What is a surface?

Torus is the surface of a donut, sphere is the surface of a ball. Need to be a bit more careful than 'the surface of any 3-dimensional object'. We really mean a 2-manifold.

Definition. A 2-manifold is a topological space $M$ (or as Adams puts it, an object) such that every point in $M$ has an open disc as a neighborhood.

That is, a surface locally looks like the plane. Here are some examples of objects which fail to be surfaces:

![Figure 4.3](image1)

*Figure 4.3* These are not surfaces.

![Figure 4.4](image2)

*Figure 4.4* Nondisk neighborhoods of points.

Figure 1: Adams, 72
Ambient Isotopy

We defined ambient isotopy as the desirable equivalence relation of knots in our first lecture. We say that two surfaces are *isotopic surfaces* if the surfaces coincide when we deform the space around them.

![Figure 4.6](image1)
*Figure 4.6*  These are isotopic surfaces.

![Figure 4.7](image2)
*Figure 4.7*  These three surfaces are all isotopic.

![Figure 4.17](image3)
*Figure 4.17*  Three genus 3 surfaces.

The first and second of these are not isotopic. The first and the third, on the other hand, are isotopic! (We can slide the end of one of the tubes along the other tube in order to unknot the knotting).
Triangulation and Homeomorphism

Another way we can try to work with surfaces is to decompose them into triangles (a disc whose boundary has been cut at 3 points) such that the triangles 'fit nicely along their edges' and such that the triangles cover the entire surface. We can rephrase the first condition a bit more rigorously as: Any two triangles must either (1) be disjoint, (2) share an edge, or (3) share one vertex and no edges.

![Figure 4.9 Triangles cannot intersect like this.](image)

The tetrahedron is a triangulation of the sphere.

![Fundamental polygons](image)

We can also use *fundamental polygons* to represent surfaces. The letters represent which sides pair up and the arrows indicate orientation.

![Figure 2: On the left we have a square scheme of the sphere. On the right, a square scheme of the torus.](image)

We can read out these polygons as $ABB^{-1}A^{-1}$ and $ABA^{-1}B^{-1}$ respectively. Drawing a center in the fundamental polygon and connecting each vertex to the center gives a triangulation for the surface.
**Definition.** Two surfaces are *homeomorphic* if we can triangulate one of them, cut along the boundary of the triangulation, move it ambient isotopically, glue the boundaries back together in the same orientation and end up with the other surface.

*Figure 4.12* These two surfaces are homeomorphic.

Remember these surfaces? The first and the second of these surfaces are homeomorphic.

**Definition.** The number of holes in a surface is its *genus*.

The sphere has genus 0. The torus has genus 1. All three of the above surfaces have genus 3.

We saw the fundamental polygon for the torus earlier. The fundamental polygon of a standard 2-genus surface is given by the octagon $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}$

An $n$-genus surface has fundamental $4n$-gon: $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})... (A_nB_nA_n^{-1}B_n^{-1})$
Euler Characteristic, $\chi$

**Definition.** The *embedding* of a surface is the choice of how to place it in space. We care about homeomorphic type of a surface, not a particular embedding. So how can we identify two surfaces as being homeomorphic short of actually performing the simplifying cutting and pasting?

**Definition.** Given a triangulation $\Delta$ of a surface, the *Euler characteristic* of the triangulation, $\chi(\Delta)$, is equal to the number of vertices ($V$) + the number of faces ($F$) - the number of edges ($E$).

For example, the Euler characteristic of the sphere given by our tetrahedral triangulation is $4 + 4 - 6 = 2$.

**Claim.** The Euler characteristic depends only on the homeomorphic type of a surface, not on the particular triangulation you choose. In particular, we will show that if $\Delta_1$ and $\Delta_2$ are two triangulations for a surface $S$, then $\chi(S) := \chi(\Delta_1) = \chi(\Delta_2)$.

**Proof.** It is clear that the Euler characteristic, if unique for a surface, is the same for homeomorphic surfaces.

**Proof.** Goal: construct a triangulation $\Delta_3$ that contains both $\Delta_1$ and $\Delta_2$ and that has the same Euler characteristic as both.

Place both $\Delta_1$ and $\Delta_2$ on the surface at the same time so they are overlapping. We will assume that each edge of $\Delta_1$ intersects each of the edges of $\Delta_2$ a finite numbers of times and that the vertices of $\Delta_1$ are not on top of the vertices of $\Delta_2$. We will not prove that these can be satisfied (the proof is quite technical), but we can accomplish the first by moving the edges of $\Delta_1$ slightly and we can accomplish the second by moving $\Delta_1$ slightly.

1. Let $\Delta_3$ begin as $\Delta_1$. Add vertices to $\Delta_3$ corresponding to all the places where the edges of $\Delta_2$ cross the edges of $\Delta_1$. Each new vertex cuts an existing edge into two edges. So, $\chi = V + F - E$ remains unchanged by this step.

2. Add in each vertex from $\Delta_2$ to $\Delta_3$, along with an edge (chosen to be a subset of an edge from $\Delta_2$) that runs from that vertex to one of the vertices in $\Delta_3$ added in the previous step. This does not change the number of faces (though it adds vertices that run into them), and adds one new edge for each vertex. So the Euler characteristic remains unchanged.

