Spin systems

Thursday, March 22, 2018 1:26 PM

__Goal:__ Find size of maximal IS of random graph.

__Definition:__ $I \subseteq V$ is an independent set (IS) of $G=(V,E)$ if $\forall u,v \in I$, $(u,v) \notin E$.

- Useful to consider a random IS $\sigma \in \{0,1\}^V$ of G.

$$\mathbb{P}[\sigma] = \frac{1}{Z} \prod_{u,v} \mathbb{I}(\sigma_u \sigma_v = 0)$$

$$Z = \# \text{IS.} = \sum_{\sigma} \prod_{u,v} \mathbb{I}(\sigma_u \sigma_v = 0)$$

__Definition:__ Hardcore Model with fugacity $\lambda > 0$,

$$\mathbb{P}[\sigma] = \frac{1}{Z_\lambda} \lambda^{\sum_u \sigma_u} \prod_{u,v} \mathbb{I}(\sigma_u \sigma_v = 0)$$

- weights independent sets by size.

__Spin System:__ $\sigma \in \mathcal{X}^V$, weights ψ_u, ψ_{uv} such that

$$\mathbb{P}[\sigma] = \frac{1}{Z_{\underline{\psi}}} \prod_{u \in V} \psi_u(\sigma_u) \prod_{(u,v) \in E} \psi_{u,v}(\sigma_u, \sigma_v)$$

__Examples:__

Examples:

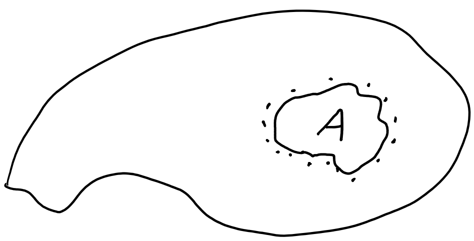
- Ising Model with inverse temperature β and external field h. $\sigma \in \{-1,+1\}^V$,

$$\mathbb{P}[\sigma] = \frac{1}{Z} \exp\Big(\beta \sum_{u \sim v} \sigma_u \sigma_v + h \sum_u \sigma_u\Big)$$

- Random k-Colouring

$$\mathbb{P}[\sigma] = \frac{1}{Z} \prod_{u \sim v} \mathbb{I}(\sigma_u \neq \sigma_v).$$

Markov Random Field Property



$$\mathbb{P}[\sigma_A = x_A \mid \sigma_{A^c} = x_{A^c}]$$
$$= \mathbb{P}[\sigma_A = x_A \mid \sigma_{\partial A} = x_{\partial A}]$$

Proof:
$$\frac{\mathbb{P}[\sigma_A = x_A \mid \sigma_{A^c} = x_{A^c}]}{\mathbb{P}[\sigma_A = x'_A \mid \sigma_{A^c} = x_{A^c}]}$$

$$= \frac{\frac{1}{Z} \prod_u \psi(x_u) \prod_{u,v} \psi_{u,v}(x_u, x_v)}{\frac{1}{Z} \prod_u \psi(x'_u) \prod \psi_{u,v}(x'_u, x'_v)}$$

$$= \prod \psi(x_u) \prod \quad \psi \quad (\quad)$$

$$= \frac{\prod_{u\in A} \psi(x_u) \prod_{(u,v)\in E(A\cup\partial A)} \psi_{u,v}(x_u, x_v)}{\prod_{u\in A} \psi(x_u') \prod_{(u,v)\in E(A\cup\partial A)} \psi(x_u', x_v')}$$

So the states outside $A \cup \partial A$ do not affect the calculation. The distribution is the spin system on $A \cup \partial A$ with the spins of ∂A fixed.

- Definition: We call a spin system permissive if given any boundary condition on ∂A, there is a configuration of A with weight > 0.

Infinite Graphs

How to define it for an infinite graph?

A measure μ on $\mathcal{X}^A$ is a Gibbs measure for a spin system with weights ψ if

$$\mu(\sigma_A = x_A \mid \sigma_{A^c} = x_{A^c}) = \mu(\sigma_A = x_A \mid \sigma_{\partial A} = x_{\partial A})$$

The DLR - Dobrushin - Lanford - Ruelle.

<u>Existence</u> (for permissive systems)

Let $D_i \subset V$ be an increasing subsequence of sets $D_i \uparrow V$. Choose τ_i a B.C. on ∂D_i.

$$\lim \mu (x_A \mid x_{\partial D_i} = \tau_i)$$

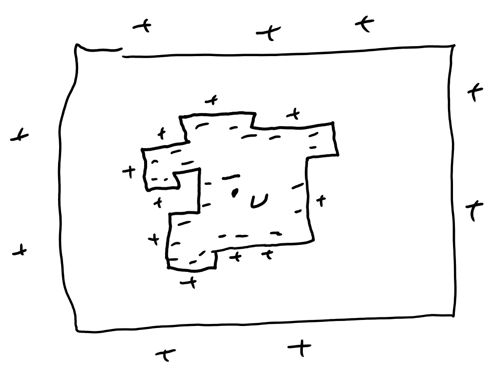
may not exist but subsequential limits exist.

