Let's investigate time homogeneous SDEs in one dimension,

$$d X_t = M(X_t) \, dt + \sigma(X_t) \, dB_t$$

Assume that $M, \sigma$ smooth, Lipschitz.

**Solution:**

Step (1) Transform it into a martingale. Find $\psi$ such that $Y_t = \psi(X_t)$ is a martingale. By Ito,

$$d Y_t = \psi'(X_t) M(X_t) \, dt + \psi'(X_t) \sigma(X_t) \, dB_t$$

$$+ \frac{1}{2} \sigma^2(X_t) \psi''(X_t) \, dt$$

So we want

$$\psi'(X_t) M(X_t) + \frac{1}{2} \sigma^2(X_t) \psi''(X_t) = 0$$

Set $h(x) = \psi'(x)$

$$\frac{d}{dx} \log(h(x)) = \frac{h'(x)}{h(x)} = -\frac{2 M(x)}{\sigma^2(x)}$$
So
\[ h(x) = B \exp \left( -\int \frac{2mcx}{\sigma c^2} \, dx \right) \]
\[ \Phi(x) = A + B \int \exp \left( -\int \frac{2mcx}{\sigma c^2} \, dx \right) \]

**Example:**
Brownian motion with drift.
\[ X_t = \mu t + B \]
\[ \Phi(x) = A + B \int \exp \left( -2\mu x \right) \, dx \]
\[ = A + B \exp(-2\mu x) \]

**Example** Bessel process
\[ dX_t = \frac{d-1}{2} \frac{1}{X_t} \, dt + dB_t \]
\[ \Phi(x) = A + B \int \exp \left( \int -\frac{d-1}{x} \, dx \right) \]
\[ = A + B \int \exp \left( - (d-1) \log x \right) \]
\[ = A + B \int x^{-(d-1)} \, dx \]
\[ = \begin{cases} A + B \frac{x^{-d+2}}{d-2} & \text{for } d > 2 \\ A + B \log(x) & \text{for } d = 2 \end{cases} \]
\\[
\begin{align*}
\{ & \log(x) \quad d = 2 \\
& A + B x^{-(d-2)} \quad d < 2.
\end{align*}
\\
\]

So
\[ d\, Y_t = h(Y_t) \sigma(e^{Y_t}) \, dB_t = g(Y_t) \, d\tilde{B}_t \\
\]
So \( Y_t \) is a time changed Brownian motion.

**Construction:**

Let \( W_t \) be Brownian motion, set
\[ Y_t = \inf \{ s : \int_0^s \frac{1}{g(W_u)} \, du = t \} \]
Set
\[ Y_t = W_{Y_t} \]

Then
\[ < Y_{T_\tau} < W_{Y_{T_\tau}} = Y_\tau \]
so
\[ d<Y_{T_\tau} = dY_t = g^2(W_{Y_t}) \, dt \]
\[ = g^2(Y_t) \, dt \]

Let
\[ Z_t = \int_0^t \frac{1}{g(Y_s)} \, dY_s \]
so \( Z_t \) is a martingale and
\[ < Z_{T_\tau} = \int_0^\tau \frac{1}{g(Y_s)}^2 \, d<Y_s \]

so \( g(t) = \int_0^t ds = t \)

So \( Z_t \) is Brownian motion and \( Y_t \) solves

\[ dY_t = g(Y_t) \, dZ_t. \]

Now set \( X_t = e^{-t} \, Y_t \) and

\[ dX_t = M(X_t) \, dt + \sigma(X_t) \, dZ_t. \]

**Hitting Probabilities**

If \( x_0 \in [a, b] \) what is the probability that \( X_t \) hits \( a \) before \( b \)?

Let \( T = T_{a,b} = \inf\{ t : X_t \in \{a, b\} \} \)

\[
E \, \Phi(X_T) = E \, \Phi(X_0) = P \left( X_T = a \right) \, \Phi(a) + P \left( X_T = b \right) \, \Phi(b).
\]

So

\[
P \left( X_T = a \right) = \frac{\Phi(b) - \Phi(x_0)}{\Phi(b) - \Phi(a)}
\]

If \( dX_t = M(X_t) \, dt + dB_t \)
under what conditions on \( c_{nw} \) is
\[ P \left[ T_x = 0 \right] > 0, \quad \text{i.e. } X_t \text{ is transient.} \]

If
\[ \lim_{b \to \infty} \phi(b) < \infty \quad \text{then} \]
\[ \lim_{b \to \infty} P \left[ X_{T_{a,b}} = a \right] = \lim_{b \to \infty} \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(c_{nw})} < 1. \]

Recall
\[ \phi(x) = \int_1^x \exp \left( -2 \int_1^y m(u) \, du \right) \, dy \]
\[ \phi(\infty) < \infty \quad \text{if } m_{nw} \text{ is positive and doesn't decay too quickly.} \]

