SDE's

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If A_t is adapted, continuous, increasing and $M_t^2 - A_t$ is a martingale then $A_t = \langle M \rangle_t$.

Proof: $M_t^2 - \langle M \rangle_t$ is a martingale and so $A_t - \langle M_t \rangle$ is a martingale and is finite variation.

So $\langle A_t - \langle M \rangle \rangle = \lim_n \sum_i \left(A_{t_{i+1}^n} - A_{t_i^n} + \langle M \rangle_{t_{i+1}^n} - \langle M \rangle_{t_i^n} \right)^2$

$= 0$

So $\mathbb{E}\left[\left(A_t - \langle M \rangle_t - (A_0 + \langle M \rangle_0) \right)^2 \right] = 0.$

$\Rightarrow A_t = \langle M \rangle_t.$

Levy's Characterization of Brownian Motion

If M_t is a continuous, L^2 martingale and $M_t^2 - t$ is a martingale then M_t is Brownian motion.

Proof: $\langle M_t \rangle = t$ by above.

We will show that

$$\mathbb{E}[\exp(iu(M_{t_{i+1}} - M_{t_i})) \mid \mathcal{F}_{t_i}] = \exp(-|u|^2/2).$$

$$e^{iuM_t} - e^{iuM_v} = \int_v^t iu e^{iuM_s} dM_s$$

$$-u^2 \int_v^t e^{iuM_s} ds$$

Dividing by e^{iuM_v} and taking expectations

$$\mathbb{E}[e^{iu(M_t - M_v)}] = 1 - u^2 \int_v^t \mathbb{E}[e^{iu(M_s - M_v)}] ds$$

if $f(t) - \mathbb{E}[e^{iu(M_t - M_v)}]$ then

$$f(t) = 1 - u^2 \int_v^t f(s) ds, \qquad f(v) = 1$$

is a deterministic integral equation whose solution is $\exp(-|u|^2(t-v))$ so $M_t - M_v \sim N(0, t-v)$.

For any $A \in \mathcal{F}_v$

$$\mathbb{E}[e^{iu M_t - M_v} 1_A \mid \mathcal{F}_v] = \mathbb{E}[1_A \mid \mathcal{F}_v] - u^2 \int_v^t \mathbb{E}\, e^{iu(M_s - M_v)} 1_A \mid \mathcal{F}_v] ds$$

$$= \mathbb{P}[A] - u^2 \int_v^t \mathbb{E}\, e^{iu(M_s - M_v)} 1_A \mid \mathcal{F}_v] ds$$

so $\mathbb{E}[e^{iu M_t - M_v} 1_A] = e^{-u^2(t-v)} \mathbb{P}[A]$

Hence $M_t - M_v$ is independent of 1_A.

So M_t is Gaussian with independent increments and $M_t - M_v \sim N(0, t-v)$. By assumption the paths are continuous so M_t is Brownian Motion

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Any continuous martingale is a time change Brownian motion.

Let M_t be a cts martingale and let $T_s = \inf\{t : \langle M \rangle_t = s\}$.

Set $W_s = M_{T_s}$, and $\mathcal{G}_s = \mathcal{F}_{T(s)}$.

$$\mathbb{E}[W_{s_2} - W_{s_1} \mid \mathcal{G}_{s_1}] = 0$$

$$\mathbb{E}[(W_{s_2} - W_{s_1})^2 \mid \mathcal{G}_{s_1}]$$

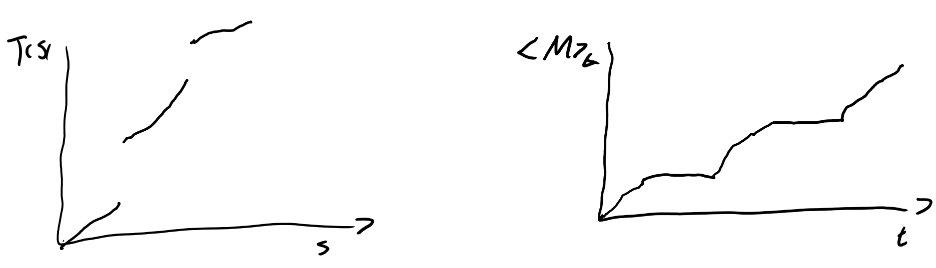
$$= \mathbb{E}[(M_{T(s_2)} - M_{T(s_1)})^2 \mid \mathcal{G}_{s_1}]$$

$$= \mathbb{E}[\langle M \rangle_{T(s_2)} - \langle M \rangle_{T(s_1)} \mid \mathcal{G}_{s_1}]$$

$$= s_2 - s_1$$

Finally need to show that W_s is continuous.