- To get full set of measures take random τ_i,

<u>Non-Uniqueness</u>

Ising Model at low temperature, β large, $h=0$.

<u>Peirls argument</u>



Plus B.C.

If $v = -$, $\exists$ a dual contour C with $-$ on inside, $+$ on exterior.

- Let $\hat{x}_u = \begin{cases} -x_u & u \in \text{Int}(C) \\ x \ldots \end{cases}$

x_n o.w.

$$\frac{\mu_+(\hat{x}_n)}{\mu(x_n)} = e^{\beta|C|}.$$

$$\mu_+(C \text{ a contour}) \le \sum_x \mu_+(x)\, I(C)$$

$$= \sum_x e^{-\beta|C|} \mu_+(\hat{x})\, I(x \sim C)$$

$$\le e^{-\beta|C|}.$$

There are $\le \ell 4 \cdot 3^{\ell-1}$ contours of length containing v so

$$\mu_+(\sigma_v = -1) \le \sum_C e^{-\beta|C|}$$

$$\le \sum_\ell \ell e^{-\beta\ell} \cdot 4 \cdot 3^{\ell-1}$$

$$= 12 e^{-\beta} / (3e^{-\beta} - 1)^2$$

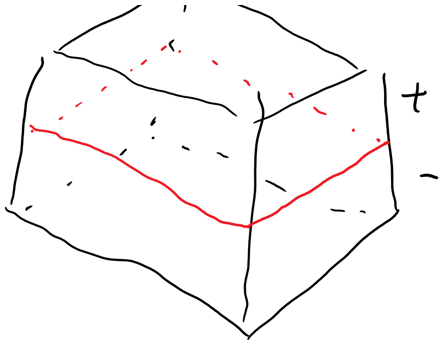
$$\le 1/3 \text{ if } \beta \text{ large.}$$

But $\mu_-(\sigma_v = -1) = 1 - \mu_+(\sigma_v = -1) \ge 2/3$

- For $d=2$ all Gibbs measures are mixtures of μ_+ and μ_- for $\beta > \beta_c > 0$.
- For $d \ge 3$ more complicated states, e.g. Dobrushin states

$+$ B.C. $x = \text{sign}(x_n)$

states



B.C. $x_u = \mathrm{sign}(x_u(1))$

- <u>Unique Gibbs measures</u>:

If interactions are weak enough then measure is unique.

We say μ stochastically dominates μ', $\mu \geq \mu'$ if for all A increasing

$\mu(A) \geq \mu'(A)$.

There exists a coupling $\sigma \sim \mu$, $\sigma' \sim \mu'$ such that $\sigma \geq \sigma'$.

- Ising model has a monotonicity property,
  - if $\tau \leq \tau'$ are B.C. on ∂A, then $\mu_\tau \leq \mu_{\tau'}$.

<u>Proof</u>: True for $A = \{v\}$.

$$\mathbb{P}[\sigma_v = 1 \mid \sigma_{\partial v} = \tau] = \frac{e^{\beta \sum_{u \in \partial v} \tau_u}}{e^{\beta \sum \tau_u} + e^{-\beta \sum \tau_u}}$$

increasing in $\sum_{u \in \partial v} \tau_u$

__Glauber dynamics__: Markov chain X_t on $\{\pm 1\}^V$

Each step

- Pick $v \in A$ uniformly at random.
- Update $X_{t+1}(v)$ with a with probability

$$\mu[\sigma_v = a \mid \sigma_{\partial v} = X_t(\partial v)]$$

Then X_t is reversible w.r.t. μ.

Let $X_0(A) = Y_0(A)$, $X_t(\partial A) = \tau$, $Y_t(\partial A) = \tau'$

So X_t is Glauber dynamics on A with B.C. τ

Y_t " " " " " " " τ'

Couple so that $X_t \leq Y_t$.

Since $X_t \xrightarrow{d} \sigma^\tau$, $Y_t \xrightarrow{d} \sigma^{\tau'}$,

$\sigma^\tau \leq \sigma^{\tau'}$.

For any Gibbs measure μ,

$$\mu_- \leq \mu \leq \mu_+$$

Enough to prove $\mu_- = \mu_+$

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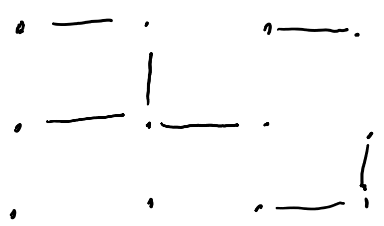
• **Ex:** Ising Model with β small.

– FK model: q-state

$$\xi \in \{0,1\}^V$$



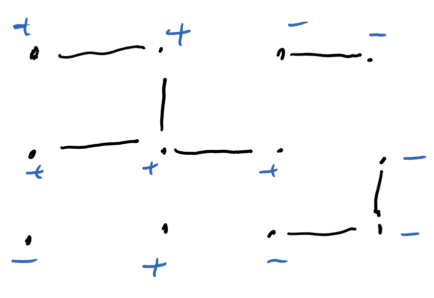
$$\mathbb{P}[\xi] = \frac{1}{Z} \cdot p^{\sum \xi_u} (1-p)^{|E| - \sum \xi_u} \cdot q^{C(\xi)}$$

where $C(\xi)$ = # connected components of ξ.

