 1$ in the sum on the right-hand side of (2.8) is the divergence of a natural 1-form.

Apply (2.8) to $\overline{g}_U$ and use (2.6) to deduce that

$$\begin{aligned}
v^{(n+1)}(\overline{g}_U) &= -\frac{1}{n+1}\sum_{k=1}^{(n-1)/2}(n+1-2k)p_{2k}^{*}v^{(n+1-2k)} \\
&= -\frac{2}{n+1}p_{n-1}^{*}v^{(2)} - \frac{1}{n+1}\sum_{k=1}^{(n-3)/2}(n+1-2k)p_{2k}^{*}v^{(n+1-2k)} \\
&= -\frac{2}{n+1}a_{n+1,(n-1)/2}\Delta^{(n-1)/2}v^{(2)} \\
&\qquad + q\,v^{(2)} - \frac{1}{n+1}\sum_{k=1}^{(n-3)/2}(n+1-2k)p_{2k}^{*}v^{(n+1-2k)},
\end{aligned}$$

where q is a natural differential operator of order less than $n-1$ which is a divergence, and all the terms on the right-hand side refer to the metric $\overline{g}_U$. Now integrate over X. The right-hand side is a divergence so can be rewritten as a boundary integral. Recalling that $v^{(2)} = -\frac{1}{2}P^k{}_k = -\frac{1}{4n}R$, one has

$$\int_X \Delta^{(n-1)/2}v^{(2)}\, dv_{\overline{g}_U} = \frac{1}{4n}\int_{\partial X}\partial_r\Delta^{(n-3)/2}R\, dv_g.$$

But (2.7) asserts that $\partial_r\Delta^{(n-3)/2}R = -2nn!B$ on ∂X. Substituting and using (2.5) gives

$$\int_X \Delta^{(n-1)/2}v^{(2)}\, dv_{\overline{g}_U} = -\frac{n!}{2}\, V_{g_+}.$$

All terms in the expression

$$q\,v^{(2)} - \frac{1}{n+1}\sum_{k=1}^{(n-3)/2}(n+1-2k)p^*_{2k}v^{(n+1-2k)}$$

involve fewer derivatives of $\overline{g}_U$. Arguing as in [CQY], the third part of Proposition 2.2 implies that the resulting integral over the boundary vanishes. Thus

$$\int_X v^{(n+1)}(\overline{g}_U)\,dv_{\overline{g}_U} = -\frac{2}{n+1}a_{n+1,(n-1)/2}\left(-\frac{n!}{2}\right)V_{g_+}.$$

Collecting the constant gives the result. □

Proof of Theorem 1.1. Let $\overline{g}$ be a totally geodesic compactification of g_+ with induced boundary metric g. Let $\overline{g}_U$ be the compactification as above with the same boundary metric. Then $\overline{g} = e^{2\omega}\overline{g}_U$ where $\omega = O(r^2)$. We will show that

$$\int_X v^{(n+1)}(\overline{g})\,dv_{\overline{g}} = \int_X v^{(n+1)}(\overline{g}_U)\,dv_{\overline{g}_U}. \tag{2.9}$$

The result then follows by Proposition 2.1.

Set $\overline{g}_t = e^{2t\omega}\overline{g}_U$. Theorem 1.5 of [G2] gives a divergence formula of the form

$$\partial_t\left(v^{(n+1)}(\overline{g}_t)\,dv_{\overline{g}_t}\right) = (-2)^{-(n+1)/2}\nabla_i\left(L^{ij}_{((n+1)/2)}(\overline{g}_t)\nabla_j\,\omega\right)\,dv_{\overline{g}_t}$$

for a particular natural symmetric 2-tensor $L^{ij}_{((n+1)/2)}$. The covariant derivatives refer to the Levi-Civita connection of $\overline{g}_t$. Integrating by parts and using $\nabla\omega|_{\partial X} = 0$ gives

$$\partial_t\int_X v^{(n+1)}(\overline{g}_t)\,dv_{\overline{g}_t} = 0.$$

Thus $\int_X v^{(n+1)}(\overline{g}_t)\,dv_{\overline{g}_t}$ is independent of t, which gives (2.9).