**Example:** Bessel Process  \( m(x) = \frac{d - 1}{2} x \)

\[ \phi(x) = \int_1^x \exp \left( - (d-1) \log y \right) \]
\[ = \int_1^x y^{-(d-1)} \, dy \]
\[ = \begin{cases} \frac{1}{(2-d)} (x^{-d-2} - 1) & d \neq 2 \\ \log x & d = 2 \end{cases} \]

So
\[ \lim_{x \to \infty} \phi(x) < \infty \quad \text{iff } d > 2. \]
Example: \( dX_t = \left( X_t^3 - \theta X_t^4 \right) dt + X_t^{\frac{3}{2}} dB_t \)

Then
\[
\psi(x) = \int_1^x \exp \left( -2S_1 \frac{u^5 - u_0^5}{u^5} \right) du
\]
\[
= \int_1^x \exp \left( 2u^4 - 2u_0^4 \right) du
\]
\[
= \int_1^x u^{2\theta} \exp (2u^4) du
\]

For any \( \gamma > -\frac{1}{2} \) \( \psi(0) \to \infty \) as \( x \to \infty \) so \( X_t \) is recurrent.

\( \psi(x) \to -\infty \) as \( x \to 0 \) so \( X_t \) never hits 0.

If \( \sigma(x) \to 0 \) as \( x \to 0 \) fast enough, \( X_t \) may never hit 0.

E.g. \( dX_t = X_t^\alpha dB_t \).

Then \( X_t = W_{\sigma_t} \), \( W_t \) B.M.,

\[
\gamma_t = \inf \{ s : \int_0^s \frac{1}{W_u^{3\theta}} du = t \}
\]

Let \( \tau_t = \inf \{ \epsilon : W_\epsilon = 2^{-\epsilon} \} \)
The time spent in \([2^{-\kappa}, 2^{1}]\) from time \(T_k\) to \(T_{k+1}\) approximately, 
\[2^{-\frac{k}{\alpha}}\]

\[
\mathbb{E} \left[ \frac{1}{W_n} dN \right] \approx 2^{2k(\alpha - 1)}
\]

So if \(T^* = \inf \{ b : W_b = 0 \} = \lim T_n\),
\[
\int_0^{T^*} \frac{1}{W_n} \, dn \geq \mathbb{E} \left[ \frac{1}{W_n} \, dn \right] \to \infty
\]

\(\alpha > 1\) so \(Y_\alpha < T^*\) for all \(\alpha\).

---

**Transition Probabilities**

Define the semigroup of operators on bounded functions:

\[
(T_k f)(x) = \mathbb{E}[f(X_k) | X_0 = x]
\]

\[
(T_0 f)(x) = f(x) \quad \text{so} \quad T_0 f = f.
\]

\[
T_{k+s} f(x) = \mathbb{E}[f(X_{k+s}) | X_k = x]
\]

\[
= \mathbb{E}[\mathbb{E}[f(X_{k+s}) | X_s] | X_k = x]
\]

\[
= \mathbb{E}[T_k(f)(X_s) | X_k = x]
\]

\[
= T_s (T_k(f))(x).
\]
The infinitesimal generator is defined as

\[ Af = \lim_{n \to 0} \frac{T_n f - f}{n} \]

By the semigroup property

\[ \frac{d}{dt} T_t f(x) = A T_t f(x). \]

\[ f(x_t) - f(x_0) = \int_0^t m(x_s) \, ds + \int_0^t \sigma(x_s) \, dB_s + \int_0^t \frac{1}{2} \sigma^2(x_s) \, ds \]

\[ \frac{1}{n} \mathbb{E}[f(x_t) - f(x_0)] \rightarrow m(x_0) \frac{d}{dx} f(x_0) + \frac{1}{2} \sigma^2(x_0) \frac{d^2}{dx^2} f(x_0). \]

\[ A = m(x_0) \frac{d}{dx} + \frac{1}{2} \sigma^2(x_0) \frac{d^2}{dx^2}. \]

If \( u(x,t) = T_t f(x) \), then \( \frac{du}{dt} = Au. \)

Solve the PoE,

(i) \( u_0 = Au \) \( u \) is \( C^{1,2} \) on \( (0, \infty) \times \mathbb{R} \)

(ii) \( u(0,x) = f(x) \), \( u \) is continuous on \( [0, \infty) \times \mathbb{R} \).

If \( u(t,x) \) solves (i) then \( Y_s = u(t-s,x) \)

\[ Y_s - Y_0 = \int_0^s u(t-r,x_r) \, dr + \int_0^s u_t(t-r,x_r) \, m(x_r) \, dr \]
\[ \begin{align*}
\frac{1}{2} \int_0^5 u_{xx}(t-r, X_r) \sigma^2(X_r) \, dr \\
+ \int_0^5 u_x(t-r, X_r) \sigma(X_r) \, dr \\
= \int_0^5 u_x(t-r, X_r) \sigma(X_r) \, dr \\
\text{ martingale}
\end{align*} \]

Hence \( \mathbb{E}[Y_6 | X_{3+}] = \mathbb{E}[f(X_6) | X_0 = x_0] \)
\[ = \mathbb{E}[Y_0 | X_0 = x_0] = u(t, x_0). \]

**Theorem:** If \( \mu, \sigma \) are bounded, Hölder continuous, and \( \sigma^2 > 0 \) then \( \exists P_\mu(x,y) \) s.t.

\[ U(t, x) = \int_{\mathbb{R}} P_\mu(x, y) \, f(y) \, dy \]

and \( \frac{dp}{dt} = A \mu \) applied in \( x \) co-ordinate.

