Ising Model on Random Graphs

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The free energy of the spin system on a sequence of graphs G_n is the limit

$$\Phi = \lim_n \frac{1}{n} \log(Z_n)$$

assuming it exists.

- We say a sequence G_n is locally treelike if

$$\frac{1}{n} \#\{v : B_\ell(v) \text{ is a tree}\} \xrightarrow{p} 0$$

as $n \to \infty$ for all ℓ.

We consider the ferromagnetic Ising model on a random d-regular graph.

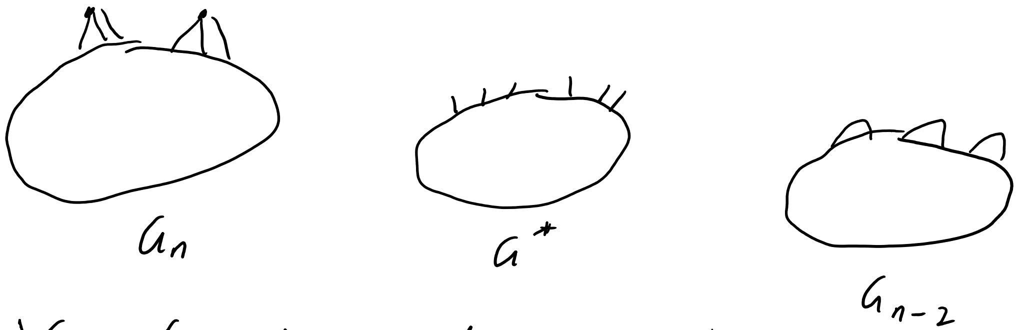
Two approaches,

- Cavity method, varying n
- Differentiating in β (or other parameters).

Start with G_n

- remove 2 random vertices

- randomly reconnect



If G_n is random regular from the configuration model, so is G_{n-2}.

If G_n is locally treelike, so is $G_{\delta n}$.

Suppose locally the measure is a TIFP.
e.g. if there is a unique Gibbs measure on the infinite tree.

Set

$$\Phi_{vertex} = \log \left(\sum_{x \in \mathcal{X}} \psi(x) \left(\sum_{x'} m(x') \psi(x', x) \right)^d \right)$$

$$\Phi_{edge} = \log \left(\sum_{x, x' \in \mathcal{X}} m(x) m(x') \psi(x, x') \right)$$

Then

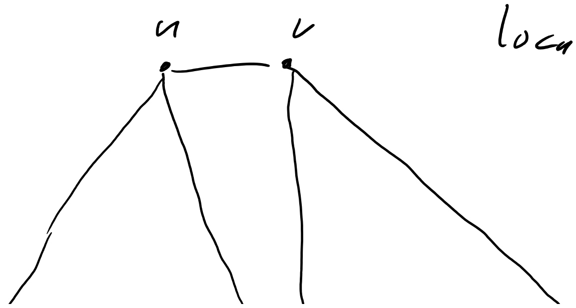
$$\log \frac{Z_{G_n}}{Z_{G^*}} = 2 \Phi_{vertex} + o(1)$$

$\log Z_{G_{n}}$ ...

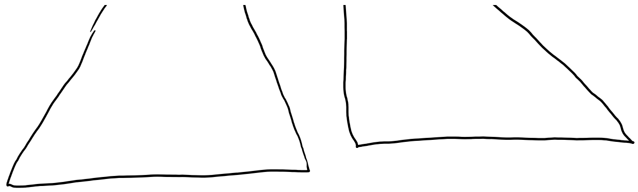
$$\log \frac{Z_{G_{n-2}}}{Z_{G^*}} = d\,\Phi_{edge} + o(1)$$

Iterating:

$$\frac{1}{n} \log Z_{G_n} = \Phi_{vertex} - \frac{d}{2}\Phi_{edge}$$

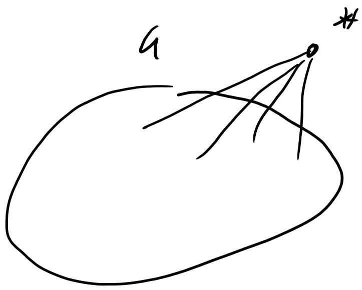


locally treelike



Let (σ, ξ) be Edwards Sokal coupling.

How to introduce an external field?



Add new vertex $*$ with $\sigma_* = +$,

$\beta_{u,*} = h$.

Cluster connected to $*$ gets $+$.

