Introduction

Sunday, February 4, 2018 3:48 PM

Plan for course

a) Random Graphs models

Erdős Renyi:

- Phase transitions, combinatorial properties

b) Stochastic Processes

- Infection (Contact, SIS, SIR)
- Voter model, gossip
- Spin systems (Ising, hardcore)

c) Random CSP's

E.G. random colouring of random graphs.

- Phase transitions

d) Time permitting dense models

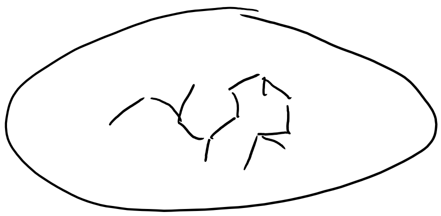
SK model.

Models:

Random graphs

Erdos - Renyi

$G(n,p)$. Sparse case $G(n, d/n)$



<u>Q's</u> - Is there a giant (linear sized) component

- Connected?
- Component size / structure
- Diameter
- Degree distribution.
- Local limit

Random regular $G(n,d)$

- Chosen uniformly from all d-regular graphs with n labeled vertices.

• Degree Distribution of ER $G(n, d/n)$:

Let d_v denote the degree of vertex v.

$$d_v = \#\{u \in G : (u,v) \in E\}.$$

- One vertex $\sim \mathrm{Bin}(n-1, d/n)$

$$\mathbb{P}[\ \mathrm{Bin}(n-1, d/n) = k\ ]$$

$$= \binom{n-1}{k} \cdot \left(\frac{d}{n}\right)^k \cdot \left(1 - \frac{d}{n}\right)^{n-1-k}$$

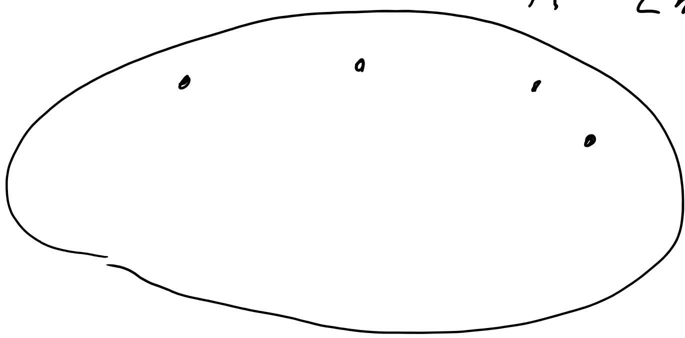
$$= \frac{(n-1)_k}{k!} \cdot \frac{d^k}{n^k} \left(1 - \frac{d}{n}\right)^{n-1-k}$$

$$\to \frac{d^k}{k!} e^{-d} = \mathbb{P}[\mathrm{Pois}(d)]$$

<u>Joint distribution</u>: Does $\frac{1}{n} \sum_{v \in V} \delta_{d_v} \to \mathrm{Pois}(d)$?

Let $N_k = \#\{v : d_v = k\}$.

A Choose $n^{1/3}$ vertices uniformly at random.



$B = \{\exists u,v \in A : (u,v) \in E\}$.

$$\mathbb{P}[B] \le \binom{n^{1/3}}{2} \cdot \frac{d}{n} \le \frac{d n^{2/3}}{n} \to 0.$$

Conditional on B, $\{d_v\}_{v \in A}$ are IID, $\mathrm{Bin}(n - n^{1/3}, d/n) \approx \mathrm{Pois}(d)$.

$$\mathbb{P}\left[\left|\frac{1}{n^{1/3}} \#\{v \in A : d_v = k\} - \frac{d^k e^{-d}}{k!}\right| > \delta\right]$$

$$\le C e^{-\delta^2 n^{1/3}/2}.$$ Azuma - Hoeffding.

$$\mathbb{P}\left[\left|\frac{1}{n} N_k - \frac{1}{n^{1/3}} \#\{v \in A : d_v = k\}\right| > \delta\right] \to 0$$

sampling without replacement.

Altogether $\frac{1}{n} N_k \to$

<u>Maximal Degree</u>: Poisson decays <u>faster</u> than exponential.

$$\max_{v \in V} d_v \quad \_\_\_\_ \xrightarrow{P} 1$$

$$\frac{\max_{v\in V} d_v}{\log n / \log\log n} \xrightarrow{P} 1.$$

exponential.