Problem, $T(s)$ may have jumps if $\langle M \rangle_t$ is flat



For $t_1 \in \mathbb{Q}$ want to show

$\langle M \rangle_{t_1} = \langle M \rangle_t$ for $t > t_1 \implies M_{t_1} = M_t$ a.s.

Let $t_* = \inf\{ t : \langle M \rangle_t > \langle M \rangle_{t_1} \}$.

Then $N_t = M_{t \wedge t_*} - M_{t_1}$ is a martingale for $t \geq t_1$

and $\langle N_t \rangle = \langle M \rangle_{t \wedge t_*} - M_{t_1}$ for $t \geq t_1$

$= 0.$

Hence $M_{t_*} = M_{t_1}$ a.s. so W_s is continuous

and hence is Brownian motion.

SDEs

Suppose

$$dX_t = f(X_t, t)\, dB_t + g(X_t, t)\, dt$$

Examples

Math Finance: Stock prices

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t$$

Biology: Genetic Drift Wright-Fisher

$$dX_t = \sqrt{X_t(1-X_t)}\, dB_t$$

Then infintesimal drift is

$$\mu(x, t) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}[X_{t+h} - X_t \mid X_t = x]$$

$$\mu(x,t) = \lim_{h\to 0} \ldots$$

$$= \lim_{h\to 0} \frac{1}{h} \mathbb{E}\left[f(X_t,t)(B_{t+h} - B_t) + g(X_t,t)\cdot h \mid X_t = x \right]$$

$$= g(x,t)$$

If X_t is a martingale then $\mu = 0$

<u>Infintesimal Variance</u>

$$\sigma^2(x,t) = \lim_{h\to 0} \frac{1}{h} Var(X_{t+h} - X_t \mid X_t = x)$$

$$= \lim \frac{1}{h} \mathbb{E}\left[f(X_t,t)^2 (B_{t+h} - B_t)^2 \mid X_t = x \right)$$

$$= f(x,t)^2.$$

Convergence of discrete processes to SDE's.

<u>Simple Random Walk</u>

$$S_{t+1} - S_t = \pm 1 \text{ w.p. } \tfrac{1}{2}$$

$$\mathbb{E}[S_{t+1} - S_t \mid S_t = x] = 0.$$

$$\mathbb{E}[(S_{t+1} - S_t)^2 \mid S_t = x] = 1$$

<u>Rescaling</u>: Set $X_{t,n} = \frac{1}{\sqrt{n}} S_{tn}$

$$n\,\mathbb{E}[X_{t+\frac{1}{n}} - X_t \mid X_t] = 0$$

$$n\,\mathbb{E}[(X_{t+\frac{1}{n}} - X_t)^2 \mid X_t = x] = 1$$

So we predict $X_{t,n} \to X_t$ with

$$dX_t = 1 \cdot dB_t \quad \text{i.e. } X_t \text{ is Brownian motion.}$$

Biased R.W.

If $\mathbb{P}[S_{t+1} - S_t = 1] = p > \frac{1}{2}$, $\mathbb{P}[S_{t+1} - S_t = -1] = 1-p$

$$\mathbb{E}[S_{t+1} - S_t \mid S_t = x] = 2p-1, \quad Var[S_{t+1} - S_t \mid S_t = x] = 1-(2p-1)^2$$

Then if $X_{t,n} = \frac{1}{\sqrt{n}} S_{tn}$, $X_{t,n} \xrightarrow[n\to\infty]{} \infty$ a.s.

