Goal: Find size of maximal IS of random graph.

Definition: \( I \subseteq V \) is an independent set (IS) of \( G=(V,E) \) if \( \forall u,v \in I, (u,v) \notin E \).

- Useful to consider a random IS \( \sigma \in \{0,1\}^V \) of \( G \).

\[
P[\sigma] = \frac{1}{Z} \prod_{u,v} I(\sigma_u \sigma_v = 0)
\]

\[
Z = \# \text{IS} = \prod_{u,v} I(\sigma_u \sigma_v = 0)
\]

Definition: Hardcore Model with fugacity \( \lambda > 0 \),

\[
P[\sigma] = \frac{1}{Z_\lambda} \lambda^{\# \sigma} \prod_{u,v} I(\sigma_u \sigma_v = 0)
\]

- weights independent sets by size.

Spin System: \( \sigma \in \{0,1\}^V \), weights \( \psi_u, \psi_{uv} \) such that

\[
P[\sigma] = \frac{1}{Z_\psi} \prod_{u \in v} \psi_u(\sigma_u) \prod_{(u,v) \in E} \psi_{uv}(\sigma_u, \sigma_v)
\]

Example:
Example:

- **Ising Model** with inverse temperature $\beta$ and external field $h$. \( \sigma \in \{ -1, 1 \} \),

\[
\Pi[\sigma] = \frac{1}{Z} \exp \left( \beta \sum_{\text{nn}} \sigma_n \sigma_m + h \sum \sigma_n \right)
\]

- **Random $k$-Colouring**

\[
\Pi[\sigma] = \frac{1}{Z} \prod_{\text{nn}} \mathbb{1} (\sigma_n \neq \sigma_m).
\]

**Markov Random Field Properties**

\[
\Pi[\sigma_A = x_A \mid \sigma_A^c = x_A^c] = \Pi[\sigma_A = x_A \mid \sigma_\partial A = x_\partial A]
\]

**Proof:**

\[
\Pi[\sigma_A = x_A \mid \sigma_A^c = x_A^c] = \frac{\Pi[\sigma_A = x_A \mid \sigma_A^c = x_A^c]}{\Pi[\sigma_A = x_A^c \mid \sigma_A^c = x_A^c]}
\]

\[
= \frac{\frac{1}{2} \prod_u \psi(x_u) \prod_{\text{nn}} \psi_{\text{nn}}(x_n, x_m)}{\frac{1}{2} \prod_u \psi(x_u^c) \prod_{\text{nn}} \psi_{\text{nn}}(x_u^c, x_m^c)}
\]

\[
= \prod_u \psi(x_u) \prod_{\text{nn}} \psi_{\text{nn}}(x_n, x_m)
\]
\[ \prod_{\mathbf{u} \in A} \Psi(x_{\mathbf{u}}) \prod_{(\mathbf{u},\mathbf{v}) \in E(\text{AVA})} \Psi(y_{\mathbf{v}}(x_{\mathbf{v}}, x_{\mathbf{u}})) \]

So the states outside AVA do not affect the calculation. The distribution is the spin system on AVA with the spins of DA fixed.

Definition: We call a spin system permissive if, given any boundary condition on DA, there is a configuration of A with weight > 0.

Infinite Graphs

How to define it for an infinite graph?

A measure \( \mu \) on \( X^A \) is a Gibbs measure for a spin system with weights \( \Psi \) if

\[ M(\sigma_A = x_A | \sigma_{\partial A} = x_{\partial A}) = M(\sigma_A = x_A | \sigma_{\partial A} = x_{\partial A}) \]
Existence (for permissive systems)

Let $D_i \subseteq V$ be an increasing subsequence of sets $D_i \uparrow V$. Choose $T_i$ a B.C. on $\partial D_i$.

$$\lim \mu(\{\omega_A \mid \omega_{\partial D_i} = T_i\})$$

may not exist but subsequential limits exist.

To get full set of measures take random $T_i$.

Non-Uniqueness

Using model at low temperature, $\beta$ large, $h = 0$.

Peirce's argument

If $u = -$, $\exists$ a dual contour $C$ with

- $u$ in inside, $+u$ on exterior.

- Let $x_n = \{ -x_\mu, u \in b_b(C) \}$
\[
\frac{m_+ (xu)}{m (xu)} = e^{\beta |C|}.
\]

\[
M_+ (C \text{ a contour}) \leq \sum_x m_+ (x u) I (C)
\]

\[
\leq \sum_x e^{-\beta |c|} m_+ (x u) I (x u | C)
\]

\[
\leq e^{-\beta |C|}.
\]

There are \(\leq \lambda^4 \cdot 3^{d-1}\) contours of length containing \(u\) so

\[
M_+ ( \sigma_v = -1) \leq \sum_C e^{-\beta |C|}
\]

\[
\leq \sum_x e^{-\beta |c|} \cdot 4 \cdot 3^{d-1}
\]

\[
\leq 12 e^{-\beta} \left( 3 e^{-\beta} - 1 \right)^2
\]

\[
\leq \frac{1}{3} \text{ if } \beta \text{ large.}
\]

But \(M_- ( \sigma_v = -1) = 1 - M_+ ( \sigma_v = -1) \geq \frac{2}{3}\)

- For \(d=2\) all Gibbs measures are mixing of \(M_+\) and \(M_-\) for \(\beta > \beta_c > 0\).
- For \(d \geq 3\) more complicated states, e.g. Dobrushin states

\[
\lambda (\gamma) = \sin (\pi \lambda) + 1 + \beta (\gamma) = \sin (\pi \lambda) + 1 + 12 e^{-\beta} \left( 3 e^{-\beta} - 1 \right)^2
\]
\[ B.C. \ x_u = \text{sign}(x_u(0)) \]

- **Unique Gibbs measures:**

If interactions are weak enough the measure is unique.

