Galton-Watson Branching Process

Offspring distribution $X$, $E[X] = \mu$.

Each vertex has independent offspring with law $X$.

$Z_n = \# \text{ in level } n,$
$Z_{n+1} = \sum_{i=1}^{Z_n} X_i$, $n$

$E[Z_{n+1}] = \mu E[Z_n] = \mu^{n+1}$.

If $\mu < 1$ then $E[Z_n] \to 0$.

Does it ever die out?

$\Pr[\text{Survive}] = p > 0 \text{ iff } \mu > 1,$

where $p$ solves $p = E[p^X]$.

Local neighbourhood

$\text{Bin}(n-1, \frac{d}{n})$

$\text{Bin}(n-3, \frac{d}{n})$

$\text{Bin}(n-8, \frac{d}{n})$

Locally close to Branching Process
with Poisson($\nu$) offspring dist.

- **Local Weak Limit**
  
  Metric on rooted graphs
  
  We say $(G,v) \approx (G',v')$ if there exists a graph homomorphism
  
  $\psi : V \rightarrow V'$ bijection, $\psi(v) = v'$
  
  $(u,v) \in E$ iff $(\psi(u), \psi(v)) \in E'$.
  
  I will write $(G,v) \approx_r (G',v')$ if $\text{Br}(G,v) \approx \text{Br}(G',v')$.

  $$d((G,v), (G',v')) = 2^{-\max \{ r : (G,v) \approx_r (G',v') \}}$$

  Let $I_n$ be uniformly chosen vertex of $G_n$
  
  $$(G_n, I_n) \xrightarrow{\text{w.c.}} (G,I)$$ weak convergence w.r.t. local metric.

  - If $\text{IP}[(G_n, I_n) \approx (G,v)] \Rightarrow \text{IP}[(G,I) \approx_r (G,v)]$
  
  - If $G$ bounded degree then space is relatively compact. Enough that
    
    $|\text{Br}(G_n, I_n)|$ is tight.
Ex: \( n\times n \) grid

\[
\begin{array}{c}
\text{L.W.C} \\
\rightarrow
\end{array}
\]

\( \mathbb{Z}^2 \)

\( G(n, \frac{d}{n}) \xrightarrow{\text{L.W.C}} GWC \text{ Pois } (d) \)

What about random d-regular?
Limit is infinite d-regular tree

Ex:

\[
\begin{array}{c}
\text{depth} n \text{ binary tree} \\
\rightarrow
\end{array}
\]

Canopy tree

What about random graph with degree sequence with law \( Y \).

\[ q_k = P \{ Y = k \} \]

- Use configuration model:

Root distribution \( Y \)

\[ P \{ \text{child degree } k \} \]

\[ = \frac{\# \text{vertices degree } k \cdot k}{\# \text{ex}} \]
\[ \text{edge} \]

\[ \frac{P \cdot n \cdot h}{\mathbb{E}Y \cdot n} = \frac{P + h}{\mathbb{E}Y} \text{ sized biased distribution } Y^* \]

Limit is a modified GWBP with root distribution \( Y^*-1 \).

Fact: If \( Y \) is Poisson then, \( Y^*-1 \preceq Y \).

- This is for the configuration model, reveal multiple local neighbourhoods to get back to the simple graph.