Either take

$$X_{t,n} = \frac{1}{n} S_{tn} \quad \text{and} \quad X_{t,n} \to (2p-1)t \quad \text{LLN.}$$

Or

$$Y_{t,n} = \frac{1}{\sqrt{n}}\left(X_{t,n} - (2p-1)t\right)$$

$$dY_{t,n} = \sqrt{(1-(2p-1)^2)}\, dB_t = \sqrt{4p(1-p)}\, dB_s$$

by CLT.

Sampling without replacement.

n objects, $\frac{n}{2}$ are blue

Draw without replacement, S_t is number of blue objects in first t draws.

$$\mathbb{E}[S_{t+1} - S_t \mid S_t = s] = \frac{\frac{n}{2} - s}{n-t} = \frac{1}{2} + \frac{t/2 - s}{n-t},$$

$$\mathbb{E}[(S_{t+1} - S_t)^2 \mid S_t = s] = \frac{1}{2} + \frac{t/2 - s}{n-t}$$

We should rescale $X_{t,n} = \frac{1}{\sqrt{n}}\left(S_{tn} - \frac{t}{2}\right)$.

Then

$$n\mathbb{E}[X_{t+\frac{1}{n},n} - X_{t,n} \mid X_t = x]$$

$$= \sqrt{n}\,\mathbb{E}[S_{tn+1} - S_{tn} \mid S_t = \tfrac{tn}{2} + x\sqrt{n}] - \tfrac{1}{2}$$

$$= \sqrt{n}\,\frac{tn/2 - (tn/2 + x\sqrt{n})}{n - tn} = \frac{-x}{1-t}$$

$$n\mathbb{E}[(X_{t+\frac{1}{n},n} - X_{t,n})^2 \mid X_t = x]$$

$$= \mathbb{E}[(S_{tn+1} - S_{tn} - \tfrac{1}{2})^2 \mid S_t = \tfrac{tn}{2} + x\sqrt{n}]$$

$$= \frac{1}{4}$$

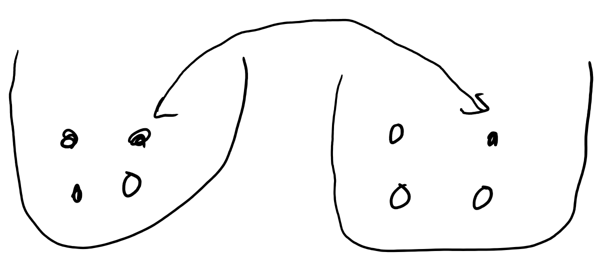
Prediction $X_{t,n} \to X_t$ and

$$dX_t = \frac{1}{2}dB_t - \frac{X_t}{1-t}dt \quad \text{for } 0 \le t \le 1.$$

This is $\frac{1}{2}$ Brownian bridge, $\frac{1}{2}B_t$ conditioned on $B_t = 0$.

Ehrenfest Urn process.

$2n$ balls in two urns, n blue + n red



Dynamics : Pick a ball from each urn and swap them

S_t = # blue balls in left urn

$$\mathbb{P}\left[S_{t+1} - S_t = 1 \mid S_t = x\right] = \frac{x}{n} \cdot \frac{x}{n} = \frac{x^2}{n^2}$$

$$\mathbb{P}\left[S_{t+1} - S_t = -1 \mid S_t = x\right] = \frac{n-x}{n} \cdot \frac{n-x}{n} = \left(\frac{n-x}{n}\right)^2$$

so

$$\mathbb{E}\left[S_{t+1} - S_t \mid S_t = \frac{n}{2} + x\right] = \frac{\left(\frac{n}{2}+x\right)^2 - \left(\frac{n}{2}-x\right)^2}{n^2}$$

$$= \frac{-x}{n}.$$

$$\mathbb{E}\left[(S_{t+1} - S_t)^2 \mid S_t = \frac{n}{2} + x\right] = \frac{\left(\frac{n}{2}+x\right)^2 + \left(\frac{n}{2}-x\right)^2}{n^2}$$

$$= \frac{1}{2} + \frac{x^2}{n^2}.$$

S_t is hypergeometric.

$$\mathbb{P}\left[|S_t - \mathbb{E} S_t| \geq t\sqrt{n}\right] \leq 2\exp(-t^2/2) \quad (\text{Azuma})$$