We have thus completed the proof of Theorem 1.1. □

3. Second Variation

If M is a connected compact manifold, consider the functional

$$\mathcal{F}_k(g) = (-2)^k\int_M v^{(2k)}(g)\,dv_g \tag{3.1}$$

on the space of metrics on M, where $v^{(2k)}(g)$ is the renormalized volume coefficient defined in §2. For notational convenience, we set

$$v_k(g) = (-2)^k v^{(2k)}(g). \tag{3.2}$$

This is the same notation as in [G2]. The coefficient is chosen so that if g is locally conformally flat, then

$$v_k(g) = \sigma_k\left(g^{-1}P\right), \qquad 0 \le k \le n$$

(see Proposition 1 of [GJ]). It will also be convenient to introduce $\rho = -\frac{1}{2}r^2$ and $g(\rho) = g_r$, where g_r is the 1-parameter family of metrics appearing in (2.1). Then the expansion (2.2) defining the $v^{(2k)}$ becomes

$$\left(\frac{\det g(\rho)}{\det g}\right)^{1/2} \sim \sum_{k\geq 0} v_k(g)\rho^k. \tag{3.3}$$

Set $v(\rho) = (\det g(\rho)/\det g)^{1/2}$.

Recall that for even n, the $v^{(2k)}(g)$ are only defined for $k \leq n/2$ for general metrics. But they are invariantly defined for all k if $n \geq 4$ and g is locally conformally flat or conformally Einstein. This is because in these cases there is an invariant determination of $g(\rho)$ to all orders; see [FG2]. If g is Einstein with $R_{ij} = 2a(n-1)g_{ij}$, one has $g(\rho) = (1 + a\rho)^2 g$. Observe that this gives $v(\rho) = (1 + a\rho)^n$, so

$$v_k(g) = a^k \binom{n}{k}, \qquad 0 \leq k \leq n. \tag{3.4}$$

For a conformal rescaling $\widehat{g} = e^{2\omega}g$ of an Einstein metric g, $\widehat{g}_r$ is defined by putting the Poincaré metric $r^{-2}\left(dr^2 + (1 - ar^2/2)^2 g\right)$ for g into normal form relative to $\widehat{g}$ by a diffeomorphism. Then one sets $\widehat{g}(\rho) = \widehat{g}_r$ with $\rho = -\frac{1}{2}r^2$ and defines $v_k(\widehat{g})$ via (3.3). This is well-defined, but a direct formula in terms of $\widehat{g}$ is not available.

The crucial ingredient in the variational analysis is the following formula for the conformal variation of the $v_k(g)$. For a Riemannian manifold (M, g) and $\omega \in C^\infty(M)$, set $g_t = e^{2t\omega}g$ and define $\delta v_k(g, \omega) = \partial_t|_{t=0} v_k(g_t)$. Then (2.4), (3.8) of [ISTY] (see also Theorem 1.5 of [G2]) show that

$$\delta v_k(g, \omega) = \nabla_i\left(L^{ij}_{(k)}(g)\nabla_j\omega\right) - 2kv_k(g)\omega, \tag{3.5}$$

where

$$L^{ij}_{(k)}(g) = -\frac{1}{k!}\partial^k_\rho\left(v(\rho)\int_0^\rho g^{ij}(u)\,du\right)\Big|_{\rho=0} = -\sum_{l=1}^{k}\frac{1}{l!}v_{k-l}(g)\,\partial_\rho^{l-1}g^{ij}(\rho)|_{\rho=0}. \tag{3.6}$$

Here $g^{ij}(u) = (g_{ij}(u))^{-1}$ and ∇ denotes the covariant derivative with respect to g.