We say \( \mu \) stochastically dominates \( \mu' \), \( \mu \geq \mu' \), if for all \( A \) increasing,

\[ M(A) \geq M'(A). \]

There exists a coupling \( \sigma \sim \mu, \sigma' \sim \mu' \) such that \( \sigma \geq \sigma' \).

- **Ising model has a monotonicity property:**

- if \( T \leq T' \) are B.C. or \( \partial A \),
  then \( M_T \leq M_{T'} \).

**Proof:** True for \( A = \{v, \bar{v} \} \).

\[ \ln \left[ \mathbb{P}(\sigma_v = 1 \mid \sigma_{\partial v} = z) \right] = \frac{e^{\beta \sum_u \nu_u}}{e^{\beta \sum_u \nu_u} + e^{-\beta \sum_u \nu_u}} \]
Glauber dynamics: Markov chain $X_\tau$ on $\Omega_{\mathbb{Z}^d}$

Each step

- Pick $v \in A$ uniformly at random.

- Update $X_{\tau v}(w)$ with $a$ with probability

$$-M \left[ \sigma_v = a \mid \sigma_{\tau v} = X_\tau(\tau v) \right]$$

Then $X_\tau$ is reversible w.r.t. $\mu$.

Let $X_0(A) = Y_0(A)$, $X_\tau(\partial A) = \tau$, $Y_\tau(\partial A) = \tau'$

So $X_\tau$ is Glauber dynamics on $A$ with B.C. $\tau$

$Y_\tau$ " " " " " " " " " " $\tau'$

Couple so that $X_\tau \leq Y_\tau$.

Since $X_\tau \leq \sigma^2$, $Y_\tau \leq \sigma^{2'}$

$\sigma^2 \leq \sigma^{2'}$

For any Gibbs measure $\mu$,

$$\mu_- \leq \mu \leq \mu_+$$

Enough to prove $\mu_- = \mu_+$. 

-
Ex: Ising Model with $\beta$ small.

- FK model: $q$-state
  $\mathbb{Z} \in \{0, 1\}^V$

\[
P[\mathbb{Z}] = \frac{1}{Z} \exp \left( \sum_{\mathcal{E}} (1 - p) |\mathcal{E}| - E_{\mathcal{E}} \right),
\]
where $C(\mathcal{E})$ = # connected components of $\mathcal{E}$.

Claim: $\mathbb{Z} \ll$ Percolation ($p$). For $q \geq 1$.

Edwards-Sokal Coupling:

Choose a uniform spin for each component to form $\sigma$.

\[
P[\mathbb{Z}, \sigma] = \frac{1}{Z} \exp \left( \sum_{\mathcal{E}} (1 - p) |\mathcal{E}| - E_{\mathcal{E}} \right),
\]
\[
P[\sigma] = \frac{1}{Z} \exp \left( \sum (1 - p) \Sigma_{\mathcal{E}} \right),
\]
\[ P[\sigma] = \frac{1}{Z} \sum_{\text{compatible with } \sigma} \left( \frac{p}{1-p} \right)^{\sum_{\sigma_{uv}} I(\sigma_{uv} = \sigma_{uv})} \]

\[ = \frac{1}{Z} \exp \left( \frac{-1}{2} \log(1-p) \sum_{\sigma_{uv}} \sum_{\sigma_{uv}} I(\sigma_{uv} = \sigma_{uv}) \right) \]

\[ = \frac{1}{Z} \exp \left( -\log(1-p) \sum_{\sigma_{uv}} \sigma_{uv} \sigma_{uv} \right) \]

\[ = \frac{1}{Z} \exp (\beta \sum_{\sigma_{uv}} \sigma_{uv}) \]

\[ \beta = -\frac{1}{2} \log(1-p) \Rightarrow p = 1 - e^{-\beta} \]

**Bd + D with + B.C.**

\[ F_k - \text{Wired B.C.} \]

\[ \text{Boundary set to } t. \]

\[ P[\sigma_u = + | \sigma_{D} = +] = P[u \leftarrow D | \text{Wired}] + \frac{1}{2}(1 - P_c) \]
\[ = \frac{1}{2} + \frac{1}{2} \mathbb{P}[C \leftrightarrow \partial D \mid \text{Wired}] \]

\[ \leq \frac{1}{2} + \frac{1}{2} \mathbb{P}[C \leftrightarrow \partial D] \text{ in percolation} \]

\[ \rightarrow \frac{1}{2} \text{ as } D \to \infty \text{ if } \beta \text{ small.} \]

So \[ M_+(\sigma_u = +) = M_-(\sigma_u = +) = \frac{1}{2} \]

and so for the coupling \[ \sigma^+ \equiv \sigma^- \]

\[ \mathbb{P}[\sigma^+_v = \sigma^+_w \mid \sigma^+_u = \sigma^-_v] = M_+(\sigma_u = +) - M_+(\sigma_u = +) = 0 \]

So \[ \sigma^+ \equiv \sigma^- \Rightarrow M_+ \equiv M_- \]

**Alternative Proof:** General spin system, maximum degree \( d \), if

\[ \max_{u,v,w} \mathbb{P}(M(x_u, x_v, x_w), \mathbb{P}(x_u, x_v, x_w)) \leq \frac{1-\epsilon}{d} \]

then unique Gibbs measure.

**Proof:** Couple \( X, Y \) as Glauber dynamics with B.C. \( T \) and \( T' \).

- Mixes in time \( O(n \log n) \)
- Time for disagreement to propagate \( \sim \) \( O(n^{2/3}) \)
\[ \text{propogate} \sim \alpha n^{21} \]