# Fundamentals

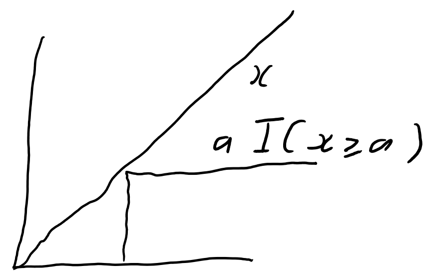
Monday, September 9, 2019 7:59 PM

## Inequalities of Random Variables

### Markov's Inequality

If $X \geq 0$ then

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$




Proof:

$$\mathbb{E}[X] \geq \mathbb{E}[a\,I(X \geq a)] = a\,\mathbb{P}[X \geq a]$$

### Union bound

If $A_1, \dots, A_n$ are events then

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$$

Proof:

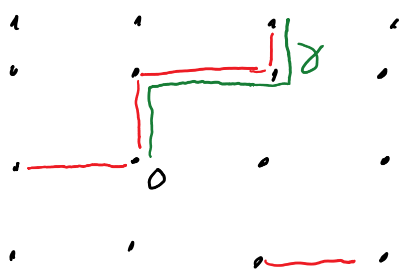
If $X = \sum_{i=1}^{n} I(A_i)$, use Markov

$LHS = \mathbb{P}[\quad \geq 1]$, $RHS = \mathbb{E}X$.

Often time we want to show some unlikely event has a small probability. Calculating the probability itself may be hard but breaking it into smaller events may simplify it.

Example: Percolation

On the grid $\mathbb{Z}^2$ place an edge between each pair of verticies at distance 1 independently with probability $p$.




Let $A$ be the event the origin is in an infinite component.

Lemma: If $p < 1/3$ then $\mathbb{P}[A] = 0$.

If $A$ holds, for each $r$ there

must be a path $\gamma$ of length $r$ in the component with distinct edges all present starting at $0$.

$A_\gamma$ means $\gamma$ is present.

$$\mathbb{P}[A_\gamma] = p^r$$

$$\mathbb{P}[\mathcal{A}] \le \mathbb{P}[\bigcup_\gamma A_\gamma] \le \sum_\gamma p^\gamma.$$

How many $\gamma$?

4 choices at step 1, at most 3 choices each subsequent step.

$\#\gamma$ of length $r \le 4 \cdot 3^{r-1}$

so $\forall r$,

$$\mathbb{P}[\mathcal{A}] \le 4 \cdot 3^{r-1} \cdot p^r = \left(\tfrac{4}{3}\right)(3p)^r \to 0.$$

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## Chebyshev's Inequality

- $\mathbb{P}[X \ge a] \le \dfrac{\mathbb{E}[(X-a)^2]}{a^2}$

- $\mathbb{P}[|X - \mathbb{E}X| \ge a] \le \dfrac{Var(X)}{a^2}$

$a^2$

Proof

$$\mathbb{P}[|X - \mathbb{E}X| \geq a]$$

$$= \mathbb{P}[|X - \mathbb{E}X|^2 \geq a^2]$$

$$\leq \frac{\mathbb{E}[(X - \mathbb{E}X)^2]}{a^2} = \frac{Var(X)}{a^2}$$

Example: Weak Law of Large Numbers

If $X_i$ IID, $\mathbb{E}X_i = \mu$, $Var(X_i) = \sigma^2$ then

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right]$$

$$\leq \frac{Var\left(\frac{1}{n}\sum X_i - \mu\right)}{\varepsilon^2} = \frac{Var(\sum X_i)}{n^2\varepsilon^2}$$

$$= \frac{n\sigma^2}{\varepsilon^2 n^2} = \frac{\sigma^2}{\varepsilon^2}\cdot\frac{1}{n}.$$

Note quite enough to immediately prove the SLLN $\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$ a.s. since $\sum_{i=1}^{n} \frac{\sigma^2}{\varepsilon^2}\cdot\frac{1}{n} = \infty$.

$$Var\left(\sum X_i\right) = \sum Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j).$$

Enough to assume $X_i$ uncorrelated.

<u>Example</u>: Sampling without replacement.

A box with $2n$ balls, $n$ green $n$ red.

