PROBLEM SET 2: DUE APRIL 21

Do as many of the problems as you can. You may work in groups of up to 3 if you like - it's fine to submit a single set of solutions for the group. Some of the problems are not so easy, feel free to come and see me if you want to discuss any of them or some hints - I'm mainly interested in you really thinking about them.

Problem 1: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Show there exists $c_d > 0$ such that with high probability γ_n, the spectral gap of the lazy simple random walk satisfies $\gamma_n \geq c_d$.

Problem 2: Consider critical bond percolation on $\mathbb{Z}^2$. Show that there are constants $c_1, c_2, c_3, c_4 > 0$ such that for all $u, v \in \mathbb{Z}^2$,

$$c_1|u-v|^{-c_2} \leq \mathbb{P}_{1/2}[u \leftrightarrow v] \leq c_3|x-v|^{-c_4}$$

where $u \leftrightarrow v$ denotes that u is connected to v.

Problem 3: Consider supercritical bond percolation on $\mathbb{Z}^2$ and let A_u denote the indicator of the event that u is in the infinite component. Show that for some constant c,

$$0 < \mathrm{Cov}[A_u, A_v] \leq \exp(-cd(u,v)).$$

Problem 4: Let G be a random d-regular graph with n vertices ($d \geq 3$ is fixed). Show that for some $\beta_d > 0$ and $c_d > 0$ that the mixing time of the Glauber dynamics for the Ising model with inverse temperature $\beta > \beta_d$ with high probability satisfies

$$t_{\mathrm{mix}}(1/4) \geq e^{c_d n}.$$

Problem 5: Let Q be an ergodic, reversible k state Markov transition matrix. A product chain of Q is a Markov chain $X_t = X_t^n$ with state space $[k]^n$ such that the projection on each co-ordinate $X_t(i)$ is an independent Markov chain with transition matrix Q (i.e. a vector of n-indecent copies of the Markov chain). Show that the family of Markov chains $\{X_t^n\}_{n\geq 1}$ has cutoff and give the cutoff location in terms of the eigenvalues of Q (and of course n).

Hint: Analyse the case that the initial state is a constant vector $X_0(1) = X_0(2) = \ldots = X_0(n)$. *Show that it is enough to consider the vector of counts* $W_x = \#\{1 \leq i \leq n : X_t(i) = x\}$ *when bounding the total variation distance.*