Random Walks on Random Graphs

Sunday, February 25, 2018 11:04 PM

- Random walk X_t
  - Pick neighbour uniformly at random
  - Lazy RW, w.p. $\frac{1}{2}$ stay still.

Stationary distribution: $\pi_v = \frac{d_v}{2|E|}$, $P_{vu} = \frac{1}{d_v} \cdot I(vu)$.

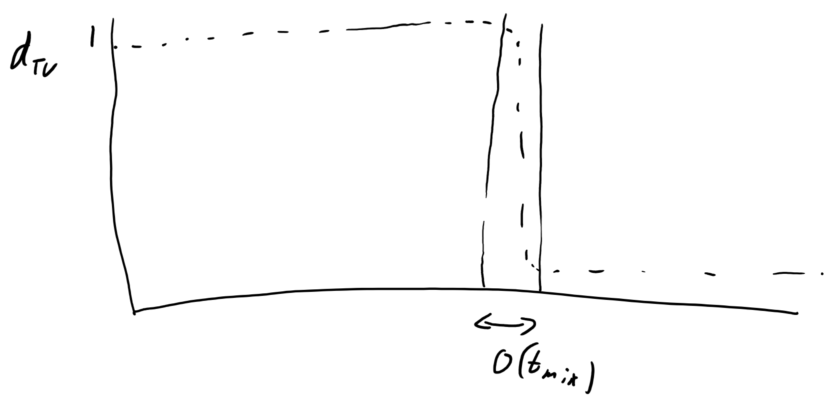
$\pi_v P_{vu} = \pi_u P_{uv}$

- <u>Questions</u>: On a random graph G, how long does it take until $X_t \sim \pi$.

<u>Mixing Time</u>

$$t_{mix}(\varepsilon) = \min\{t : \max_v d_{TV}(X_t^v, \pi) \le \varepsilon\}.$$

<u>Cutoff</u> If $t_{mix}^{(n)}(\varepsilon) / t_{mix}^{(n)}(1-\varepsilon) \to 1$



Spectral Gap:

$D^{\frac{1}{2}} P D^{-\frac{1}{2}}$ is symmetric so

$D_\pi P D_\pi$

the eigenvalues of P

$1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$

are real. $\lambda_n \geq 0$ for lazy walk.

$$\frac{P_t(x,y)}{\pi_y} = \sum_{j=1}^{n} f_j(x) f_j(y) \lambda_j^t$$

$$= 1 + \sum_{j=2}^{n} f_j(x) f_j(y) \lambda_j^t.$$

Spectral Gap $\gamma = 1 - \lambda_2$.

Relaxation time $t_{rel} = 1/\gamma$.

$$d_{\ell^2}(X_t^x, \pi) := \sqrt{\sum_y \left(\frac{P_t(x,y)}{\pi_y} - 1\right)^2 \cdot \pi(y)} \geq \sum_y \left|\frac{P_t(x,y)}{\pi(y)} - 1\right| \pi(y)$$

$$= 2 d_{TV}(X_t^x, y)$$

Lemma:

$$(t_{rel} - 1) \log\left(\frac{1}{2\varepsilon}\right) \leq t_{mix}(\varepsilon) \leq t_{rel} \cdot \log\left(\frac{1}{\varepsilon \pi_{min}}\right)$$

Conductance

$$Q(A, B) = \sum_{\substack{x \in A \\ y \in B}} \pi(x) P(x,y) = \mathbb{P}_\pi(X_0 \in A, X_1 \in B).$$

$$\Phi_* = \min_{S : \pi(S) \leq 1/2} Q(S, S^c)/\pi(S)$$

Lemma: Jerrum - Sinclair

$$\frac{\Phi^2}{2} \le \gamma \le 2\Phi$$

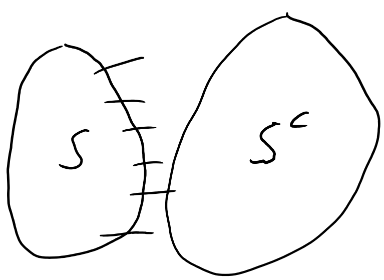
A graph is a δ-expander if

$$\min_{|S| \le \frac{n}{2}} \frac{|\partial S|}{|S|} \geq \delta .$$

__Lemma:__ A random d-regular graph ($d \geq 3$) is w.h.p. a δ-expander and therefore,

$$t_{mix} \le C \log n.$$

__Proof:__



recall

$$M_n = \sqrt{2}\, n^{n/2} e^{-n/2}$$

$|S| = k$, B_ℓ = # edges S to $S^c = \ell$

$\mathbb{P}[G \text{ not } \delta\text{-expander}]$

$$= \sum_{k=1}^{n/2} \sum_{\ell=1}^{\delta k} \binom{n}{k} \mathbb{P}[B_{S,\ell}]$$