We first review the identification of the critical points of $\mathcal{F}_k$ from [CF, G2]. By (3.2), (3.1) can be re-written as

$$\mathcal{F}_k(g) = \int_M v_k(g)\,dv_g.$$

By (3.5) and the fact that $\delta dv_g = n\omega dv_g$, one deduces that the conformal variation of $\mathcal{F}_k$ is given by

$$\delta\mathcal{F}_k = (n - 2k)\int_M v_k\omega\,dv_g. \tag{3.7}$$

For n even and $2k = n$, this recovers the fact that $\mathcal{F}_{n/2}$ is conformally invariant. For $2k \neq n$, we are interested in the restriction $\mathcal{F}_k|_{\mathcal{C}_1}$, where $\mathcal{C}_1$ denotes the space of unit volume metrics in a conformal class $\mathcal{C}$ of metrics on M. We use the Lagrange multiplier method. The critical points are the metrics $g \in \mathcal{C}_1$ which satisfy for some constant λ that

$$\delta\left(\mathcal{F}_k - \lambda \operatorname{Vol}(M)\right)(g,\omega) = 0 \quad \text{for all } \omega.$$

By (3.7), this is

$$(n-2k)\int_M v_k\omega\, dv_g - n\lambda\int_M \omega\, dv_g = 0 \quad \text{for all } \omega,$$

which gives $v_k(g) = n\lambda/(n-2k)$. Thus the critical points are precisely the unit volume metrics for which $v_k(g)$ is constant.

The following theorem identifies the second variation of $\mathcal{F}_k|_{\mathcal{C}_1}$ at a critical point. Suppose g is a unit volume metric for which $v_k(g)$ is constant. The tangent space of $\mathcal{C}_1$ at g is given by

$$T_g\mathcal{C}_1 = \left\{2\omega g : \int_M \omega\, dv_g = 0\right\}.$$

For such an ω, set

$$\left(\mathcal{F}_k|_{\mathcal{C}_1}\right)''(\omega) = \partial_t^2|_{t=0}\mathcal{F}_k(\gamma_t),$$

where γ_t is a curve in $\mathcal{C}_1$ satisfying $\gamma_0 = g$ and $\gamma_0' = 2\omega g$.

Theorem 3.1. *Let $n \geq 3$, $k \geq 1$ and $k \leq n/2$ if n is even. Let (M,g) be a connected compact Riemannian manifold and suppose g satisfies $v_k(g) = c$ for some constant c. Let $\omega \in C^\infty(M)$ satisfy $\int_M \omega\, dv_g = 0$. Then*

$$\left(\mathcal{F}_k|_{\mathcal{C}_1}\right)''(\omega) = -(n-2k)\int_M \left[L_{(k)}^{ij}(g)\omega_i\omega_j + 2kv_k(g)\omega^2\right] dv_g.$$

Proof. We can assume that $k \neq n/2$. Define λ by $c = n\lambda/(n-2k)$, so that $\delta\left(\mathcal{F}_k - \lambda\operatorname{Vol}(M)\right) = 0$. Since the Hessian at a critical point is invariantly defined on the tangent space, we have

$$\left(\mathcal{F}_k|_{\mathcal{C}_1}\right)''(\omega) = \partial_t^2|_{t=0}\left(\mathcal{F}_k - \lambda\operatorname{Vol}(M)\right)(g_t),$$

with $g_t = e^{2t\omega}g$ as above. Now (3.7) gives

$$\partial_t\mathcal{F}_k(g_t) = (n-2k)\int_M v_k(g_t)\omega\, dv_{g_t}.$$

Combining this with

$$\partial_t \operatorname{Vol}_{g_t}(M) = n\int_M \omega\, dv_{g_t}$$

shows that

$$
\begin{aligned}
\partial_t^2|_{t=0}\left(\mathcal{F}_k-\lambda\,\mathrm{Vol}(M)\right)(g_t)=&\,\partial_t|_{t=0}\left[(n-2k)\int_M v_k(g_t)\omega\,dv_{g_t}-\lambda n\int_M\omega\,dv_{g_t}\right]\\
=&(n-2k)\int_M\left[\delta v_k(g,\omega)+nv_k(g)\omega\right]\omega\,dv_g-\lambda n^2\int_M\omega^2\,dv_g\\
=&(n-2k)\int_M\delta v_k(g,\omega)\omega\,dv_g+n^2\lambda\int_M\omega^2\,dv_g-\lambda n^2\int_M\omega^2\,dv_g\\
=&(n-2k)\int_M\delta v_k(g,\omega)\omega\,dv_g\\
=&-(n-2k)\int_M\left[L^{ij}_{(k)}(g)\omega_i\omega_j+2kv_k(g)\omega^2\right]\,dv_g,
\end{aligned}
$$