So if $X_{t,n} = \frac{1}{\sqrt{n}}\left(S_{nt} - \frac{n}{2}\right)$

then

$$n\,\mathbb{E}\left[X_{t+\frac{1}{n},n} - X_{t,n} \mid X_t = x\right]$$

$$= \sqrt{n}\,\mathbb{E}\left[S_{tn+1} - S_{tn} \mid S_t = \frac{n}{2} + x\sqrt{n}\right]$$

$$= \sqrt{n} \cdot \frac{-x\sqrt{n}}{n} = -x$$

$$n\,\mathbb{E}\left[(X_{t+\frac{1}{n},n} - X_{t,n})^2 \mid X_t = x\right]$$

$$= \mathbb{E}\left[(S_{tn+1} - S_{tn})^2 \mid S_t = \tfrac{n}{2} + x\sqrt{n}\right]$$

$$\cong \frac{1}{2}.$$

So $\quad dX_t = \frac{1}{\sqrt{2}}\, dB_t - X_t\, dt$

Ornstein - Ulenbeck Process (rescaled)

Same as $\quad e^{-t/2} B_{e^t}$.

<u>Strong solutions to an SDE</u>

Given x_0, $\mu(x,t)$, $\sigma(x,t)$,

X_t is a <u>strong solution</u> to

$(*)$ $dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dB_t, \quad X_0 = x_0$

if

(a) X_t is adapted to $\mathcal{F}_t$ the filtration generated by B_t

$$B_t, x_0 \longrightarrow \boxed{\mu, \sigma} \longrightarrow X_t.$$

(b) $\mathbb{P}[X_0 = x_0] = 1$

(c) $\int_0^t |\mu(X_s, s)| + \sigma^2(X_s, s)\, ds < \infty$ a.s.

(d) $X_t = X_0 + \int_0^t \mu(X_s, s)\,ds + \int_0^t \sigma(X_s, s)\,dB_s$

<u>Theorem</u>: If $\mu(x,t)$, $\sigma(x,t)$ satisfy

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K|x-y|$$

$$|\mu(x,t)|^2 + |\sigma(x,t)|^2 \leq K(1+x^2).$$

then (*) has a unique strong solution.

<u>Proof</u>:

<u>Uniqueness</u>

If $X_t = t^2$ then X_t solves

$dX_t = 2\sqrt{X_t}\,dt$ since $t^2 = \int_0^t 2\sqrt{s^2}\,ds$

but $X_t = 0$ is also a solution

so some conditions on μ, σ are necessary for uniqueness.

<u>Gronwall Inequality</u> (100th Anniversary, Princeton Faculty).

If $0 \leq g(t) \leq \alpha(t) + \beta \int_0^t g(s)\,ds$

then $g(t) \leq \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)}\,ds$

Let $h(t) = \beta e^{-\beta t} \int_0^t g(s)\,ds$

$h'(t) = \left(g(t) - \beta \int_0^t g(s)\,ds \right) \beta e^{-\beta t}$

$$\leq \alpha(t)\beta e^{-\beta t}$$

$$h(t) \leq \int_0^t \alpha(s)\beta e^{-\beta s} ds$$

$$\beta \int_0^t g(s)\, ds = e^{\beta t} h(t)$$
$$\leq \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds$$

Suppose $X_t, \tilde{X}_t$ are both solutions

$$X_t - \tilde{X}_t = \int_0^t \mu(X_s, s) - \mu(\tilde{X}_s, s) ds$$
$$+ \int_0^t \sigma(X_s, s) - \sigma(\tilde{X}_s, s)\, dB_s$$