Claim: $\xi \preccurlyeq$ Percolation(p). for $q \geq 1$.

Edwards–Sokal Coupling:

Choose a uniform spin for each component to form σ.



$$\mathbb{P}[(\xi, \sigma)] = \frac{1}{Z} p^{\sum \xi_u} (1-p)^{|E| - \sum \xi_u} \cancel{2^{C(\xi)} \cdot 2^{-C(\xi)}}$$

$$\mathbb{P}[\sigma] = \frac{1}{Z'} \sum \qquad \left(\frac{p}{\quad}\right)^{\sum \xi_u}$$

$$\mathbb{P}[\sigma] = \frac{1}{Z'} \sum_{\substack{\xi : \xi \text{ compatible} \\ \text{with } \sigma}} \left(\frac{p}{1-p}\right)^{\sum \xi_e}$$

$$= \frac{1}{Z'} \left(1 + \frac{p}{1-p}\right)^{\sum_{u \sim v} I(\sigma_u = \sigma_v)}$$

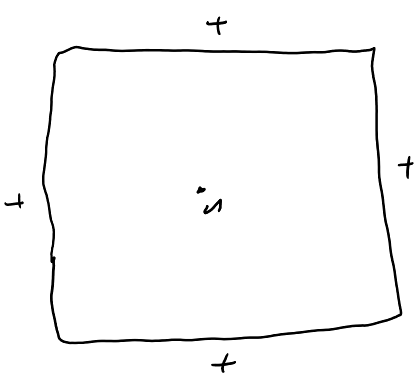
$$= \frac{1}{Z'} \exp\left(-\tfrac{1}{2} \log(1-p) \sum_{u \sim v} 2 I(\sigma_u = \sigma_v)\right)$$

$$= \frac{1}{Z''} \exp\left(-\log(1-p) \sum_{u \sim v} \sigma_u \sigma_v \cancel{-1}\right)$$

$$= \frac{1}{Z''} \exp\left(\beta \sum \sigma_u \sigma_v\right)$$

$$\beta = -\tfrac{1}{2} \log(1-p) \iff p = 1 - e^{-\beta}$$

Box D with $+$ B.C.



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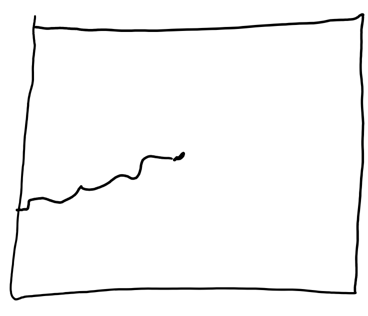


FK-Wired B.C.
Boundary set to +.

$$\mathbb{P}[\sigma_u = + \mid \sigma_{\partial D} \equiv +] = \mathbb{P}[u \leftrightarrow \partial D \mid \text{Wired}] + \tfrac{1}{2}(1 - \mathbb{P}[\dots])$$

$$= \frac{1}{2} + \frac{1}{2}\mathbb{P}[u \leftrightarrow \partial D \mid \text{Wired}]$$

$$\le \frac{1}{2} + \frac{1}{2}\mathbb{P}_p[u \leftrightarrow \partial D] \quad \text{in percolation}$$

$$\to \frac{1}{2} \quad \text{as } D \to \infty \text{ if } \beta \text{ small.}$$

So $\mu_+(\sigma_u = +) = \mu_-(\sigma_u = +) = \frac{1}{2}$.

and so for the coupling $\sigma_+ \geq \sigma_-$

$$\mathbb{P}[\sigma_+(u) \neq \sigma_-(u)] = \mu_+(\sigma(u) = +) - \mu_+(\sigma(u) = +) = 0$$

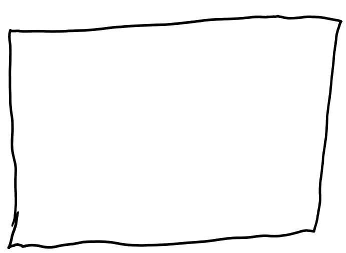
So $\sigma_+ \equiv \sigma_- \Rightarrow \mu_+ \equiv \mu_-$.

<u>Alternative Proof:</u> General spin system, maximum degree d, if

$$\max_{v, x_{\partial v}, x'_{\partial v}} d_{TV}(\mu(x_v \mid x_{\partial v}), \mu(x_v \mid x'_{\partial v})) \le \frac{1-\varepsilon}{d}$$

then unique Gibbs measure.

<u>Proof:</u> Couple X_t, Y_t as Glauber dynamics with B.C. τ and τ'.



- mixes in time $O(n \log n)$
- time for disagreement to propogate $\sim \Omega(n^2)$

propogate ~ $O(n^2)$