Draw $n$ balls without replacement.

Let $Y$ be number of green balls.

$\mathbb{E}\, Y = n/2.$

$$\mathbb{P}\left[ \left| \tfrac{1}{n} Y - \tfrac{1}{2} \right| > \varepsilon \right] \le \frac{Var(Y)}{\varepsilon^2 n^2}$$

Write $Y = \sum_{i=1}^{n} X_i$ $X_i$ indicator of green on draw $i$.

$$Var(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

$$Cov(X_i, X_j) = Cov(X_1, X_2)$$

$$= \mathbb{E}\, X_1 X_2 - \mathbb{E}[X_1]\,\mathbb{E}[X_2]$$

$$= \frac{n}{2n} \cdot \frac{n-1}{2n-1} - \frac{1}{4}$$

$$= \frac{n \cdot (n - \frac{1}{2})}{2n \cdot (2n-1)} - \frac{1}{4} + \frac{n}{2n} \frac{(-\frac{1}{2})}{2n-1}$$

$$= -\frac{1}{4(2n-1)}$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \underset{n}{n\, Var(X_1)} + \underset{(n-1)}{n(n-1)\, Cov(X_1, X_2)}$$

$$\operatorname{var}\left(\sum_{i=1} X_i\right) = n \operatorname{var}(X_1) \text{ [illegible] } \operatorname{cov}(X_1, X_2)$$

$$= \frac{n}{4} - \frac{n \quad (n-1)}{4 \ (2n-1)}$$

$$= \frac{n}{8} + \frac{n}{8\ (2n-1)} = \frac{n}{8} + o(n).$$

$$\mathbb{P}\left[\ \left|\tfrac{1}{n} Y - \tfrac{1}{2}\right| > \varepsilon\ \right] \le \frac{1 + o(1)}{8\varepsilon^2 n}.$$

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Chernoff Bounds / Large deviation theory

If $\mathbb{E}\, e^{\alpha X} < \infty$ then

$$\mathbb{P}[\ X > t\ ] = \mathbb{P}[\ e^{\alpha X} > e^{\alpha t}\ ]$$

$$\le \frac{\mathbb{E}[e^{\alpha X}]}{e^{\alpha t}}. \qquad \text{exponentially small in } t.$$

Let $\varphi(\theta) = \mathbb{E}\, e^{\theta X}$ be the moment generating function.

and $\kappa(\theta) = \log \varphi(\theta)$

If $S_n = \sum X_i \qquad X_i$ IID

$$\mathbb{P}[S_n > tn] \le \frac{\mathbb{E}[e^{\theta \sum X_i}]}{e^{\theta t n}} = \frac{\prod \mathbb{E}[e^{\theta X_i}]}{\exp(\theta t n)}$$

$$= \exp(n(\kappa(\theta) - \theta t))$$

<u>Theorem (Cramer)</u>

If $X_1, \dots$ IID $\mathbb{E}X_i = \mu$, and $\varphi(\theta) < \infty$ for some $\theta > 0$, $S_n = \sum_{i=1}^{n} X_i$ then for $t > \mu$,

$$\lim_n \frac{1}{n} \log \mathbb{P}[S_n > tn] = \inf_\theta K(\theta) - \theta t < 0.$$

<u>Proof:</u> Upper bound is by optimizing over $\theta$.

Lower bound.

We will assume that $\mathbb{E}e^{\theta X} < \infty$ for all $\theta$ and $\mathbb{P}[X > t] > 0$ for all $t$.

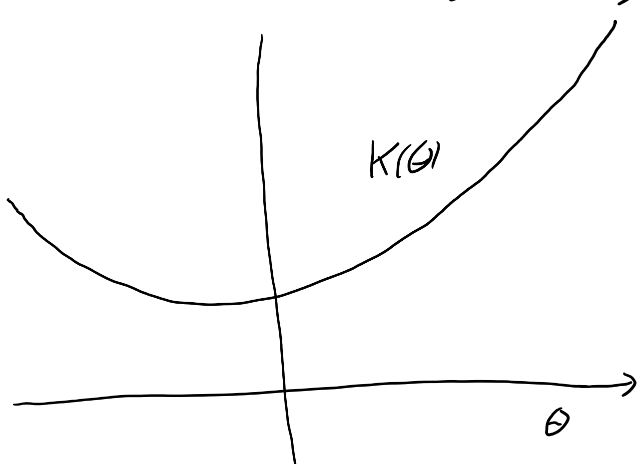
1) Limit exists:

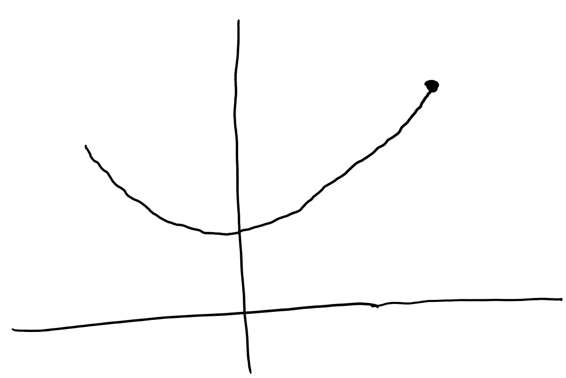
If $a_n = \log \mathbb{P}[S_n > tn]$ then

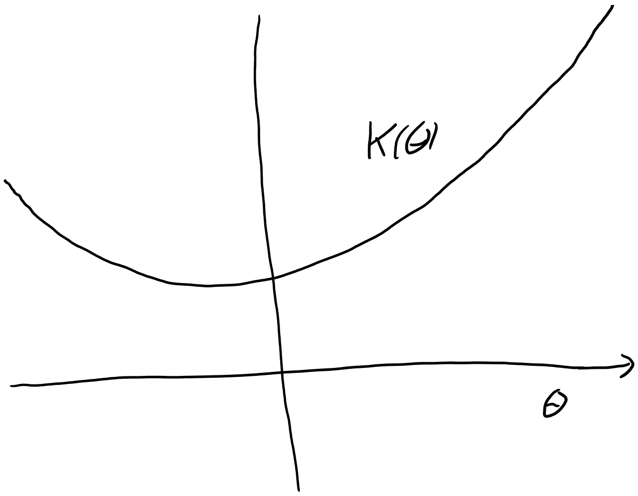
$$\begin{aligned} a_{n+m} &= \log \mathbb{P}[S_{n+m} > t(n+m)] \\ &\geq \log \mathbb{P}[S_n > tn,\ S_{n+m} - S_n > tm] \\ &= \log\left(\mathbb{P}[S_n > tn]\,\mathbb{P}[S_m > tm]\right) \\ &= a_n + a_m \end{aligned}$$

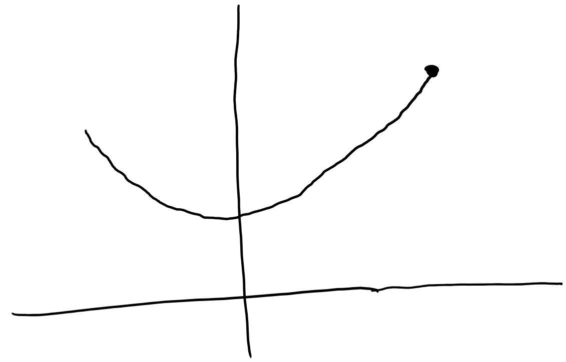
So $a_n$ is superadditive and $\frac{a_n}{n} \to c$.

Two cases.





$$\varphi(\theta) = \mathbb{E}\exp(\theta X) \quad \text{is} \quad \text{smooth}$$

$$\varphi'(0) = \frac{d}{d\theta}\mathbb{E}\exp(\theta X)\Big|_{\theta=0} = \mathbb{E}\,X\exp(\theta X)\Big|_{\theta=0}$$
$$= \mathbb{E}X = \mu.$$

$$K'(0) = \frac{\varphi'(0)}{\varphi(0)} = \mu$$

So

$$\frac{d}{d\theta} K(\theta) - \theta t \Big|_{\theta=0} = \mu - t < 0.$$

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## Change of measure:

Suppose $X_i$ was integer valued, $f(x) = \mathbb{P}[X_i = x]$

Want to calculate

$$\mathbb{P}\Big[\sum_{i=1}^{n} X_i = \mu n\Big] \approx \frac{1}{\sqrt{2\pi n}\,\sigma}$$