So

$$\mathbb{E}\left[|X_t - \tilde{X}_t|^2\right] \leq 2\, \mathbb{E}\left(\int_0^t |\mu(X_s, s) - \mu(\tilde{X}_s, s)| ds\right)^2$$
$$+ 2\, \mathbb{E}\left(\int_0^t \sigma(X_s, s) - \sigma(\tilde{X}_s, s)\, dB_s\right)^2$$

$$\leq 2t \int_0^t \mathbb{E}|\mu(X_s, s) - \mu(\tilde{X}_s, s)|^2 ds$$

$$+ 2\int_0^t \mathbb{E}\left[|\sigma^2(X_s, s) - \sigma^2(\tilde{X}_s, s)|^2\right] ds$$

$$\leq 2K^2(t+1) \int_0^t \mathbb{E}|X_s - \tilde{X}_s|^2 ds.$$

Set $g(t) = \mathbb{E}|X_t - \tilde{X}_t|^2$ then for $0 \leq t \leq T$

$$g(t) \leq 2K^2(T+1)\int_0^t g(t)\, dt$$

so $g(t) \le 0$ by Gronwall.

For uniqueness it is enough that μ, σ are locally Lipschitz.

Existence

Construction:

$$X_t^{(0)} = x_0,$$

$$X_t^{(n+1)} = x_0 + \int_0^t \mu(X_s^{(n)}, s)\,ds + \int_0^t \sigma(X_s^{(n)}, s)\,dB_s.$$

– $X_t^{(n)}$ is adapted.

Set

$$Y_t = \int_0^t \mu(X_s^{(n)}, s) - \mu(X_s^{(n-1)}, s)\,ds$$

$$M_t = \int_0^t \sigma(X_s^{(n)}, s) - \sigma(X_s^{(n-1)}, s)\,dB_s$$

$$\mathbb{E}\left[\max_{0\le s\le t} |M_s|^2\right] \le 4\,\mathbb{E}|M_t|^2$$

$$= 4\int_0^t \mathbb{E}\left(\sigma(X_s^{(n)}, s) - \sigma(X_s^{(n-1)}, s)\right)^2 ds$$

$$\le 4K^2 \int_0^t \mathbb{E}\left[|X_s^{(n)} - X_s^{(n-1)}|^2\right] ds$$

$$\mathbb{E}\left(\max_{0\le s\le t} (|Y_t|^2)\right) \le \mathbb{E}\left[\int_0^t K\,|X_s^{(n)} - X_s^{(n-1)}|\,ds\right.$$

$$\le K^2 t \int_0^t \mathbb{E}\,|X_s^{(n)} - X_s^{(n-1)}|^2\,ds$$

Set $L = K^2(4+T)$

$$\Rightarrow \mathbb{E}\left[\max_{0\le s\le t} |X_s^{(n+1)} - X_s^{(n)}|^2\right] \le L \int_0^T \mathbb{E}|X_s^{(n)} - X_s^{(n-1)}|^2 ds$$

By induction

$$\mathbb{E}\left[\max_{0\le s\le t} |X_s^{(n+1)} - X_s^{(n)}|^2\right] \le C\frac{(Lt)^n}{n!} \quad \text{for } 0\le t\le T.$$

$\Rightarrow X_t^{(n)} \to X_t$ uniformly a.s. on $[0,T)$

Example:

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t, \quad X_0 = 1$$

Since $\frac{df}{dx} = \mu x$ has solution $Ce^{\mu x}$ natural to guess an exponential form of the solution. Let $Y_t = \log X_t$. Then by Ito,

$$dY_t = \frac{1}{X_t}\,dX_t + \frac{1}{2}\frac{-1}{X_t^2}\,d\langle X\rangle_t$$

$$= \mu\,dt + \sigma\,dB_t - \tfrac{1}{2}\sigma^2\,dt$$

since $d\langle X\rangle_t = \sigma^2 X_t^2\,dt$.

So $Y_t = (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t$

Let's check that

$X_t = \exp((\mu - \sigma^2/2)t + \sigma B_t)$ is a solution.

This is called exponential Brownian motion.