$$= \sum_{k=1}^{n/2} \sum_{\ell=1}^{\delta k} \binom{dk}{\ell} \cdot \binom{d(n-k)}{\ell} \binom{n}{k} \cdot \ell! \cdot M_{dk-\ell}\, M_{d(n-k)-\ell}$$

expand with stirling

$\to 0.$

Conductance of S:

$$Q(S,S^c) = \sum_{a\in S, b\in S^c} \pi(a)\cdot P(a,b)$$

$$= \frac{1}{n} \sum_{a\in S, b\in S^c} \frac{1}{d}\cdot I(a\sim b)$$

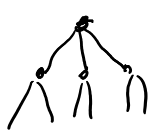
$$\geq \frac{|S|\cdot \delta}{dn}$$

So $\Phi \geq \min_{|S|\leq \frac{n}{2}} \frac{Q(S,S^c)}{\pi(S)} \geq \frac{\delta}{d} > 0.$

$\Rightarrow \mathbb{P}[\gamma \geq \frac{\delta}{d}] \to 1]$

$\mathbb{P}[t_{mix} \leq C \log n] \to 1$

<u>Cutoff</u>:



Random walk on a tree

D_t = distance to the root is a biased RW with drift $\frac{d-2}{d}$.

- Non back tracking RW. (NBRW) Y_t

$Y_{t+1} \neq Y_{t-1}$

<u>Claim</u>: On a d-regular graph.

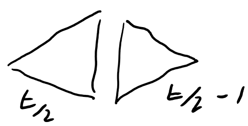
$X_t \overset{d}{=} Y_{D_t}$

- on a tree both are uniform given the level.
- Embedding $T_d \to G$ cover tree

Mixing for Y_t

$$\mathbb{P}[Y_t^v = u] = \frac{\#\text{path } v \text{ to } u}{d(d-1)^{t-1}} \text{ length } t$$

We showed that



$$\#\text{paths } v \text{ to } u \text{ length } t \sim \mathrm{Pois}\left(\frac{d^2(d-1)^{t-1}}{dn}\right)$$

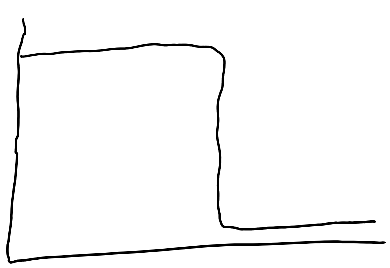
concetrated if $t - \log_{d-1} n \to \infty$

so $\mathbb{P}[Y_t^v = u] \approx 1/n$ for most u.

$$\Rightarrow d_{TV}(Y_t^v, \pi) = \frac{1}{2}\sum_u \left|\mathbb{P}[Y_t^v = u] - \frac{1}{n}\right|$$

$$= \sum_u \left(\frac{1}{n} - \mathbb{P}[Y_t^v = u]\right) \vee 0. \to 0$$

Cutoff for Y_t at $\log_{d-1} n$



For SRW:

$$d_{TV}(X_t^v, \pi) \le \sum \mathbb{P}[D_t = h] \cdot d_{TV}(Y_h, \pi)$$

$$d_{TV}(X_t^v, \pi) \le \sum_k \mathbb{P}[D_t = k] \cdot d_{TV}(Y_k, \pi)$$

$$\le \mathbb{P}[D_t \le \log_{d-1} n + \log\log n] + o(1)$$

For $t \ge (1+\delta)\frac{d}{d-2}\log_{d-1} n$,

$\mathbb{P}[D_t \le \log_{d-1} n] \to 0$

so $d_{TV}(X_t^v, \pi) \to 0$.

