July 8: Uhlenbeck's gauge fixing theorem
and compactness theorem

Lemma: Let $P$ be the trivial $SU(2)$-bundle over $S^4$, let $E$ be the associated $C^2$-bundle, let $A_0$ be the trivial connection on $E$. Then there exists a constant $C > 0$, $C > 0$ such that the following holds:

Suppose $\{A(t)\}$ is a 1-parameter family of $L^2$-connections with $\|FA(t)\|_{L^2} < \varepsilon$ and $A(0) = A_0$. Then for each $t$, there exists $g(t) \in G_\mathbb{C}$, s.t.

1. $d^* (g(t) A(t)) = 0$.
2. $\|g(t) A(t)\|_{L^2} \leq C \cdot \varepsilon \cdot \|FA(t)\|_{L^2}$

Remark: The constants $\varepsilon$ and $C$ depend on the metric on $S^4$.

Proof: Let $S$ be the set of $t$ s.t. the desired result holds. We show that $S$ is both open and closed. (Since $0 \in S$, the desired lemma would follow.)

(i) $S$ is closed. If $t_i \to t$,

\[
\begin{cases}
    d^* (g(t_i) A(t_i)) = 0, \\
    \|g(t_i) A(t_i)\|_{L^2} \leq C \|FA(t_i)\|_{L^2}
\end{cases}
\]
we show that $\mathbf{g}_t$ has a weak limit in $L^2$.

Let $B_i := g(t_i) A(t_i), \quad A_i := A(t_i), \quad g(t_i) := \mathbf{g}$.

then $B_i = A_i - (dg_i) g_i^{-1}$

$$= A_i - [A_i, g_i] g_i^{-1} - (dg_i) g_i^{-1}$$

$$= -B_i + g_i + g_i A_i = (dg_i)$$

Since $\|g_i\|_{L^\infty} \leq C \Rightarrow \|B_i - g_i + g_i A_i\|_{L^4} \leq C$

$$\Rightarrow \|g_i\|_{L^4} \leq C$$

$$\Rightarrow \|B_i - g_i + g_i A_i\|_{L^2} \leq \|B_i\|_{L^2} + \|g_i\|_{L^2} + \|g_i A_i\|_{L^2} \leq C$$

Similarly, $\|g_i\|_{L^2} \leq C$

$$\Rightarrow \|g_i\|_{L^2} \leq C.$$

So $\{g_i\}$ has a subsequence that is weakly convergent in $L^2$ (and hence strongly convergent in $L^1$).

By the assumptions, $A_i \to A_{\infty}$ in $L^2$, after taking a subsequence, $B_i \to B$ in $L^2$, $g_i \to g$ in $L^2$.

$$\Rightarrow B_i \to B g_i \text{ in } L^p \text{ (for all } p < 2).$$

$g_i A_i \to g A$ in $L^p$. $dg_i \to dg$ in $L^2$. 
\[ -B^* \beta + \gamma A = dB \]

We also have \( d^* B = 0 \) if \( B \in L^2 \) by similar arguments.

(ii) \( S \) is open. Suppose \( t \in S \). WLOG, assume \( \frac{d^* \theta(t)}{dt} = i d^* \).

Consider the map

\[ \Phi : L^2(\text{ad} P) \times L^2 \left( T^* S^4 \otimes \text{ad} P \right) \to L^2(\text{ad} P) \]

defined by

\[ \Phi (\beta, \alpha) := d^* \left( \exp (\beta) \left( A + \alpha \right) \right) \]

Then \( d^* \beta \bigg|_{(0,0)} (\beta, \alpha) = d^* \left( \exp (\beta) (A + \alpha) - d^* \beta \right) \).

Notice that \( S \in \ker d^* d \implies d^* d s = 0 \)

\[ \Rightarrow \left( d^* d s, s \right) = 0 \]

\[ \Rightarrow \left( ds, ds \right) = 0 \]

\[ \Rightarrow ds = 0. \]

So \( \ker d^* d = \ker d \).

\[ \Rightarrow \text{Im} (d^* d) = \text{Im} d^* \subseteq L^2(\text{ad} P) \]

Therefore \( \text{Im} \left( \beta \mapsto (d^* \exp (\beta) (A) - d^* d \beta) \right) \)

\[ = \text{Im} d^* \] when \( h \in H \) is sufficiently small.

So statement (ii) is an open condition.
To show that (2) is an open condition, notice that if \( d^* A = 0 \), then

\[
\overline{F}_A = (d + d^*) A + A \wedge A
\]

Since \( \ker (d + d^*) = 0 \) on \( S^4 \), we have

\[
\| A \|_{L^2_1} \leq C_1 \cdot \| \overline{F}_A \|_{L^2} + C_1 \cdot \| A \wedge A \|_{L^2}
\]

\[
\leq C_1 \cdot \| \overline{F}_A \|_{L^2} + C_2 \cdot \| A \|_{L^2}
\]

Therefore, \( \| A \|_{L^2_1} < \frac{1}{2} C_2 \Rightarrow \| A \|_{L^2_1} \leq 2 C_1 \cdot \| \overline{F}_A \|_{L^2} \)

and the first inequality is an open condition. \( \square \).

Cor. If \( A \) is a connection on \( B^k(c) \) s.t.

\[
\int_{B^k(c)} \| \overline{F}_A \|_2 < \epsilon.
\]

\[
\then \exists \ g \in C^2_\infty (B^k(1-\delta))
\]

s.t.