Since $X_t = f(B_t, t)$ where

$$f(x,t) = \exp((\mu - \sigma^2/2)t + \sigma x)$$

so by Itô's formula

$$X_t - X_0 = \int_0^t (\mu - \sigma^2/2) X_s \, ds$$

$$+ \int_0^t \sigma X_s \, dB_s + \frac{1}{2} \int_0^t \sigma^2 X_s \, ds \quad \checkmark$$

<u>Integration by Parts</u>

Quadratic Covariation of M_t, N_t is

$$\langle M, N \rangle_t = \lim \sum_i (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i})$$

so $\langle M \rangle = \langle M, M \rangle$

$$\langle M, N \rangle_t = \frac{1}{2} \left(\langle M+N, M+N \rangle - \langle M \rangle - \langle N \rangle \right).$$

<u>Lemma</u> If $X_t = X_0 + M_t + C_t$

$Y_t = Y_0 + N_t + D_t$ then

$$\int_0^t X_s \, dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \, dX_s - \langle X, Y \rangle_t$$

<u>Proof:</u> Let $Z_t = X_t + Y_t$

$$Z_t^2 - Z_0^2 = \int_0^t Z_s \, dZ_s - \langle Z \rangle_t \quad (1)$$

$$X_t^2 - X_0^2 = \int_0^t X_s \, dX_s - \langle X \rangle_t \quad (2)$$

$$Y_t^2 - Y_0^2 = \int_0^t Y_s \, dY_s - \langle Y \rangle_t \qquad (3)$$

$\frac{1}{2}\big((1)-(2)-(3)\big)$ gives

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s - \langle X, Y \rangle_t.$$

Local Times

If $f(x) = |x|$ then $f'(x) = \mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$

and $f''(x)$ "=" $2\delta(x)$. Itô's formula would say

$$|B_t| - |B_0| = \int_0^t \mathrm{sgn}(B_s) \, dB_s$$

$$+ \int_0^t \delta(B_s) \, ds \leftarrow \text{time of B.M. at } 0.$$

We will introduce local times to make formal sense of this equation.

Lemma:

Let f be convex. Then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \, dB_s + K_t^f$$

where K_t^f is an increasing adapted process.

Proof: If f is C^2 then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \, dB_s + \int_0^t \tfrac{1}{2} f''(B_s) \, ds$$

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\, dB_s + \int_0^t \tfrac{1}{2} f''(B_s) ds$$

so $K_t^f = \int_0^t \frac{1}{2} f''(B_s) ds.$

Let φ be smooth supported on $(0,1)$, $\int_0^1 \varphi(x) dx = 1.$

Set $f_n(x) = \int_0^1 f(x + \frac{y}{n}) \varphi(y) dy$

so $f_n(x) \to f(x)$, $f_n'(x) \uparrow f'(x)$ and $f_n(x)$ is smooth.

$$\mathbb{E}\left(\int f_n'(B_s) - f'(B_s) dB_s\right)^2 = \int_0^t \mathbb{E}[f_n'(B_s) - f'(B_s)]^2 ds$$

$$\to 0$$

so $\int_0^t f_n'(B_s)\, dB_s \xrightarrow{L^2} \int_0^t f'(B_s) dB_s$

and on some subsequence n_k converges uniformly. So

$$f_{n_k}(B_t) - f_{n_k}(B_0) - \int_0^t f_{n_k}'(B_s) dB_s = K_t^{f_{n_k}}$$

converges uniformly to

$$f(B_t) - f(B_0) - \int_0^t f'(B_s) dB_s$$

and so $K_t^{f_{n_k}}$ converges to K_t^f.

Since $K_t^{f_n}$ are increasing + adapted so is K_t^{f}.

Define $L_t^a = K_t^f$ with $f(x) = |x-a|$.

We won't prove it but $L(t,a)$ has a version that is continuous.

Let $f(x) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |x-a|\,da$.

Then f_n is (almost) twice differentiable and

$$f''(x) = \frac{1}{\varepsilon} I(|x|<\varepsilon)$$

so

$$f_\varepsilon(B_t) - f_\varepsilon(B_0) - \int_0^t f_\varepsilon'(B_s)\,dB_s$$

$$= \frac{1}{2\varepsilon}\int_0^t I(|B_s|<\varepsilon)\,ds$$

$$= \frac{1}{2\varepsilon}\left|\{0 \le s \le t : |B_s| < \varepsilon\}\right|.$$

$$LHS = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |B_t - a| - |B_0 - a|\,ds$$

$$- \int_0^t \frac{1}{2\varepsilon}\int_0^t \operatorname{sgn}(B_s - a)\,da\,dB_s$$

Use Fubini

$$= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |B_t - a| - |B_0 - a| - \int_0^t \operatorname{sgn}(B_s-a)\,dB_s\,da$$

$$= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} L_t^a\,da.$$

$\to L_t^0$ by continuity as $\varepsilon \to 0$

$\to L_t^0$ by continuity. as $\varepsilon \to 0$.

