Problem Session 2

1. Suppose $M$ is a closed oriented Riemannian $n$–manifold, let $G$ be a Lie group endowed with a bi-invariant metric, let $\mathfrak{g}$ be the Lie algebra of $G$. Let $P$ be a principal $G$–bundle over $M$, let $\text{ad} P := P \times_{\text{ad}} \mathfrak{g}$ be the vector bundle associated with the adjoint representation. The ad-invariant inner product on $\mathfrak{g}$ induces an inner product structure on $\text{ad} P$. Let $A$ be the affine space of connections on $P$. Consider the following functional defined over $A$:

$$F(A) := \frac{1}{2} \int_M |F_A|^2.$$ 

(a) Let $a$ be an ad $P$–valued 1-form on $M$. Show that

$$\frac{d}{dt} F(A + ta) |_{t=0} = \int_M \langle *a, d_A(*F_A) \rangle.$$ 

(b) We say that $A$ is a critical point of $F$ if and only if

$$\frac{d}{dt} F(A + ta) |_{t=0} = 0$$

for all $a$. Show that $A$ is a critical point of $F$ if and only if $d_A^* F_A = 0$.

(c) For $A \in A$, define

$$\text{grad} F(A) := d_A^* F_A.$$ 

Let $a$ be an ad $P$–valued 1-form. Verify that

$$\langle \frac{d}{dt} F(A + ta), b \rangle = \langle d_A a, d_A b \rangle + \langle F_A, [a \wedge b] \rangle$$

for all $b \in \Gamma(T^* M \otimes \text{ad} P)$, and compute

$$\frac{d}{dt} F(A + ta) |_{t=0} \text{grad} F(A + ta).$$

(d)* Suppose $A$ is a fixed smooth connection that is a critical point of $F$. Consider the following second-order differential operator:

$$\text{Hess}_A : a \mapsto \frac{d}{dt} F(A + ta) |_{t=0}.$$ 

Show that $\text{Im} \ d_A \subset \ker \text{Hess}_A$.

2. Show that the quotient space of $\mathfrak{sl}_2(\mathbb{C})$ by the adjoint action of $\text{SL}_2(\mathbb{C})$ is not Hausdorff.

3. Suppose $G$ is a compact Lie group acting smoothly on a compact manifold $M$. Show that the quotient topological space $M/G$ is Hausdorff.