where for the last equality we use (3.5) and integration by parts. □

We remark that Theorem 3.1 and its proof remain valid for all $k\geq 1$ when $n\geq 4$ is even if g is Einstein or locally conformally flat. This is because the main ingredient, (3.5), just uses that the Poincaré metrics arising from conformally related metrics on the boundary are related by a diffeomorphism.

Proof of Theorem 1.2. Let g be Einstein with $R_{ij}=2a(n-1)g_{ij}$. We use Theorem 3.1 to evaluate $(\mathcal{F}_k|_{\mathcal{C}_1})''$. Recall that $g_{ij}(\rho)=(1+a\rho)^2g_{ij}$ and $v(\rho)=(1+a\rho)^n$. So $g^{ij}(\rho)=(1+a\rho)^{-2}g^{ij}$. Hence

$$
v(\rho)\int_0^\rho g^{ij}(u)\,du=\rho(1+a\rho)^{n-1}g^{ij}.
$$

Therefore (3.6) gives

$$
L^{ij}_{(k)}(g)=-a^{k-1}\binom{n-1}{k-1}g^{ij},\qquad 1\leq k\leq n. \tag{3.8}
$$

Recalling (3.4), Theorem 3.1 gives

$$
\begin{aligned}
\left(\mathcal{F}_k|_{\mathcal{C}_1}\right)''(\omega)=&(n-2k)a^{k-1}\binom{n-1}{k-1}\int_M\left(|\nabla\omega|_g^2-2na\omega^2\right)\,dv_g\\
=&(n-2k)a^{k-1}\binom{n-1}{k-1}\int_M\left(|\nabla\omega|_g^2-R\omega^2/(n-1)\right)\,dv_g.
\end{aligned}
$$

If $R<0$, this has the same sign as the leading coefficient $(n-2k)a^{k-1}$, which gives the desired conclusion.

If $R>0$, we use Obata's estimate [O] for the first eigenvalue of $-\Delta$ for an Einstein metric: $\lambda_1(-\Delta)\geq R/(n-1)$ with equality only for S^n. This leads to the desired result. For S^n, the equality holds if and only if ω is an eigenfunction corresponding to λ_1. This is the $(n+1)$-dimensional space of infinitesimal conformal factors corresponding to conformal diffeomorphisms. □

It is possible to formulate a result also for non-Einstein critical points. It is clear from Theorem 3.1 that if $L^{ij}_{(k)}(g)$ is definite and $v_k(g)$ is a constant of the same sign, then $(\mathcal{F}_k|_{\mathcal{C}_1})''$ is definite. This generalizes the result of Viaclovsky that negative k-admissible critical points are local extrema when $k = 1$ or 2 or g is locally conformally flat.

Consider finally the second variation of the renormalized volume when n is even. Let (X, g_+) be AHE and let $\mathcal{C} = (M, [g])$ be one of the connected components of its conformal infinity. Fix a representative of the conformal infinity on each of the other connected components and view $V_{g_+}(g)$ as a function on $\mathcal{C}$. As discussed in the introduction, its conformal variation is

$$\delta V_{g_+} = \int_M v^{(n)}(g)\omega \, dv_g.$$

Upon introducing a Lagrange multiplier exactly as above for $\mathcal{F}_k$, one deduces that the critical points of $V_{g_+}|_{\mathcal{C}_1}$ are the unit volume metrics for which $v^{(n)}(g)$ is constant. For such a g and for ω satisfying $\int_M \omega \, dv_g = 0$, we define the second variation by

$$\left(V_{g_+}|_{\mathcal{C}_1}\right)''(\omega) = \partial_t^2|_{t=0} V_{g_+}(\gamma_t),$$

where γ_t is a curve in $\mathcal{C}_1$ satisfying $\gamma_0 = g$ and $\gamma_0' = 2\omega g$.