So

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\{0 \le s \le t : |B_s| \le \varepsilon\}|.$$

measure the time close to 0.

Tanaka's Formula is

$$L_t^a = |B_t - a| - |B_0 - a| - \int_0^t sgn(B_s - a)\, dB_s$$

Weak solutions of SDE's.

We say X_t is a solution of

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t$$

if there exists (X_t, W_t, G_t)

where (X_t, W_t) are adapted to G_t,

W_t is Brownian motion and

$$X_t - X_0 = \int_0^t \mu(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dB_s,$$

$X_0 = x_0$.

Suppose $\mu \equiv 0$, $\sigma(x,t) = sgn(x)$.

If (X, W) is a solution,

$$X_t = \int_0^t sgn(X_s)\, dW_s$$

then $\int_0^t sgn(-X_s)\, dW_s = -X_s$

so $(-X, W)$ is also a solution so there is pathwise non-uniqueness.

But X_t is a martingale and

$$\langle X \rangle_t = \int_0^t (sgn(X_s))^2 ds = t$$

so X_t is Brownian motion so the law of X_t is unique.

The existence of a solution is found by taking X_t as B.M., $W_t = \int_0^t sgn(X_s) dX_s$
Then W_t is B.M. and

$$sgn(X_s)\, dX_s = sgn(X_s)^2 dW_s = dW_s$$

so (X, W) solves the SDE.

Now $W_t = \int_0^t sgn(X_s) dX_s$.

$$= |X_t| - L_t^0$$

$$= |X_t| - \lim \tfrac{1}{2\varepsilon} \{ 0 \le s \le t : |X_s| < \varepsilon \}$$

So W_t is measurable w.r.t. $\mathcal{F}_t^{|X|}$.

If X_t is a strong solution then

X_t is $\mathcal{F}_t^W$ measurable and therefore $\mathcal{F}_t^{|X|}$

measurable but $\mathrm{sgn}(X_t)$ is not measurable

w.r.t. $\mathcal{F}_t^{|X|}$. Hence there are no

strong solutions.

Girsanov's Theorem

Change of measure.

If X_t is adapted on $[0,T]$ w.r.t. $\mathcal{F}_t$,

$$Z_t = \exp\left(\int_0^t X_s \, dB_s - \frac{1}{2}\int_0^t X_s^2 \, ds \right)$$

then if $M_t = \int_0^t X_s \, dB_s$, $d\langle M\rangle_t = X_t^2 \, dt$

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t \right)$$

Now with $F(x) = e^x$, apply Itô's formula

to the semi-martingale $M_t - \frac{1}{2}\langle M\rangle_t$

$$Z_t - 1 = \int_0^t \exp\left(M_t - \tfrac{1}{2}\langle M_t\rangle \right) dM_s$$

$$+ \frac{1}{2}\int_0^t \exp\left(M_t - \langle M\rangle_t \right) d\langle M_t - \tfrac{1}{2}\langle M_t\rangle\rangle_t$$

$$\cdot \int^t \exp\left(M_t - \langle M_t\rangle \right) d\left(-\tfrac{1}{2}\langle M\rangle_t \right)$$

$$+ \int_0^t \exp(M_t - \langle M_t \rangle) \, d\left(-\tfrac{1}{2}\langle M \rangle_t\right)$$

$$= \int_0^t Z_s \, dM_s$$

So $Z_t = 1 + \int_0^t Z_s X_s \, dB_s$

and Z_t is a martingale, $\mathbb{E} Z = 1$, $Z \geq 0$.