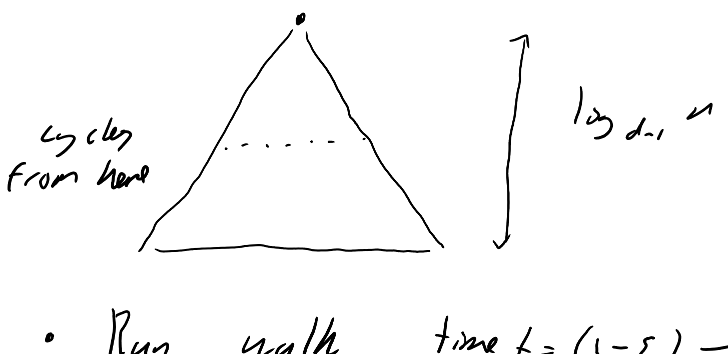
For $t \le (1-\delta)\frac{d}{d-2}\log_{d-1} n$,

$\mathbb{P}[X_t^v \in B_{(1-\frac{\delta}{2})\log_{d-1} n}(v)] \to 1$

but $\pi(B_{(1-\frac{\delta}{2})\log_{d-1} n}) \le O(n^{-\delta/2})$

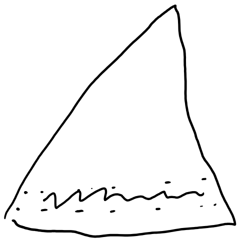
Hence cutoff at $\frac{d}{d-2}\log_{d-1} n$.

Alternate proof:



- Run walk time $t = (1-\varepsilon)\frac{d}{d-2}\log_{d-1} n$.

$\mu_t \sim \mathbb{P}[X_t^v \in \cdot\,]$



Most of μ_t on vertices distance

$\approx (1-\varepsilon)\log_{d-1} n$

$\mu_t(u) \approx n^{-\varepsilon+o(1)}$ in this range so

$$\mu_t = p\hat{\mu}_t + (1-p)\check{\mu}_t$$

where $\hat{\mu}_t(u) \le n^{-\varepsilon+o(1)}$, $\quad 1-p \le o(1)$.

$$d_{L^2}(\hat{\mu}_t(u), \pi) \le n^{-2\varepsilon+o(1)}.$$

$$d_{L^2}(\hat{\mu}_t(u)P^r, \pi) \le \gamma^r n^{-2\varepsilon+o(1)} = o(1).$$

if $r \ge \frac{3\varepsilon}{\gamma}\log_{d-1} n$

$$d_{TV}(X^u_{t+r}, \pi) \le p\, d_{TV}(\hat{\mu}P_r, \pi) + (1-p)\cdot 1$$

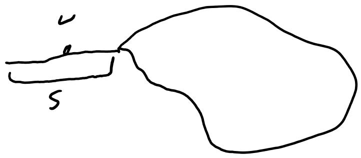
$$\le d_{L^2}(\hat{\mu}P_r, \pi) + o(1) = o(1).$$

So mixed at time

$$\left((1-\varepsilon)\frac{d}{d-2} + \frac{3\varepsilon}{\gamma}\right)\log_{d-1} n.$$

Erdos - Renyi: $\quad d \ge 1$.

- Not connected so consider the Giant
- Path length $\delta \log n$



- time to exit $O(\log^2 n)$, SRW on path

$\mathbb{P}[X^v_{\alpha \log n} \in S] \geq e^{-c\alpha}$

So t_{mix} at least $\log^2 n$.

Theorem:

a) Fountoulakis, Reed '06
Benjamini, Kozma, Wormald

$t_{mix} \sim \log^2 n$ (no cutoff).

b) For typical starting point: $\exists\, S_d$ s.t.
Berestycki, Lubetzky, Peres, S.

$$\frac{1}{n} \#\{v : d_{TV}(X^v_{(S-\varepsilon)\log n}, \pi) < 1-\varepsilon\} \xrightarrow{P} 0.$$

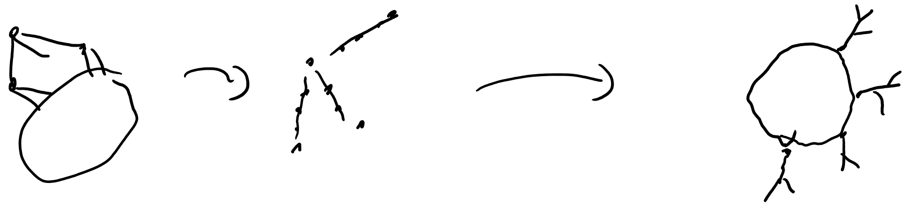
$$\frac{1}{n} \#\{v : d_{TV}(X^v_{(S+\varepsilon)\log n}, \pi) < \varepsilon\} \xrightarrow{P} 1$$

Structure Theorem

Let $\theta e^{-\theta} = d e^{-d}$, $\theta < 1 < d$.