Recall

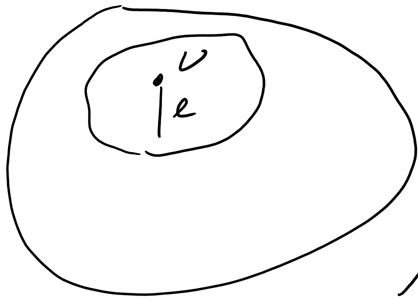
$$\mathbb{P}[\xi = w] = \frac{1}{Z} \prod p_e^{w_e} (1-p_e)^{1-w_e} \, q^{C(w)}$$

For $e = (u,v) \in E$,

$$\mathbb{P}[\xi_e = 1 \mid \xi_{E \setminus e}] = \begin{cases} p_e & \text{if } u \leftrightarrow v \text{ in } \xi_{E\setminus e} \\ \dfrac{p_e}{p_e + q(1-p_e)} & \text{o.w.} \end{cases}$$

Increasing in both $\xi_{E\setminus e}$ and p_e so the Glauber dynamic argument implies that if $p, p' \in [0,1]^E$, $p \le p'$ then

$$\mu_p(\xi \in \cdot) \preccurlyeq \mu_{p'}(\xi \in \cdot)$$



$\tilde{\mu}_\ell$ model in ball of radius ℓ, free B.C.

$\hat{\mu}_\ell$ ball of radius ℓ, + B.C.

Then $\tilde{\mu}_\ell(\xi_{B_\ell} \in \cdot) \preccurlyeq \mu(\xi_{B_\ell} \in \cdot)$

$\preccurlyeq \hat{\mu}_\ell(\xi_{B_\ell} \in \cdot)$

since $\tilde{\mu}_\ell$ corresponds to removing edges outside B_ℓ

and $\hat{\mu}_\ell$ corresponds to increasing edges from $*$ to ∂B_ℓ to $p=1$.

- Now $\tilde{\mu}_\ell \to \mu_f$ and $\hat{\mu}_\ell \to \mu_+$.

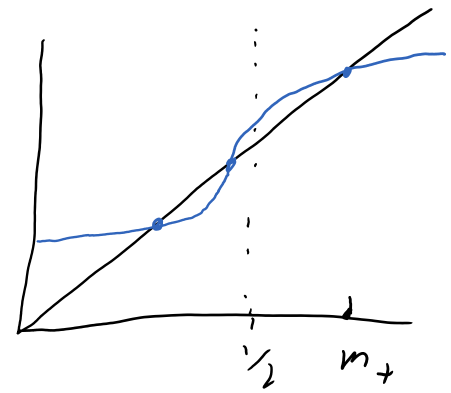
When $h=0$ and $d \tanh \beta > 1$, $\mu_f \neq \mu_+$.

But if $h>0$, $\mu_f = \mu_+$

$$m^{(n+1)} = \frac{e^h\left(m^{(n)}e^{\beta} + (1-m^{(n)})e^{-\beta}\right)^d}{e^h\left(m^{(n)}e^{\beta} + (1-m^{(n)})e^{-\beta}\right)^d + e^{-h}\left(m^{(n)}e^{-\beta} + (1-m^{(n)})e^{\beta}\right)^d}$$

$$= f(m^{(n)}).$$

For $h>0$,



If $m^{(0)} = \frac{1}{2}$,

$$f(\cdots f(\tfrac{1}{2})\cdots) \to m_+$$

and

$$f(\cdots(f(1))\cdots) \to m_+$$

Hence local weak limit is μ_+.

Free energy.

$$\Phi_{vertex} = \log\left(e^h(me^\beta + (1-m)e^{-\beta})^d + e^{-h}(me^{-\beta} + (1-m)e^\beta)^d\right)$$

$$\Phi_{edge} = \log\left(m^2 e^\beta + (1-m)^2 e^\beta + 2m(1-m)e^{-\beta}\right)$$

$$\Phi = \Phi_{\beta,h} = \Phi_{vertex} - \frac{d}{2}\Phi_{edge}.$$

By taking the limit

$$\lim_{h \downarrow 0} \Phi_{\beta,h}$$

we get the free energy of the Ising model with no external field.

Q: What is its local weak limit?