Local Central Limit Theorem says

$$\mathbb{P}\Big[\sum_{i=1}^{n} X_i = x\Big] = \frac{1}{\sqrt{2\pi n}\,\sigma}\exp\left(-|x-\mu n|^2/2\sigma^2 n\right)$$
$$+ o\left(\frac{1}{\sqrt{n}}\right) \quad \text{for } x \in \mathbb{Z}.$$

Only useful when $|x - \mu n| = O(\sqrt{n})$.
What if we could modify $f(x)$ so that $\mathbb{E} X_i \approx t$?

Set $f_\theta(x) = f(x) \cdot \frac{e^{\theta x}}{\varphi(x)}$.

$$\sum_x f_\theta(x) = \frac{1}{\varphi(\theta)} \sum_x f(x) \exp(\theta x) = \frac{\mathbb{E} e^{\theta x}}{\varphi(\theta)} = 1.$$

So $f_\theta(x)$ is a probability distribution.

Let $\mathbb{E}_\theta[\cdot]$, $\mathbb{P}_\theta[\cdot]$ denote expectation and probability when $X_i \sim f_\theta$.

$$\mathbb{P}_\theta\left[\sum_{i=1}^n X_i = x\right]$$

$$= \sum_{\substack{x_1, \dots, x_n \\ x_1 + \dots + x_n = x}} \prod_i \mathbb{P}_\theta[X_i = x_i]$$

$$= \sum_{x_i, \sum x_i = x} \prod_i \frac{e^{\theta x_i} f(x_i)}{\varphi(\theta)} = \frac{e^{\theta x}}{\varphi(\theta)^n} \sum_{x_i : \sum x_i = n} \prod f(x_i)$$

$$= \frac{e^{\theta x}}{\varphi(\theta)^n} \mathbb{P}\left[\sum_{i=1}^n X_i = x\right].$$

- Now choose $\theta^*$ such that $\mathbb{E}_{\theta^*}[X_i] = t$.

Note that $\mathbb{E}_\theta[X_i] = \sum_x x f(x) \frac{e^{\theta x}}{\varphi(\theta)}$

$= \frac{1}{\varphi(\theta)} \mathbb{E}[X e^{\theta x}]$

$$= \frac{1}{\varphi(\theta)} \mathbb{E}[X e^{\theta X}]$$

$$= \frac{\varphi'(\theta)}{\varphi(\theta)} = K'(\theta).$$

$$K''(\theta) = \frac{\varphi''(\theta)}{\varphi(\theta)} - \left(\frac{\varphi'(\theta)}{\varphi(\theta)}\right)^2$$

$$= \sum_x x^2 \frac{e^{\theta x}}{\varphi(\theta)} f(x) - \left( \sum_x x \frac{e^{\theta x}}{\varphi(\theta)} f(x) \right)$$

$$= \mathbb{E}_\theta[X^2] - (\mathbb{E}_\theta[X])^2$$

$$= Var_\theta[X] > 0.$$

So $\mathbb{E}_\theta[X]$ is increasing in $\theta$.

With $\mathbb{E}_{\theta_t}[X_i] = t$,

$$\mathbb{P}\left[\sum_{i=1}^n X_i = tn\right] = \frac{\varphi(\theta^*)^n}{e^{\theta^* tn}} \mathbb{P}_{\theta^*}\left[\sum_{i=1}^n X_i = tn\right]$$

$$= \frac{\varphi(\theta^*)^n}{e^{\theta^* tn}} \left( \frac{1 + o(1)}{\sqrt{2\pi n}\, \sigma_{\theta^*}} \right)$$

$$\Rightarrow \frac{1}{n} \log \mathbb{P}\left[\sum_{i=1}^n X_i = tn\right]$$

$$= K(\theta^*) - \theta^* t + \left( \frac{1}{n} \left( \log \frac{1 + o(1)}{\sqrt{2\pi}\, \sigma_{\theta^*}} - \frac{1}{2} \log n \right) \right)$$

$\to 0.$

## General Change of measure

$F(x) = \mathbb{P}[X \le x]$ the CDF.

