Problem Session 3.

1. Let $E$ be an Hermitian vector bundle over a Riemannian manifold $M$. Let $A, B$ be two Hermitian connections on $E$. Show that $d_A^*(A - B) = 0$ if and only if $d_B^*(A - B) = 0$.

2. Suppose $V, W$ are two Banach spaces. Consider a map

$$f : V \to W$$

$$x \mapsto L(x) + Q(x, x),$$

where $L$ is a linear isomorphism, $Q$ is a quadratic map, and there exists a constant $C > 0$ such that $\|Q(x_1, x_2)\| \leq C \cdot \|x_1\| \cdot \|x_2\|$ for all $x_1, x_2 \in V$.

(a) Let $r := \frac{1}{4C \cdot \|L^{-1}\|}$, show that $f$ is injective on the open ball $B(r)$ centered at zero with radius $r$.

(b) Show that $f^{-1} : f(B(r)) \to B(r)$ is continuous.

(c)* Show that $f^{-1} : f(B(r)) \to B(r)$ is a smooth map. (Hint: By the inverse function theorem, we only need to show that $df$ are isomorphisms.)

3. (a) Suppose $P$ is an SU(2) bundle on a closed oriented 3-manifold $M$. Consider the associated $\text{SL}_2(\mathbb{C})$ bundle

$$\tilde{P} := P \times_{\text{SU}(2)} \text{SL}_2(\mathbb{C})$$

and the associated Lie algebra bundle

$$\text{ad} P := P \times_{\text{SU}(2)} \mathfrak{su}(2).$$

Suppose $A$ is a connection on $P$ and $\phi$ is a section of $T^*M \otimes \text{ad} P$, then $A + i\phi$ defines a connection on $\tilde{P}$ by adding the $T^*M$–valued matrices in local coordinates.

Show that every connection $\hat{A}$ of $\tilde{P}$ decomposes uniquely as

$$\hat{A} = A + i\phi$$

as described above.

(b) Let $\hat{A} = A + i\phi$ be an $\text{SL}_2(\mathbb{C})$–connection. Show that $F_{\hat{A}} = 0$ if and only if

$$F_A = \phi \wedge \phi,$$

$$d_A \phi = 0.$$

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(c)* Let \((A_n, \phi_n)\) be a sequence of solutions to the following system of equations:

\[
F_A = \phi \wedge \phi,
\]

\[
d_A \phi = 0,
\]

\[
d^\ast_A \phi = 0.
\]

Suppose \(U\) is a small open ball on \(M\), and suppose that after choosing a local trivialization and applying suitable gauge transformations, we have

\[
d^\ast A_n = 0,
\]

\[
\|A_n\|_{L^3(U)} \leq C,
\]

\[
\|\phi_n\|_{L^3(U)} \leq C,
\]

for all \(n\). Show that \((A_n, \phi_n)\) has a convergent subsequence in \(C^\infty\) on compact subsets of \(U\).