Let $\tilde{P}$ be the measure on $\mathcal{F}$

$$\tilde{P}(A) := \mathbb{E}[\, I(A) \cdot Z \,].$$

<u>Girsanov's Theorem</u>

If B_t is B.M. w.r.t. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$,

$$\tilde{B}_t = B_t - \int_0^t X_s \, ds \qquad 0 \leq t \leq T$$

is B.M. in the space $(\Omega, \mathcal{F}, \mathcal{F}_t, \tilde{P})$.

<u>Proof</u>: Want to show $\tilde{B}_t$ is a $\tilde{P}$ martingale so

$$\tilde{\mathbb{E}}[\, \tilde{B}_t \mid \mathcal{F}_s \,] = \tilde{B}_s$$

<u>Claim</u> If Y is $\mathcal{F}_t$ meas,

$$\tilde{\mathbb{E}}[\, Y \mid \mathcal{F}_s \,] = \frac{1}{Z_s} \mathbb{E}[\, Y Z_t \mid \mathcal{F}_s \,]$$

Then need to show for all $A \in \mathcal{F}_s$,

$$\tilde{\mathbb{E}}[\, 1_A \cdot Y \,] = \tilde{\mathbb{E}}\left[\, 1_A \cdot \frac{1}{Z_s} \mathbb{E}[Y Z_t \mid \mathcal{F}_s] \,\right]$$

$$\tilde{\mathbb{E}}[1_A \; Y] = \tilde{\mathbb{E}}\left[1_A \frac{1}{Z_s} \mathbb{E}[Y Z_t \mid \mathcal{F}_s]\right]$$

$$RHS = \mathbb{E}\left[Z_T \; 1_A \frac{1}{Z_s} \mathbb{E}[Y Z_t \mid \mathcal{F}_s]\right]$$

$$= \mathbb{E}\left[1_A \; \mathbb{E}[Y Z_t \mid \mathcal{F}_s]\right]$$ since Z_t is a martingale

$$= \mathbb{E}[1_A \; Y Z_t]$$

$$= \mathbb{E}[Z_T \; 1_A Y] = \tilde{\mathbb{E}}[1_A Y] \checkmark$$

Now by integration by parts,

$$\tilde{B}_t Z_t = \int_0^t \tilde{B}_s \, dZ_s + \int_0^t Z_s \, d\tilde{B}_s + \langle \tilde{B}, Z \rangle_t$$

$$= \int_0^t \tilde{B}_s \, dZ_s + \int_0^t Z_s \, dB_s - \int_0^t X_s Z_s \, ds + \langle \tilde{B}, Z \rangle_t$$

$$\langle \tilde{B}, Z \rangle_t = \langle B, Z \rangle$$

$$= \lim \sum_i (B_{t_{i+1}} - B_{t_i})(Z_{t_{i+1}} - Z_{t_i})$$

$$= \lim \sum_i (B_{t_{i+1}} - B_{t_i}) \cdot Z_{t_i} X_{t_i} (B_{t_{i+1}} - B_{t_i})$$

$$= \int_0^t Z_s X_s \, ds$$

so $\tilde{B}_t Z_t$ is a P-martingale, so

$$\tilde{\mathbb{E}}[\tilde{B}_t \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[\tilde{B}_t Z_t \mid \mathcal{F}_s]$$

$$= \frac{1}{Z_s} \tilde{B}_s Z_s = \tilde{B}_s$$

Hence $\tilde{B}_t$ is a $\hat{P}$ martingale.

Now $\langle \tilde{B} \rangle_t = t$ under both P and P'

so $\tilde{B}$ is a Brownian motion under $\tilde{P}$.

Weak solution to

$$dY_t = \mu(Y_t, t)\,dt + dB_t.$$

Let Y_t be Brownian motion,

$$B_t = Y_t - \int_0^t \mu(Y_s, s)\,ds.$$

Then B_t is Brownian motion on

$(\Omega, \mathcal{F}, \mathcal{F}_t, \tilde{P})$ where

$$\tilde{P}(A) = \mathbb{E}[Z_T 1_A]$$

with $X_t = \mu(Y_t, t)$ and

$$Z_T = \exp\left(\int_0^T X_s\,dY_s - \frac{1}{2}\int_0^T X_s^2\,ds \right)$$

by Girsanov's Theorem.