1. \( d^* (g(A)) = 0 \) on \( B^k(1-\delta) \)

2. \( \| g(A) \|_{L^2_1} (B^k(1-\delta)) \leq C \cdot \| \overline{F}_A \|_{L^2} (B^k) \)

pf. Define \( A(t) = f(t) A \), where

\[
f(t) : S^4 \to B^k \text{ is a smooth family of maps, s.t.}
\]

\[
f(0) = \text{the zero map, and } f(t) \text{ is an isometry from the northern hemisphere to } B^k(1-\delta)
\]
Proposition. If $A$ is an $L^3$-connection on $B^4(1)$ s.t. $F_A^+ = 0$, then $A$ is gauge equivalent to a $C^\infty$-connection on $B^4(1-\frac{5}{2})$.

Proof. After gauge transformation, assume $d^*A = 0$ on $B^4(1-\frac{5}{2})$. Then

$$(d^* + d^+) A + (A \wedge A)^+ = 0.$$ 

$A \in L^2(B^4(1-\frac{5}{2})) \Rightarrow (A \wedge A)^+ \in L^2(B^4(1-\frac{3}{2}))$

$\Rightarrow A \in L^2(B^4(1-\frac{3}{2}))$ (induction)

$\Rightarrow A \in L^k(B^4(1-\frac{5}{2}))$

for all $k$.

$\Rightarrow A \in C^\infty(B^4(1-\frac{5}{2}))$

Remark. If $A \in A_k^+$ for $(n+2)$, the same argument shows that $A \in C^\infty$ on $B^4(1-\frac{5}{2})$. 

Proposition. If $A \in A_3^+$ and $F_A^+ = 0$, then

$\exists g \in G_4$ s.t. $g(A)$ is smooth on $X$.

Proof. $\exists$ an open covering $\{U_i\}$ of $X$ s.t.

on each $U_i$, $\exists g_i$ w/ $g_i(A_i)$ smooth.

We need to "patch" $\{g_i\}$ together.
Suppose \( X = U_1 U_2 U_3 U_4 \ldots U_n \).

Let \( V_m := U_1 U_2 \ldots \hat{U}_m \hat{U} \).

We use induction to find a desired gauge transformation on \( V_m \).

Suppose \( \tilde{g}_{V_m} \) is defined on \( V_m \), \( \tilde{g}_{U_{m+1}} \) is defined on \( U_{m+1} \), s.t.

\[
\tilde{g}_{V_m}(A) \in C^\infty(V_m)
\]

\[
\tilde{g}_{U_{m+1}}(A) \in C^\infty(U_{m+1})
\]

We may assume that \( \tilde{g}_{V_m}, \tilde{g}_{U_{m+1}} \) are close to \( \text{id} \) in \( C^0 \) by replacing \( \tilde{g}_{V_m} \) with \( \tilde{g}_{V_m}^{\sim} \tilde{g}_{V_m} \), where \( \tilde{g}_{V_m}^{\sim} \) is a \( C^0 \) approximation of \( \tilde{g}_{V_m} \) (and similar for \( \tilde{g}_{U_{m+1}} \)).

Then \( g := \tilde{g}_{V_m} \circ \tilde{g}_{U_{m+1}} \in C^\infty(V_m \cap U_{m+1}) \)

(In fact, let \( B_1 = \tilde{g}_{V_m}(A) \), \( B_2 = \tilde{g}_{U_{m+1}}(A) \),

then \( dg = B_1 g - g B_1 \).)

and \( g \) is close to \( \text{id} \) in \( C^0 \). So \( g \) extends to \( U_{m+1} \) as a \( C^0 \)-gauge transformation (after shrinking \( U_{m+1} \) if necessary).

\[ \Box \]
Lemma. Suppose $A$ is an ASD connection on $B^*(1)$. Then
\[ \|F_A\|_{L^2(B^*(1))} \leq E \]
and
\[ \|d^A\| = 0 \quad \text{on} \quad B(1) \]
\[ \|A\|_{L^1(B(1-\delta))} \leq C \cdot \|F_A\|_{L^2(B^*(1))} \]

Then for $k$, $\exists$ a polynomial $f$ with zero constant term, s.t.
\[ \|A\|_{L_k^1(B(1-2\delta))} \leq f\left(\|F_A\|_{L^2(B^*(1))}\right) \]

Proof. Let $\eta$ be a cut-off function.
\[ (d^* + d^*) \eta A = (d^* + d^*) \eta A + \eta (A \wedge A) \]
\[ d^* \eta A \leq C \cdot \|A\|_{L_{k+1}(B(1-\delta))} \]
\[ + C \cdot \|A\|_{L_k^1(B(1-\delta))} \]

when $k \leq 3$.

So, we only need to bound $L_k^1$ and $L_k^2$ norms of $A$.

For that, we have
\[ \|\eta A\|_{L_k^2(B(1-\delta))} \leq C \cdot \|A\|_{L_k^2(\overline{B(1-\delta))}} \]
\[ + C \cdot \|A\|_{L_k^1(\overline{B(1-\delta))}} \cdot \|\eta A\|_{L_k^2(B(1-\delta))} \]

If $\epsilon$ is sufficiently small, we obtain $L_k^2$ and $L_k^1$ estimates by rearrangement.
Theorem (Uhlenbeck). If \( \{ A_i \} \) is a sequence of ASD connections with instanton number \( k \), then there exists a subsequence that is convergent on \( X \) finite self in \( C^0 \) after gauge transformations.