Let $F_\theta(x) = \frac{1}{\varphi(\theta)} \int_{-\infty}^{x} e^{\theta y} \, dF(y)$

$= \mathbb{E}[e^{\theta X} I(X \le x)] / \varphi(\theta)$

$F_\theta$ is a probability measure

$$\frac{dF_\theta(x)}{dF(x)} = \frac{e^{\theta x}}{\varphi(\theta)}.$$ Radon - Nikodym Derivative

It means

$$\mathbb{P}_\theta[X \in A] = \mathbb{E}\left[\frac{e^{\theta X}}{\varphi(\theta)} I(X \in A)\right].$$

For more $X_i$,

$$\mathbb{P}_\theta[(X_1, \dots, X_n) \in A] = \mathbb{E}\left[\frac{e^{\theta \Sigma X_i}}{\varphi(\theta)^n} I((X_1, \dots, X_n) \in A)\right]$$

Check that under $\mathbb{P}_\theta$, $X_i \sim F_\theta$ IID

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Choose $\theta^*$ such that $\mathbb{E}_{\theta^*}[X_i] = t.$

$\sigma = Var_{\theta^*}[X_i]$

$$\mathbb{P}_{\theta^*}\left[\sum_{i=1}^{n} X_i \in (tn, tn + \sigma^* \sqrt{n})\right]$$

$$= \mathbb{P}\left[\frac{\sum_{i=1}^{n} X_i - tn}{\dots} \in (0, 1)\right]$$

$$= \mathbb{P}\left[ \frac{\sum_{i=1}^n X_i - tn}{\sigma^* \sqrt{n}} \in (0,1) \right]$$

$$\to \mathbb{P}[ N(0,1) \in (0,1) ] \approx 0.34.$$

$$\mathbb{P}_{\theta_*} \left[ \sum_{i=1}^n X_i \in (tn, tn + \sigma^* \sqrt{n}) \right]$$

$$= \mathbb{E}\left[ \frac{e^{\theta_* \sum X_i}}{\varphi(\theta_*)^n} I\left( \sum_{i=1}^n X_i \in (tn, tn + \sigma^* \sqrt{n}) \right) \right]$$

$$\leq \frac{e^{\theta_*(tn + \sqrt{n}\sigma)}}{\varphi(\theta_*)^n} \mathbb{P}\left[ \sum X_i > tn \right]$$

$$\lim \frac{1}{n} \log \mathbb{P}\left[ \sum X_i > tn \right]$$

$$\geq \lim \frac{1}{n} \log \varphi(\theta_*)^n \exp\left( \theta^* (tn + \sqrt{n}\sigma) \right) \times 0.34$$

$$= K(\theta^*) - \theta^* t.$$

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## Strong Law of Large Numbers

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If $X_i$ IID, $\mathbb{E}X_i = \mu$ and $\mathbb{E}e^{\theta|X_i|} < \infty$

then $\frac{1}{n} \sum_{i=1}^n X_i \longrightarrow \mu$ a.s.

Proof: Enough to show that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left[ \left| \sum_{i=1}^{n} X_i - \mu n \right| > \varepsilon n \right] < \infty$$

$$\leq \sum_{n=1}^{\infty} 2 \exp(-cn) < \infty$$

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The empirical distribution

$$F_n(x) = \frac{1}{n} \# \{ 1 \leq i \leq n : X_i \leq x \}$$

- $\mathbb{E}\, F_n(x) = F(x)$

Since $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$

$$\mathbb{P}\left[ \left| F_n(x) - F(x) \right| > \varepsilon \right] \leq \exp(-cn)$$

HW: Show that

$$\mathbb{P}\left[ \max_x \left| F_n(x) - F(x) \right| > \varepsilon \right] \leq \exp(-cn).$$

So what does this say about our question, what is the most likely way to have

$$\sum X_i > n(\mu + \delta) ?$$

The answer is, <u>it depends.</u>

<u>Light Tail Case</u> Example Gaussian, Poisson

If for all $\theta > 0$, $\mathbb{E}\, e^{\theta X_i} < \infty$ then

Set $\theta^*$ such that $\mathbb{E}_{\theta^*}[X_i] = \mu + \delta$,

then $F_n \approx F_{\theta,n}$ conditional on $\sum_{i=1}^{n} X_i > n(\mu+\delta)$.

