Random walk on a group
$G_\infty$ IID group elements

$x_n = G_n \ldots G_0 \cdot x_0 = G_n x_{n-1}.$

$\pi$ is uniform distribution.

Lazy RW on Hypercube $G = \{0, 1\}^n$, $\mathbb{Z}^n_2$

- Pick $i \in \{1, \ldots, n\}$
- $x_t(i) = x_{t-1}(i) + Z_i \quad P[Z_i = 1] = P[Z_i = 0] = \frac{1}{2}.$

Coordinates are random after update.

$\Delta TV \left( X_t, \pi \right) \leq 1$ if all coordinates updated by $t$

Coupon collector problem

Sample $U_1, \ldots, U_\infty$ IID with dist $U(\{1, \ldots, n\}$. Let $T$ be the first time such that all $j \in \{1, \ldots, n\}$ have appeared. Then

$$\lim_{n \to \infty} \frac{T_n}{n \log n} = 1 \quad \text{in probability}.$$

Let $S_k = \text{time until } k \text{ distinct } U_i \text{ chosen.}$
Then \( S_k - S_{k-1} \sim \text{Geom} \left( \frac{n+1-k}{n} \right) \) independent.

\[ \mathbb{E} S_k - S_{k-1} = \frac{n}{n+1-k} \]

So \( \mathbb{E} T = \mathbb{E} S_n = \sum_{k=1}^{n} \frac{n}{n+1-k} = n \sum_{k=1}^{n} \frac{1}{k} \approx n \log n. \)

\[ \text{Var} T = \sum \text{Var} (S_k - S_{k-1}) = \frac{(k-1)n}{(n+1-k)^2/n^2} \]

\[ \text{Var} \text{ Geom} (p) = \frac{1-p}{p^2} \]

Now \( \frac{(k-1)n}{(n+1-k)^2/n^2} \leq \frac{n^2}{(n+1-k)^2} \) so

\[ \text{Var} T \leq n^2 \sum_{k} \frac{1}{(n+1-k)^2} = n^2 \sum_{k} \frac{1}{k^2} = O(n^2). \]

So by Chebyshev's inequality,

\[ P \left( |T - \mathbb{E} T| \geq C n \log n \right) \leq \frac{C n^2}{\epsilon^2 n^2 \log^2 n} \rightarrow 0. \]

So \( \frac{T}{n \log n} \xrightarrow{p} 1. \)

So \( d_{TV} (X_{(1+\varepsilon)n \log n}, \pi) \rightarrow 0. \)

**Lower Bound:** Homework.
Split into two decks of size $\text{Bin}(n, \frac{1}{2})$.

- Interlace drop with probability $\frac{M_L}{M_L + M_R}$.

New permutation

If $g$ corresponds to one such split into $N_L + N_R = n - N_L$ deck and a choice of $(L, L, R, R, \ldots, L, R)$ or interleaving, the interleavings have equal probability $\frac{1}{(n)}$. sina
\[ \text{IPC} (L, L, R, R, \ldots) | N_L \]
\[ = \frac{N_L}{n} \cdot \frac{N_L - 1}{n-1} \cdot \frac{N_R}{n-2} \cdot \frac{N_R - 1}{n-3} \ldots \cdot \frac{1}{1} \]
\[ = \frac{N_L! \cdot N_R!}{n!} = \binom{n}{N_L}^{-1} \]

\[ \text{IPC} (g) = \text{IPC} \text{ Bin} (n, \frac{1}{2}) = N_L \] \[ \cdot \binom{n}{N_L}^{-1} \]
\[ = \binom{n}{N_L} \left(\frac{1}{2}\right)^{n-N_L} \cdot \binom{n}{N_L}^{-1} = 2^{-n} \]

**Inverse Map**

Assign each card L or R.

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Split into L & R decks preserving the order.

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Place L deck on top.

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For the same sequence (L, L, R, \ldots) this operation is \( g^{-1} \) easier to analyze.
Claim: Random walk and inverse walk have the same distribution.

\[ X_t = g_t, g_{t-1}, \ldots, g_1, \]
\[ Y_t = H_t, \ldots, H_1. \]

\[ \mathbb{P}(X_t = x) = \sum_{g_1, \ldots, g_t} \mathbb{P}(X_t = g_1, \ldots, g_t = x) \]
\[ = \sum_{g_1, \ldots, g_t} \mathbb{P}(g_t = g_t, \ldots, g_1 = g_1) \mathbb{P}(g_t = x) \]
\[ = \sum_{g_1, \ldots, g_t} \mathbb{P}(Y_t = g_t, \ldots, H_1 = g_1) \mathbb{P}(g_t = x) \]
\[ = \mathbb{P}(Y_t = x). \]

\[ d_{TV}(X_t, Y_t) = \frac{1}{2} \sum_x |\mathbb{P}(X_t = x) - \frac{1}{1+1}| \]
\[ = \frac{1}{2} \sum_x |\mathbb{P}(Y_t = x) - \frac{1}{1+1}| \]
\[ = \frac{1}{2} \sum_x |\mathbb{P}(Y_t = y) - \frac{1}{1+1}| \]
\[ = d_{TV}(Y_t, \pi). \]

Analysis of inverse chain:

\[ \text{Suiten R/L with I/O.} \]
Will be mixed once each card has a unique string.

\[ P[ A, B \text{ same string}] = 2^{-6} \]

Union bound over \( \binom{n}{2} \) pairs,

\[ d_{tv}(\alpha, \pi) \leq P[ \text{all strings unique}] \leq \binom{n}{2} 2^{-6} \]

\[ \Rightarrow 0 \quad \text{if} \quad \ell = (2 + \epsilon) \log_2 n. \]

Right bound \( \ell = \frac{3}{2} \log_2 n. \)

For 52 cards:

<table>
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<th>\ell</th>
<th>1</th>
<th>...</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>1.000</td>
<td>1.000</td>
<td>0.924</td>
<td>0.611</td>
<td>0.334</td>
<td>0.167</td>
<td>...</td>
</tr>
</tbody>
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