$\Lambda = N(d-\theta, 1/n)$, D_u iid Pois(Λ)

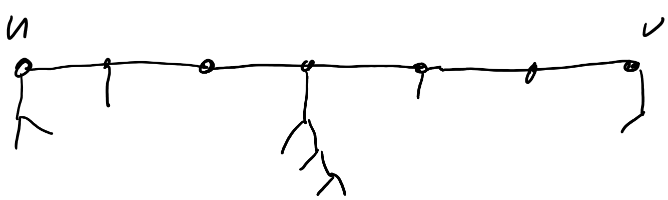
- Form M the configuration model on $\{D_u\}_{u: D_u \geq 3}$,
- Replace each edge with path length Geom$(1-\theta)$ IID.
- Add GWBP offspring Pois(θ) to each vertex



Graph C_n^*, then C_n^* contiguous w.r.t. Giant

$\mathbb{P}[C_n^* \in A_n] \to 0 \Rightarrow \mathbb{P}[\text{Giant}_n \in A_n] \to 0.$

Upper bound of $O(\log^3 n.)$



$R_{u,v}$ = vertices in between $(u,v) \in M$.

- $\mathbb{P}[|R_{u,v}| \geq r] \leq e^{-cr}$. max $R_{u,v} \leq O(\log n)$ w.h.p.
- $\mathbb{P}[\sum_{u: u \sim v} |R_{u,v}| \geq r] \leq e^{-cr}$

$\sum R_{u,v} \sim \sum^{D_v} \sum^{\text{Geom}} Y_i \sim$ BP

$$\sum R_{u,v} \sim \sum_{i=1}^{DV} \sum_{j=1}^{Geom} Y_i \sim BP.$$

Conductance:

Let $S \subset C_n^*$, connected.

- $|S \cap M| \geq \frac{|S|}{C \log n}$.
- If $|S \cap M| \leq M/2$ then $E(S, S^c) \geq \delta |S \cap M|$

  so $Q(S, S^c) \geq \frac{c}{\log n} |S|$
- If $|S \cap M| > M/2$ then either $|S^c \cap M| \geq \frac{cn}{\log n}$

  and so $E(S, S^c) \geq \frac{c' n}{\log n}$

  or $A = \{(u,v) \in E_m : S^c \cap R_{u,v} \neq \emptyset\}$

  $|A| \geq \frac{cn}{\log n}$. Check that

- So $\Phi \geq \frac{c}{\log n}$, $\gamma \geq \frac{1}{\log^2 n}$

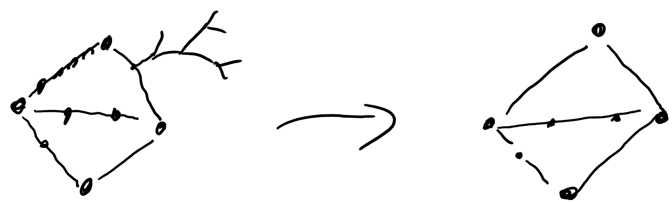
  $t_{mix} \leq \frac{1}{\gamma} \log\left(\frac{1}{1/n}\right) \leq \log^3 n.$

From a random vertex.

Let $\tilde{C}_n$ be C_n^* where

all $R_{u,v}$ with $|R_{u,v}| \geq C \log\log n$

are replaced with a single edge.



$$\mathrm{Gap}(\tilde{C}_n)^{-1} \leq C(\log\log n)^2.$$

So $t_{mix}(\tilde{C}_n) \leq C \log n \cdot (\log\log n)^2$

Coupling the two.

Let $A = \bigcup \{R_{u,v} : |R_{u,v}| \geq C \log\log n\}$.

Then $\pi(A) \leq 1/\log^{10} n$.

Let I be a uniform vertex.

$$\sum_{t=1}^{\log^2 n} \mathbb{P}[X_t^I \in A] = \sum_{t=1}^{\log^2 n} \frac{1}{n} \sum_{v \in V} \mathbb{P}[X_t^v \in A]$$

$$\leq \sum_{t=1}^{\log^2 n} \frac{2|E|}{n} \frac{d_v}{2|E|} \sum_{v \in V} \mathbb{P}[X_t^v \in A]$$

$$= \sum_{t=1}^{\log^2 n} O(1) \cdot \mathbb{P}[X_t^\pi \in A]$$

$$= \sum_{t=1}^{\log^2 n} O(1)\, \pi(A) \leq \frac{1}{\log^8 n} = o(1).$$

- So w.h.p X_t^I never enters A.

- Coupling this with the walk on $\tilde{C}_n$,

$$d_{TV}\left(\mathbb{P}[X_t^I \in \cdot \mid I], \pi_{\tilde{C}_n} \right) = o(1)$$

$$\Rightarrow d_{TV}\left(\mathbb{P}[X_t^I \in \cdot \mid I], \pi \right) = o(1).$$