Also $\exists C$ such that

$$\mathbb{P}\left[\max_{1\le i\le n} X_i > C\log n \;\middle|\; \sum_{i=1}^{n} X_i > n(\mu+\delta)\right] \to 0.$$

Heavy Tailed Case

If $\mathbb{P}[X_i > x] \sim x^{-\alpha}$ then

$$\mathbb{P}\Big[\underbrace{\max_{1\le i\le n} |F_n(x) - F(x)| > \varepsilon}_{B} \;\Big|\; \underbrace{\sum X_i > n(\mu+\delta)}_{A}\Big] \to 0.$$

$$\mathbb{P}[X_1 > 2\delta n] \approx (2\delta n)^{-\alpha}$$

$$\mathbb{P}\left[\sum_{i=2}^{n} X_i > n\mu - \delta n\right] \to 1 \qquad \text{C.L.T.}$$

so $\mathbb{P}\left[\sum_{i=1}^{n} X_i > n\mu + \delta n\right] \geq (2\delta n)^{-\alpha} + o(n^{-\alpha})$

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[A]} \le \frac{\mathbb{P}[B]}{\mathbb{P}[A]}$$

$$\le \frac{\exp(-cn)}{(2\delta n)^{-\alpha}} \to 0.$$

Weakening the assumption of Independence

A filtration $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ is an increasing family of "information" more formally an increasing family of $\sigma$-algebras we can condition on.

Example $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$

We say that $X_n$ is a martingale with respect to $\mathcal{F}_n$ if

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n$$

If $\mathcal{F}_n$ is not specified assume $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$

$$\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] = X_n.$$

<u>Examples</u>: • If $X_n = \sum_{i=1}^{n} Y_i$, $Y_i$ IID, $\mathbb{E} Y_i = 0$.

• If $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for some $X, \mathcal{F}_n$.

Azuma - Hoeffding Inequality

If $M_n$ is a martingale and

$|M_i - M_{i-1}| \leq k_i$ then

$$\mathbb{P}[M_n - M_0 \geq t] \leq \exp\left(-\frac{t^2}{\sum_{i=1}^n k_i^2}\right).$$

Proof:

<u>Jensen's Inequality</u>: If $\varphi$ is convex

$$\mathbb{E}\,\varphi(X) \geq \varphi(\mathbb{E}[X]), \quad \mathbb{E}[\varphi(X)|\mathcal{F}] \geq \varphi(\mathbb{E}[X|\mathcal{F}]).$$

In reverse, if $a \leq X \leq b$ then

$$\mathbb{E}\,\varphi(X) \leq \varphi(a)\frac{b-\mathbb{E}X}{b-a} + \varphi(b)\frac{\mathbb{E}X - a}{b-a}$$

Construct $Y$ s.t.

$$\mathbb{P}[Y = a \mid X = x] = \frac{b-x}{b-a},$$

$$\mathbb{P}[Y = b \mid X = x] = \frac{x-a}{b-a}$$

So $\mathbb{E}[Y \mid X = x] = a\frac{b-x}{b-a} + b\frac{x-a}{b-a} = x.$

$$\begin{aligned}
\varphi(X) &= \varphi(\mathbb{E}[Y|X]) \\
&\leq \mathbb{E}[\varphi(Y) \mid X] \\
&= \varphi(a)\frac{b-X}{b-a} + \varphi(b)\frac{X-a}{b-a}.
\end{aligned}$$

Back to Azuma:

$$\begin{aligned}
\mathbb{E}\, e^{\theta M_i - M_0} &= \mathbb{E}\left[\mathbb{E}\left(e^{\theta(M_i - M_0)} \mid \mathcal{F}_{i-1}\right)\right] \\
&= \mathbb{E}\, e^{\theta(M_{i-1} - M_0)}\,\mathbb{E}\left[e^{\theta(M_i - M_{i-1})} \mid \mathcal{F}_{i-1}\right]
\end{aligned}$$