3. Add all of the pieces of edges from $\Delta_2$ that have not yet been added to $\Delta_3$. Each one of these becomes a new, separate edge in $\Delta_3$. Each time we add such an edge, the number of edges and the number of faces goes up by one, leaving $\chi$ unchanged.

4. Almost done! Some of the faces that we now have in $\Delta_3$ may not be triangles. So we add edges to cut the faces into triangles. Every time we add such an edge, it cuts an existing face into two faces, so we add one face and one edge. So $\chi$ remains unchanged.

5. So, $\Delta_3$ has the same $\chi$ as $\Delta_1$. And we could have gotten to $\Delta_3$ by starting with $\Delta_2$ and performing the above sequence of steps. So $\Delta_3$ and $\Delta_2$ have the same $\chi$. So $\Delta_1$ and $\Delta_2$ have the same $\chi$. 

---

5
Illustration of the proof

Figure 3: From left to right: $\Delta_1$, $\Delta_2$, $\Delta_1 \cup \Delta_2$

Figure 4: Steps 1 and 2

Figure 5: Steps 3 and 4
Claim. We can define a cellular decomposition of a surface using cells of any kind, and the Euler characteristic will be consistent.

a. Choose some number of 0-cells on your surface. Points, vertices.

b. Connect these with 1-cells (1-cells whose boundaries are 0-cells). Edges.

c. Fill in the faces with 2-cells whose boundaries are 1-cells. Faces.

For example, we can create a simple cellular decomposition of the sphere. We can see that the $\chi = 2 + 2 - 2 = 2$.
Connected sum

**Definition.** Given a cellular decomposition of two surfaces, we can remove the interior of one cell on each surface, and fuse the two surfaces together along the boundary of the two chosen cells. This is called taking the *connected sum* of the surfaces.

![Connected sum of two tori](image)

*Figure 4.21*  The connected sum of two tori is a genus 2 surface.

Let’s call the connected sum $S$, and the two original surfaces $T_1$, $T_2$. We can express $\chi(S)$ in terms of $\chi(T_1)$ and $\chi(T_2)$ by noting that:

a. $V(S) = V(T_1) + V(T_2) - V(C)$, where $V(C)$ is the number of vertices in the connecting cell.

b. $E(S) = E(T_1) + E(T_2) - E(C)$, where $E(C)$ is the number of edges in the connecting cell.

c. $F(S) = F(T_1) + F(T_2) - 2$.

We further note that for any type of cell, $V(C) = E(C)$. Putting this together, we get that

$$\chi(S) = \chi(T_1) + \chi(T_2) - 2$$

We can calculate the Euler characteristic of a genus $n$ surface by considering the connected sum of tori.

**Claim.** The Euler characteristic of a genus $n$ surface is $2 - 2n$

**Proof.** For a sphere, genus 0, $\chi = 2$. A torus has $\chi = 0$.

If we have a genus $n$ surface, we can obtain a genus $n + 1$ surface by taking the connected sum of our original surface and a torus. So, if $\chi(S_n) = A$, then $\chi(S_{n+1}) = A - 2$.

Starting from our genus 0 sphere, this immediately gives us that $\chi(S_n) = 2 - 2n$. 

Some more properties of surfaces

Definition. A surface is orientable if it has 2 sides. Otherwise, it is non-orientable.

The sphere and the torus are orientable. The klein bottle, a 4 dimensional surface whose projection into $\mathbb{R}^3$ is shown here,

![Klein Bottle](image)

is not orientable.

Definition. A surface is compact if it has a finite triangulation (cellular decomposition).

Classification of surfaces

Theorem. Any closed, compact, orientable surface is homeomorphic either to a sphere or to a connected sum of tori
Surfaces with boundary

**Definition.** We can obtain a *surface with boundary* by removing the interior of some number of discs from a surface without boundary

For example, we can obtain an annulus by removing two discs from the sphere.

![Surfaces with boundary](image)

*Figure 4.33* Surfaces with boundary.

**Definition.** We will call the boundary circles of the removed discs *boundary components*.

**Definition.** When we fill in the boundary components of a surface with boundary by attaching discs, we are *capping off* that surface with boundary.

Let \( \hat{S} \) denote the capped off surface with boundary \( S \).

**Claim.** \( \chi(\hat{S}) = \chi(S) - b \), where \( b \) is the number of boundary components of \( S \).

*Proof.* We can form a cellular decomposition of \( \hat{S} \) using the boundary components of \( S \) as cells. Then we see that each time we remove the interior of a face from \( \hat{S} \) to return to \( S \), we lose one face from the decomposition.

**Definition.** We define the genus of a surface with boundary \( g(S) := g(\hat{S}) \).

**Claim.** \( g(S) = (2 - \chi(S) + b)/2 \).

*Proof.* \( \chi(\hat{S}) = 2 - 2g(\hat{S}) \Rightarrow g(\hat{S}) = (2 - \chi(\hat{S}))/2 \)

\[ g(S) = g(\hat{S}) = (2 - \chi(\hat{S}))/2 = (2 - \chi(S) + b)/2 \]