<u>Proof:</u> Check $\mathbb{P}\left[d_v > \frac{(1+\varepsilon)\log n}{\log\log n}\right] = o(n)$.

$$\left(\frac{\alpha \log n}{\log\log n}\right)! \approx e^{-\alpha \log n / \log\log n} \cdot \left(\frac{\log n}{\log\log n}\right)^{\log n / \log\log n}$$

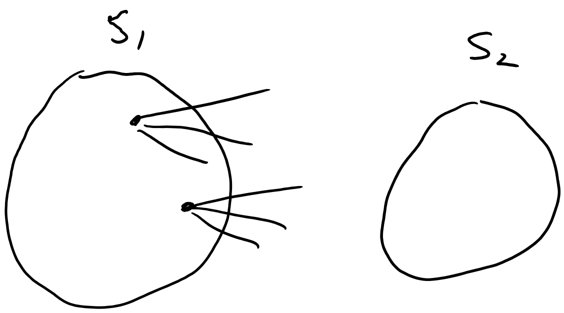
$$= n^{o(1)} \cdot n^{\alpha - \frac{\alpha \log\log\log n}{\log\log n}}$$

$$d^{\log n / \log\log n} = n^{o(1)}$$

So $\mathbb{P}\left[d_v \geq (1+\varepsilon)\frac{\log n}{\log\log n}\right] = n^{-(1+\varepsilon)+o(1)}$

So $\mathbb{P}\left[\max d_v \geq (1+\varepsilon)\frac{\log n}{\log\log n}\right] = o(1)$.

<u>Lower bound</u> split verticis in two have



Let $\tilde{d}_v = \#\{u \in S_2 : (u,v) \in E\}$

$\sim \mathrm{Bin}(n/2, d/n)$. IID

$$\mathbb{P}\left[\tilde{d}_v = (1-\varepsilon)\frac{\log(n/2)}{\log\log(n/2)}\right] = \left(n/2\right)^{-(1-\varepsilon)+o(1)}$$

$$\mathbb{P}\left[\max_{v \in S_1} \tilde{d}_v < (1-\varepsilon)\frac{\log n}{\log\log n}\right]$$

$$\leq \left(1 - \left(\tfrac{n}{2}\right)^{(1-\varepsilon)+o(1)}\right)^{n/2} \leq \exp\left(-n^{\varepsilon + o(1)}\right)$$

Cycles: Number of k-cycles:

Let C_k be # of k-cycles.

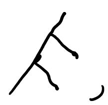
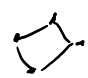
$$\mathbb{E}\, C_k = \frac{n_k}{2k} \cdot \left(\frac{d}{n}\right)^k \to \frac{d^k}{2k} =: \mu_k$$

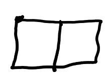
Locally treelike!

Law of C_k? Poisson.

Let H be a connected graph

$$\mathbb{E}\, C_H \approx n^{V(H)} \cdot \left(\frac{d}{n}\right)^{E(H)}.$$

So lots of trees , some cycles ,

no more complicated graphs 

Moments

$$\mathbb{E}\,(C_k)_2 = \mathbb{E}\, C_k(C_k - 1)$$

$$= \mathbb{E} \sum_{\substack{A, A' \\ A \neq A'}} I(A, A' \text{ } k\text{-cycles in } G)$$

$$= \mathbb{E} \sum_{\substack{A, A' \\ \text{disjoint}}} I(A, A' \text{ } k \text{ cycles in } G) \quad (1)$$

$$+ \mathbb{E} \sum_{\substack{A \neq A' \\ A \cap A' \neq \emptyset}} I(A, A' \text{ } k\text{-cycle}) \quad (2)$$

$$(1) = \mathbb{E} \sum_{A} I(A) \sum_{A' \subset V \setminus A} I(A')$$

$$= \mu_k \cdot \left(\mathbb{E} \# \text{ } k\text{-cycles in } G(n-k, \tfrac{d}{n})\right)$$

$$\approx \mu_k^2$$

If $A \cap A' \neq \emptyset$ then $V(A \cup A') - E(A \cup A') \leq -1$.

so $(2) \to 0$.

So $\mathbb{E}(C_k)_2 \to \mu_k^2$ and in general

$\mathbb{E}(C_k)_\ell \to \mu_k^\ell$

- Fact: $\mathbb{E}(\mathrm{Pois}(\mu))_\ell = \mu^\ell$ so $\mathbb{E} C_k^\ell \to \mathbb{E}\, \mathrm{Pois}(d)^\ell$.

This implies $C_k \xrightarrow{d} \mathrm{Pois}(d)$ since Pois has better than exponentially decaying tails.

- More generally $(C_3, C_4, \ldots) \xrightarrow{d}$ Independent $\mathrm{Pois}(\mu_k)$.
  - Use joint moments.

Configuration Model Bollobas '79

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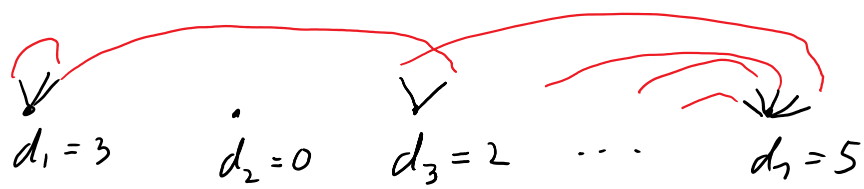
How do you build a uniform d-regular random graph?