Free energy via interpolation

$$\frac{d}{d\beta}\frac{1}{n}\log(Z_{n,\beta,h})$$

$$= \frac{d}{d\beta}\frac{1}{n}\log\Big(\sum_{x\in\mathcal{X}^V} \exp\big(\sum_{u\sim v}\beta\sigma_u\sigma_v + \sum_{u\in V} h\sigma_u\big)\Big)$$

$$= \frac{1}{n}\frac{\sum_{x\in\mathcal{X}^V} \sum \sigma_u\sigma_v \exp(\sum \beta\sigma_u\sigma_v + \sum h\sigma_u)}{Z}$$

$$= \sum_{u\sim v}\frac{1}{n}\mathbb{E}\,\sigma_u\sigma_v.$$

So $\frac{1}{n}\sum_{u\sim v}\mathbb{E}\,\sigma_u\sigma_v \to \frac{d}{2}\mu_{+,\beta,h}(\sigma_u\sigma_v)$.

and so

$$\frac{\partial}{\partial\beta}\Phi_{\beta,h} = \frac{d}{2}\mu_{+,\beta,h}(\sigma_u\sigma_v)$$

and $\Phi_{\beta,h} = \int_0^\beta \frac{d}{2}\mu_{+,s,h}(\sigma_u\sigma_v)\,ds. + \Phi_{0,h}$

for $h>0$. We have $\Phi_{0,h} = e^h + e^{-h}$

Then taking limit $h\downarrow 0$,

$$\Phi_{\beta,0} = 2 + \int_0^\beta \frac{d}{2}\mu_{+,s,0}(\sigma_u\sigma_v)\,ds.$$

* Can check that $\mu_{+,\beta,h}(\sigma_u\sigma_v)$ is continuous in β, h.

Under a Gibbs measure μ_m from m a BP fixed point,

$$\mathbb{E}_m[\sigma_u\sigma_v]$$
$$= \frac{(m_u m_v + (1-m_u)(1-m_v))e^{\beta}}{(m_u m_v + (1-m_u)(1-m_v))e^{\beta} + (m_u(1-m_v)+(1-m_u)m_v)e^{-\beta}}$$

where $m_u = m_{u\to v}(+)$, $m_v = m_{v\to u}(+)$.

- Maximized at $m_u = m_v = m_+$
  or $m_u = m_v = (1-m_+)$

The only such Gibbs measures are μ_+, μ_-.

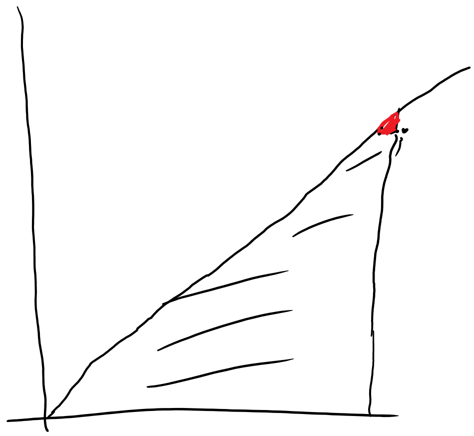
So we will have

$$\mathbb{E}_{G_n,\beta}\left[\frac{1}{n}\sum \sigma_u\sigma_v\right] \le \frac{d}{2}\mu(\sigma_u\sigma_v) + o(1).$$

For any graph $\mathbb{E}_{G,\beta}\sum \sigma_u\sigma_v$

is increasing in β.

If $\liminf_n \mathbb{E}_{G_n,\beta}\left[\frac{1}{n}\sum \sigma_u\sigma_v\right] \le \frac{d}{2}\mu(\sigma_u,\sigma_v) - \delta.$



then

$$\liminf \frac{1}{n} \log Z_{a_n}$$

$$\liminf_n \int_0^\beta \mathbb{E}_{a_n, s} \left[\frac{1}{n} \Sigma(\sigma_u \sigma_v)\right] ds + \log 2$$

$$\leq \int_0^\beta \frac{d}{2} \mu_{+, s}(\sigma_u \sigma_v)\, ds + \log 2 - \delta'$$

$$= \Phi_{\beta, 0} - \delta'.$$

Contradiction.

So $\lim \Sigma\, \mathbb{E}\, \sigma_u \sigma_v = \mu_+(\sigma_u \sigma_v)$

so local weak limit is a mixture of μ_+ + μ_-.

By symmetry it is $\frac{1}{2}(\mu_+ + \mu_-)$.

What about the anti-ferromagnetic model?

- Recall the Gibbs measures μ_F and two semi-translation invariant Gibbs measures

In each case $\lim_{\beta \to -\infty} \mu(\sigma_u \sigma_v) = -1$.

- But w.h.p. the graph is an expander, no set $S \subset V$ has $|E(S, S^c)| > (1-\delta) \frac{dn}{2}$.

So at least for large β something else must be happening!