This is the fundamental solution to the PDE and gives the transition probability

\[ P(X_6 \in B | X_0 = x_0) = T_6 1_B(x_0) \]
\[ = \int_B P_\mu(x_0, y) \, dy \]

**Example:** Ornstein-Uhlenbeck

If \( dX_t = -\Theta X_t \, dt + \sigma \, dB_t \),

\[ P_\mu(x, y) = \frac{1}{\sqrt{2\pi(\sigma^2 \Theta)}} (1-e^{-2\Theta t}) \exp \left( -\frac{(y-e^{-\Theta t}x_0)^2}{\sigma^2 (1-e^{-2\Theta t})} \right) \]

\[ \sim N(e^{-\Theta t} x, \sigma^2 (1-e^{-2\Theta t})) \]
Converges to \( N(0, \frac{\sigma^2}{2b}) \).

What is the stationary distribution in general?

Find \( \pi \) such that

\[
\int \pi(x) \, p(x, y) \, dx = \pi(y).
\]

If \( \pi \) is stationary,

\[
\mathbb{E}_{\pi} f(x_0) = \mathbb{E} f(x_t)
\]

So

\[
\frac{d}{dt} \mathbb{E}_{\pi} f(x_t) = 0
\]

\[
= \int \pi(x) f(x) \, dx
\]

\[
= \int \pi(x) \left[ M(x) + \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} f(x) \right] \, dx
\]

\[
= \int \left[ \frac{d}{dx} \left[ \pi(x) M(x) \right] + \frac{1}{2} \sigma^2(x) \pi(x) \right] f(x) \, dx
\]

So

\[
\frac{d}{dx} \left[ - \pi(x) M(x) + \frac{1}{2} \sigma^2(x) \pi(x) \right] = 0
\]

Solution
\[- \frac{d \ln(\pi(x))}{dx} = \frac{\pi'(x)}{\pi(x)} = \frac{2(M(x) - \sigma'(x) \sigma(x))}{\sigma^2(x)}.
\]

So
\[
\pi(x) = \frac{1}{2} \exp \left( \int_0^x \frac{2(M(y) - \sigma'(y) \sigma(y))}{\sigma^2(y)} dy \right).
\]

For the Ornstein-Uhlenbeck process, \( \mu = -\theta x \)
\[
\pi(x) = \frac{1}{2} \exp \left( -\frac{\theta x^2}{2\sigma^2} \right) = \frac{1}{2} \exp \left( -\frac{\theta x^2}{2(\sigma \sqrt{16})} \right)
\]

For \( dX_t = (X^{3}_t - \theta X^{4}_t) dt + X^{5/4} dB_t \),
\[
\pi(x) = \exp \left( \int_1^x \frac{2(y^3 - \theta y^4 - \frac{5}{2} y^{3/2} y^{5/4})}{y^5} dy \right)
\]

\[
= \exp \left( \int_1^x 2y^2 - (2\theta + 5)y + dy \right)
\]

\[
= \exp \left( -2x^{-1} - (2\theta + 5) \log x \right)
\]

\[
= x^{-(2\theta + 5)} \exp (-2 x^{-1})
\]

Inverse Gamma \((2\theta + 4, 2)\).

Connection to PDE's
Dirichlet Problem on $\mathbb{R}^d$

$\Delta u = 0$ in $D$, $u$ is $C^2$.

$u = g$ on $\partial D$, and continuous.

Assume $D$ is open, boundal, $\partial D$ is smooth, nice.

Let $\tau$ be the first exit time of $D$.

$u(x) = \mathbb{E}[f(B_\tau) | X_0 = x]$.

Then $u(B_{\tau \wedge \tau})$ is a martingale.

By the mean value property

If $B$ ball around $x_0$,

$u(x_0) = \mathbb{E}[f(B_\tau) | \tau_B]$

$= \mathbb{E}[u(B_{\tau_B})]$

$= \int_{\partial B} u(y) dy$.

So by the mean value property

$u$ is $C^\infty$, $\Delta u = 0.$
Without PDE's

- Smoothness:

Let \( u : [0, \infty) \to \mathbb{C} \)

supported on \([s, \bar{s}]\), positive on \((\frac{s}{2}, \bar{s})\).

Then

\[
\int_{\mathbb{R}^2} u(y) \frac{1}{1|x-y|} \, dy = Cu,
\]

so \( u \) is \( C^\infty \).

By Itô's formula,

\[
U(B_{t+\Delta t}) - U(B_t) = \sum_{i \in \mathbb{Z}} \int_0^t u_{x_i}(B_s) \, dB_s^i + \frac{1}{2} \int_0^t \Delta U(B_s) \, ds,
\]

and since \( U(B_s) \) is a martingale.