Configuration Model

Choosing G given sequence d_v.
Common model is d_v IID from some law.
Assume Σd_v is even



Each edge has d_v "half edges".

- Choose a uniform perfect matching of the half edges.

The number of perfect matchings of n objects is:

$$M_n = \frac{n!}{2^{n/2}(n/2)!} \approx \frac{\sqrt{2\pi n}\; n^n e^{-n}}{\sqrt{2\pi n/2}\; n^{n/2} e^{-n/2}} = \sqrt{2}\; n^{n/2} e^{-n/2}$$

Fact: $n! \approx \sqrt{2\pi n}\; n^n e^{-n}$ Stirling's Formula

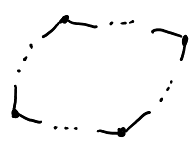
- Configuration model is a multi-graph, it includs self-loops and multiple edges e.g. C_1, C_2 may be non-zero.

- <u>Advantages</u>: Given a set of edges of the graph the remaining edges are still a uniform matching.

  On d-regular random graph:

  $$\mathbb{E} C_k = \frac{n_k \cdot (d \cdot (d-1))^k}{2k} \cdot \frac{1}{dn-1} \cdot \frac{1}{dn-3} \cdot \ldots \cdot \frac{1}{dn-2k+1}$$

  $$\to \frac{d^k}{2k}$$



As in ER case

$$(C_1, \ldots) \xrightarrow{d} \text{Independent } \mathrm{Pois}(\mu_k).$$

$$\mathbb{P}[G \text{ simple}] = \mathbb{P}[C_1 = 0, C_2 = 0]$$

$$\to \exp(-\mu_1 - \mu_2) > 0.$$

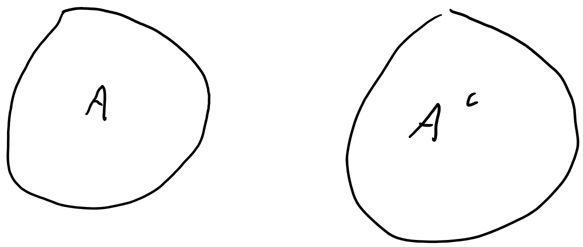
<u>Lemma</u>: Conditional on G Simple, G is uniformly distributed given d_v.

$$\mathbb{P}_{d\text{-reg}}[A] = \mathbb{P}_{\text{config}}[A \mid \text{Simple}]$$

$\Rightarrow$ If $\mathbb{P}_{\text{con}}[A] \to 0$, $\mathbb{P}_{d\text{-reg}}[A] \to 0$.

Example: If $d \geq 3$,

$\mathbb{P}_{d\text{reg}}[\text{Connected}] \to 1.$



Size of $|A| = k$.

$$\mathbb{P}_{\text{Config}}[\text{no edges from } A \text{ to } A^c] = \frac{M_{dk} \, M_{d(n-k)}}{M_{dn}}$$

So

$$\mathbb{P}_{\text{Config}}[\text{Not Connected}] = \mathbb{P}\Big[\bigvee_A \{\text{no edges from } A \text{ to } A^c\}\Big]$$

$$\leq \sum_{k=1}^{n/2} \binom{n}{k} \frac{M_{dk} \, M_{d(n-k)}}{M_{dn}}$$

$$\approx \sum_{k=1}^{n/2} \frac{\sqrt{2\pi n} \cdot n^n e^{-n}}{\sqrt{2\pi k}\sqrt{2\pi(n-k)} \; k^k n^n e^{-n}} \cdot \frac{2 \cdot (dk)^{dk/2} \cdot e^{-dk/2} \, (d(n-k))^{\frac{d(n-k)}{2}} e^{-\frac{d(n-k)}{2}}}{\sqrt{2} \, (dn)^{dn/2} \, e^{-dn/2}}$$

$$= O(1) \sum_k \frac{1}{\sqrt{k}} \cdot \frac{n^n}{k^k (n-k)^{n-k}} \cdot \frac{k^{dk/2} \cdot (n-k)^{d(n-k)/2}}{n^{dn/2}}$$

$$= O(1) \sum_k \frac{1}{\sqrt{k}} \left(\frac{k^k (n-k)^{(n-k)}}{n^n} \right)^{\frac{d-2}{2}}$$

$$\leq O(1) \sum_k \frac{1}{\sqrt{k}} \left(\frac{k}{n}\right)^{\left(\frac{d-2}{2}\right)k} \longrightarrow 0$$

since $\left(\frac{k}{n}\right)^k \leq \Big\{ n^{-k/2} \quad k \leq \sqrt{n}$

since $\left(\frac{k}{n}\right)^k \leq \begin{cases} n^{-k/2} & k \leq \sqrt{n} \\ 2^{-\sqrt{n}} & k > \sqrt{n} \end{cases}$

So $\mathbb{P}_{config}[\text{Connected}] \to 1$ so $\mathbb{P}_{d\text{-reg}}[\text{Connected}] \to 1.$