Theorem 3.2. *Let $n \geq 2$ be even. Let (X, g_+) be AHE and let $(M, [g])$ be one of the connected components of its conformal infinity. Suppose that g satisfies that $v^{(n)}(g) = c$ for some constant c and let $\int_M \omega \, dv_g = 0$. Then*

$$\left(V_{g_+}|_{\mathcal{C}_1}\right)''(\omega) = (-1)^{n/2+1} 2^{-n/2} \int_M \left[L^{ij}_{(n/2)}(g)\omega_i\omega_j + n v_{n/2}(g)\omega^2 \right] dv_g.$$

Proof. We argue exactly as in the proof of Theorem 3.1. Define λ by $c = n\lambda$ so that $\delta\left(V_{g_+} - \lambda \operatorname{Vol}(M)\right) = 0$. Then

$$\left(V_{g_+}|_{\mathcal{C}_1}\right)''(\omega) = \partial_t^2|_{t=0}\left(V_{g_+} - \lambda \operatorname{Vol}(M)\right)(g_t)$$

with $g_t = e^{2t\omega} g$. And

$$\begin{aligned}
\partial_t^2|_{t=0}\left(V_{g_+} - \lambda \operatorname{Vol}(M)\right)(g_t) &= \partial_t|_{t=0}\left[\int_M v^{(n)}(g_t)\omega \, dv_{g_t} - \lambda n \int_M \omega \, dv_{g_t}\right] \\
&= \int_M \left[\delta v^{(n)}(g,\omega) + n v^{(n)}(g)\omega\right]\omega \, dv_g - \lambda n^2 \int_M \omega^2 \, dv_g \\
&= (-2)^{-n/2} \int_M \delta v_{n/2}(g,\omega)\omega \, dv_g \\
&= (-1)^{n/2+1} 2^{-n/2} \int_M \left[L^{ij}_{(n/2)}(g)\omega_i\omega_j + n v_{n/2}(g)\omega^2 \right] dv_g.
\end{aligned}$$

□

Proof of Theorem 1.3. This follows exactly as in the proof of Theorem 1.2 above. If $n \geq 4$ and g is Einstein with $R_{ij} = 2a(n-1)g_{ij}$, or if $n = 2$ and g has constant scalar curvature $R = 4a$, then $L^{ij}_{(n/2)}(g)$ is given by (3.8) and $v_{n/2}(g)$ by (3.4). Substituting into Theorem 3.2 gives

$$\left(V_{g+}|_{\mathcal{C}_1}\right)''(\omega) = -(-a)^{n/2-1}2^{-n/2}\binom{n-1}{n/2-1}\int_M \left(|\nabla\omega|_g^2 - R\omega^2/(n-1)\right)\,dv_g.$$

The conclusion is now clear if $R < 0$. If $R > 0$, it follows from the same argument as in the proof of Theorem 1.2 using Obata's estimate on $\lambda_1(-\Delta)$. $\square$

The sign of the second variation can also be deduced from Theorem 3.2 for certain non-Einstein critical points of V_{g+}. It is clear that $\left(V_{g+}|_{\mathcal{C}_1}\right)''$ is definite if $L^{ij}_{(n/2)}(g)$ is definite and $v_{n/2}(g)$ is a constant of the same sign. For instance, one concludes that $\left(V_{g+}|_{\mathcal{C}_1}\right)''$ is negative definite if g is a negative $n/2$-admissible solution of $\sigma_{n/2}(g^{-1}P) = c$ and $n = 4$ or $n \geq 6$ with g locally conformally flat. Under these conditions, $v_{n/2}(g) = \sigma_{n/2}(g^{-1}P)$ and $L^{ij}_{(n/2)}(g) = -T^{ij}_{(n/2-1)}(g^{-1}P)$ is the negative of the corresponding Newton tensor (see [G2]).

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