Now $-k_i \leq M_i - M_{i-1} \leq k_i$, $\mathbb{E}[M_i - M_{i-1} \mid \mathcal{F}_{i-1}] = 0$, so

$$\mathbb{E}\, e^{\theta(M_i - M_{i-1})} \le \tfrac{1}{2} e^{\theta k_i} + \tfrac{1}{2} e^{-\theta k_i} = \cosh(\theta k_i).$$

$$\cosh(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \le \sum_{j=0}^{\infty} \frac{(x^2)^j}{2^j j!} \le \exp(x^2/2).$$

So $\mathbb{E}\, e^{\theta(M_i - M_0)} \le e^{\theta^2 k_i^2/2}\, \mathbb{E}\, e^{\theta(M_{i-1} - M_0)}$

$$\mathbb{E}\, e^{\theta(M_i - M_0)} \le \exp\left(\tfrac{1}{2} \sum k_i^2\right)$$

By Markov,

$$\mathbb{P}[M_n - M_0 > t] = \mathbb{P}[e^{\theta(M_n - M_0)} > e^{\theta t}]$$

$$\le \exp\left(\tfrac{1}{2}\theta^2 \sum_{i=1}^{n} k_i^2 - \theta t\right)$$

$$(\text{set } \theta = t/\textstyle\sum k_i^2)$$

$$\le \exp\left(-\frac{t^2}{2\sum_{i=1}^{n} k_i^2}\right).$$

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<u>Theorem</u>: If $g: \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|g(x_1, \dots, x_i, \dots, x_n) - g(x_1, \dots, x_i', \dots, x_n)| \le 1$$

for all $i$, and $W_i$ are independent and

$X = g(W_1, \dots, W_n)$ then

$$\mathbb{P}[|X - \mathbb{E}X| \ge t\sqrt{n}] \le 2\exp(-t^2/2)$$

<u>Proof</u>

Let $X_i = \mathbb{E}[X \mid W_1, \dots, W_i]$ martingale

Suppose $W_i^*$ independent copy of $W_i$

$$X_i = \mathbb{E}[g(W_1,\dots,W_i,W_{i+1},\dots,W_n) \mid W_1,\dots,W_i]$$

$$X_{i-1} = \mathbb{E}[g(W_1,\dots,W_i,W_{i+1},\dots,W_n) \mid W_1,\dots,W_{i-1}]$$

$$= \mathbb{E}[g(W_1,\dots,W_i^*,W_{i+1},\dots,W_n) \mid W_1,\dots,W_{i-1},W_i]$$

$$|g(\dots,W_i,\dots) - g(\dots,W_i^*,\dots)| \le 1$$

$\Rightarrow |X_i - X_{i-1}| \le 1.$

$\Rightarrow \mathbb{P}[X - \mathbb{E}X \ge t\sqrt{n}] \le e^{-t^2/2}$ (A-H).

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## Knapsack Problem

You can choose from $n$ objects with weights $W_i$, value $V_i$. Assume $(W_i, V_i)$ IID, $V_i \le 1$. You can take at most weight $M$.

$$\Phi = \max_{S \subseteq \{1,\dots,n\}} \{\sum_{i \in S} V_i : \sum W_i \le M\}.$$

Then

$$\mathbb{P}[|\Phi - \mathbb{E}\Phi| \ge t] \le \exp(-\tfrac{t^2}{2n}).$$

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Concentration for the Chromatic Number for Random Graphs

Let $G$ be an Erdos-Renyi random graph $G(n,p)$.

- $n$ verticies
- Independently an edge between each vertex with probability $p$.

Set $X$ = chromatic number of $G$.

$X_i = \mathbb{E}[X \mid \mathcal{F}_i]$ - Doob martingale

$\mathcal{F}_i$ - edges connected to vertices $1, \dots, i$.

$X$ monotone decreasing in edge set.

Let $G^{i,+}$ be graph with all edge from $i$ to $\{i+1, \dots, n\}$ present.

