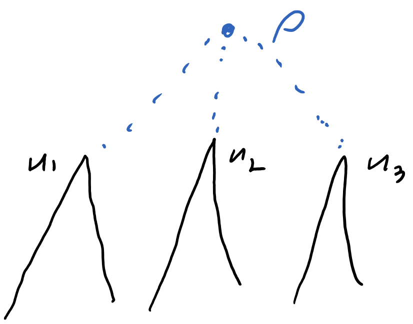
Spin systems on trees

Friday, March 23, 2018 1:26 PM

Recursive structure



- Initially we have d trees rooted at $u_1, \ldots, u_d$ with marginals $m_1, \ldots, m_d$

- We add a new layer to the tree rooted at ρ, what is the marginal of ρ?

$$Z = \left(\prod Z_i\right) \cdot \sum_x \psi(x) \sum_{\{x_i\}} \prod_{i=1}^{d} \psi(x, x_i)\, m_i(x_i)$$

$$= \left(\prod Z_i\right) \sum_x \psi(x) \prod_{i=1}^{d} \left(\sum_{x_i} \psi(x, x_i)\, m_i(x_i) \right)$$

and

$$m(x) = \frac{\psi(x) \prod_{i=1}^{d} \left(\sum_{x_i} \psi(x, x_i)\, m_i(x_i) \right)}{\sum_{x'} \psi(x') \prod_{i=1}^{d} \left(\sum_{x_i} \psi(x', x_i)\, m_i(x_i) \right)}$$

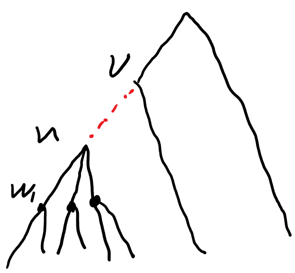
To calculate marginals recursively from the leaves up write

$$m_{u \to v}(x) = \mathbb{P}_{T \setminus (u,v)}[\sigma_u = x]$$

and

$$m_{u \to v}(x) = \frac{\psi(x) \prod_{\substack{w \sim u \\ w \neq v}} \sum_{x_w} \psi(x, x_w)\, m_{w \to u}(x_w)}{\sum_{x'} \psi(x') \prod_{\substack{w \sim u \\ w \neq v}} \sum_{x_w} \psi(x', x_w)\, m_{w \to u}(x_w)}$$

$$=: BP(\{m_{w \to u}\})(x)$$

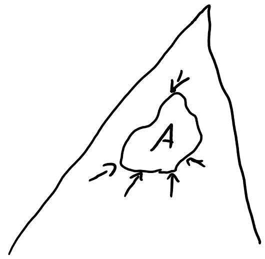


This is the __belief propagation__ function.

The marginal at v is

$$BP(\{m_{u \to v}\}_{u \sim v}).$$

A fixed point of $\{m_{u \to v}\}$ of BP equations gives rise to a Gibbs measure as it gives a consistent set of distributions on finite sets



a consistent set of distributions on finite sets

A fixed point $m = BP(m, \dots, m)$ corresponds to a translation invariant Gibbs measure on the regular tree.

<u>Example:</u> Ising model, $h=0$, d-ary tree

Let $m = m(+)$,

$$BP(m) = \frac{(m e^{\beta} + (1-m)e^{-\beta})^d}{(m e^{\beta} + (1-m)e^{-\beta})^d + (m e^{-\beta} + (1-m)e^{\beta})^d}$$

$$m e^{\beta} + (1-m)e^{-\beta} = \tfrac{1}{2}\cosh\beta + (m-\tfrac{1}{2})\sinh\beta$$

$$= \frac{1+2(m-\frac{1}{2})\tanh\beta}{\cosh\beta}$$

so

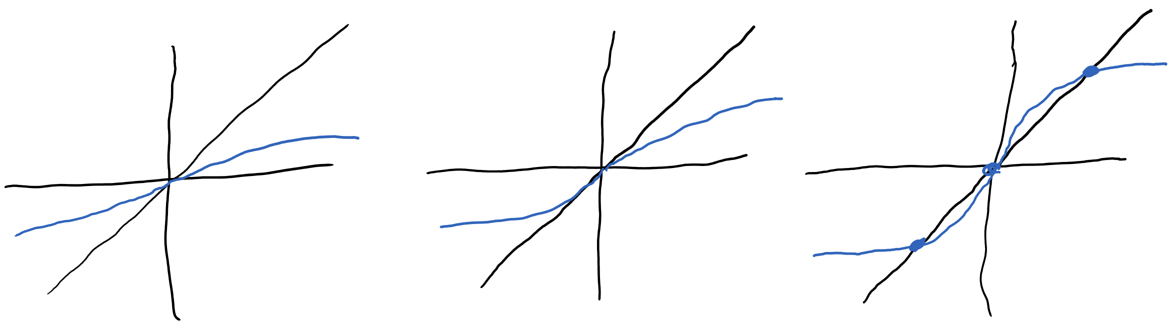
$$BP(m) - \tfrac{1}{2} = \frac{\frac{1}{2}(1+2(m-\frac{1}{2})\tanh\beta)^d - \frac{1}{2}(1-2(m-\frac{1}{2})\tanh\beta)^d}{(1+2(m-\frac{1}{2})\tanh\beta)^d + (1-2(m-\frac{1}{2})\tanh\beta)^d}$$

with $y = m - \frac{1}{2}$,

$$f(y) = \frac{\frac{1}{2}(1+2y\tanh\beta)^d - \frac{1}{2}(1-2y\tanh\beta)^d}{(1+2y\tanh\beta)^d + (1-2y\tanh\beta)^d}$$

$f(0) = 0$,

$$f'(0) = \frac{2d(\tanh\beta)\cdot 2}{4} = d\tanh\beta.$$



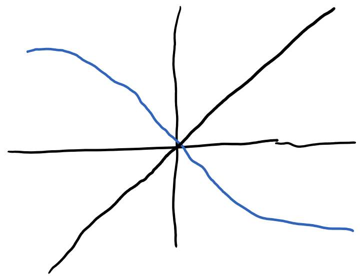
$0 < d\tanh\beta < 1$ $\qquad$ $d\tanh\beta = 1$ $\qquad$ $d\tanh\beta > 1$

one fixed point $\qquad$ 3 fixed point

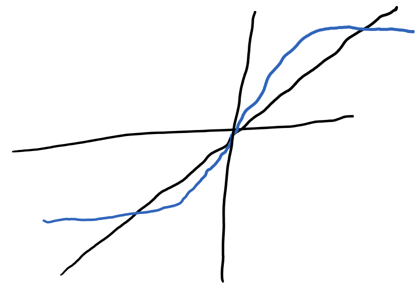
uniqueness $\qquad$ non-uniqueness

When $d > \tanh\beta$ there are 3 measures, the symmetric "free measure" plus μ_+, μ_-.

If $\beta < 0$, $\qquad$ $f(f(y))$



One fixed point



Semi-translation invariant measure.