$G^{i,-}$ - no edges $i$ to $\{i+1, \dots, n\}$

Then

$\mathcal{F}_{i-1}$ measurable

$$\mathbb{E}[X^{i,+} \mid \mathcal{F}_i] \leq X_{i-1} \leq \mathbb{E}[X^{i,-} \mid \mathcal{F}_i]$$

$$0 \leq X^{i,-} - X^{i,+} \leq 1$$

So $|X_i - X_{i-1}| \leq 1$.

$$\Rightarrow \mathbb{P}[X_n - X_0 > t\sqrt{n}] \leq \exp\left(\frac{-t^2 n}{2n}\right)$$

$$\Rightarrow \quad \mathbb{P}[X_n - X_0 > t\sqrt{n}] \leq \exp\left(\frac{-t^2 n}{2n}\right)$$

$$\mathbb{P}[|X - \mathbb{E}X| > t\sqrt{n}] \leq 2e^{-t^2/2}$$

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<u>Theorem</u>: If $p_n < n^{-\frac{5}{6}-\delta}$ then there exist $\varphi_n$ s.t.

$$\mathbb{P}[\varphi_n \leq X \leq \varphi_n + 3] \to 1.$$

Set $\varepsilon > 0$, and choose

$$\varphi_n = \min\{\varphi : \mathbb{P}[X(n) \leq \varphi] > \varepsilon/3\}$$

So $\mathbb{P}[X(n) < \varphi_n] \leq \varepsilon/3$.

Let $U$ be the size of the minimal set of vertices such that $G \setminus U$ is $\varphi_n$ colourable.

- $\mathbb{P}[U = 0] \geq \frac{\varepsilon}{3}$.

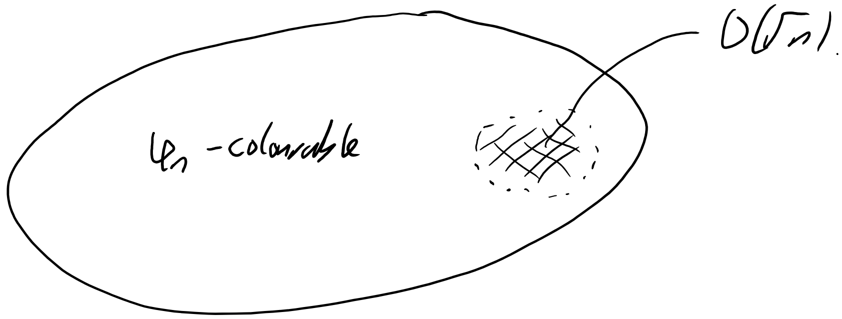
Let $U_i = \mathbb{E}[U \mid \mathcal{F}_i]$.

$|U_i - U_{i-1}| \leq 1$ so

$$\mathbb{P}[|U_i - \mathbb{E}U_i| > t\sqrt{n}] \leq 2e^{-t^2/2}.$$

$$\Rightarrow \quad \mathbb{E}U_i \leq 2\sqrt{-\log(\varepsilon/6)}\ \sqrt{n}$$

$$\mathbb{P}[U_i > c(\varepsilon)\sqrt{n}] \leq \frac{\varepsilon}{3}.$$




Colour remaining vertices with 3 colours

- Let $S$ be smallest non-3 colourable subset.

$\Rightarrow$ all vertices of $S$ degree $\geq 3$.

$\Rightarrow$ at least $\frac{3|S|}{2}$ edges

$$\mathbb{P}\left[\ \exists\, S,\ |S| \leq C\sqrt{n},\ E(S) \geq \tfrac{3s}{2}\right]$$

$$\leq \sum_{s=4}^{C\sqrt{n}} \binom{n}{s} \cdot \binom{\binom{s}{2}}{\frac{3s}{2}} \cdot p^{3s/2} \qquad \boxed{\binom{a}{b} \leq \left(\frac{e\,a}{b}\right)^{b}}$$

$$\leq \sum_{s} \left(\frac{en}{s}\right)^{s} \left(\frac{e(s-1)}{3}\right)^{3s/2} p^{3s/2}$$

$$\leq \sum_{s} \left(C\, n\, p^{3/2} s^{1/2}\right)^{s}$$

$$\leq \sum_{s} \left(C\, n^{\frac{5}{4}} p^{\frac{3}{2}}\right)^{s} \to 0.$$

since $n^{\frac{5}{4}} p^{\frac{3}{2}} \leq n^{\frac{5}{4}} \cdot n^{\frac{5}{6}\cdot\frac{3}{2} - \frac{3}{2}\delta}$