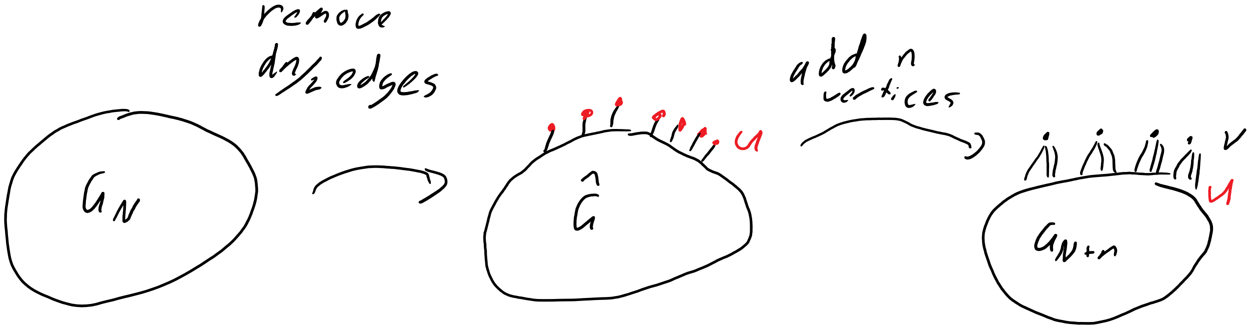
Interpolation

Sunday, April 29, 2018 10:34 PM

We want to move from G_N to G_{N+n}, how does the partition function change?

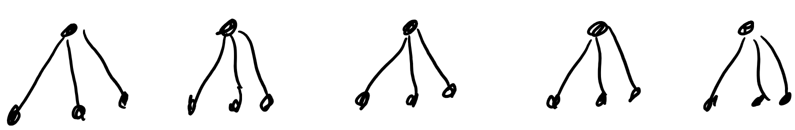


If we know $\hat{\mu}(\sigma_u)$ we could calcate

$$\frac{Z_{N+n}}{Z_N} = \frac{Z_{N+n}}{\hat{Z}} \Big/ \frac{Z_N}{\hat{Z}}$$

Interpolation Method and Upper Bounds on free energy

G^*



V n vertices

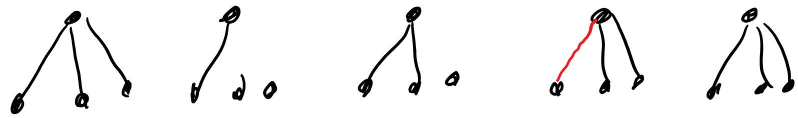
U dn vertices

Model:

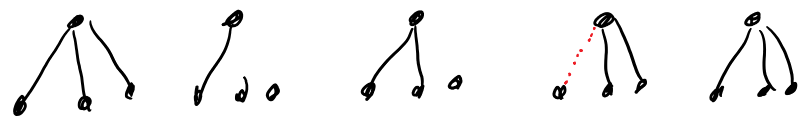
$$\mathbb{P}_m[\sigma] = \frac{1}{Z^*} \lambda^{\sum_v \sigma_v} \prod_{\substack{u \sim v \\ u \in U, v \in V}} \left(1 - (1-\delta)\sigma_u \sigma_v\right) \cdot \mu(\sigma_1, \ldots, \sigma_n)$$

* absorb $\lambda^{\Sigma \sigma_u}$ into $\prod m_u(\sigma_u)$.

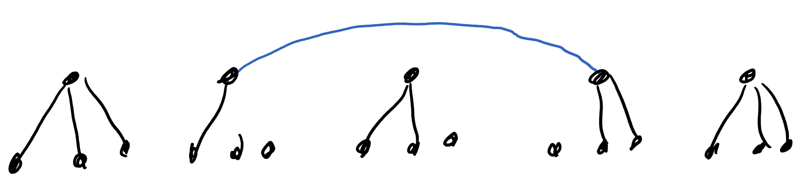
↳ remove $n^{2/3}$ edges

$G_0 =$ 

- For $r = 0, \dots, n - n^{2/3} - 1$
  - Pick an edge (u,v) in G_r, $u \in U, v \in V$. Remove it to form G_r'

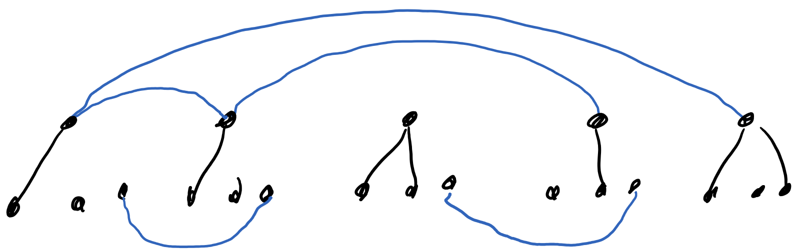
$G_0 =$ 

- With prob $\frac{1}{2}$, connect two half edges $v_1, v_2 \in V$.
- Otherwise connect two half edges $u_1, u_2 \in U$.

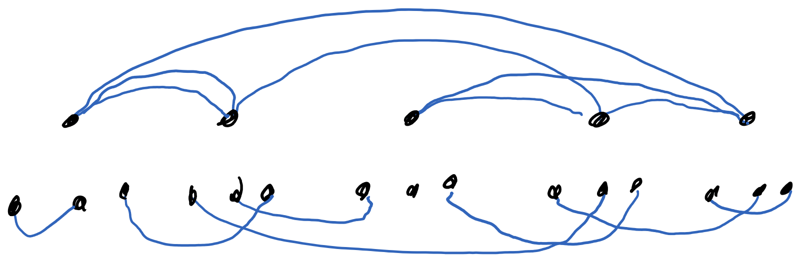
$G_1 =$ 

This is G_{r+1}.

G_m

$G_\ell =$ 

- Form G_n by matching remaining half edges within U and within V.

$G_n =$ 

Let $\Delta_r = \mathbb{E}\log\frac{Z_{r+1}}{Z_r} = \mathbb{E}\log\frac{Z_{r+1}}{Z_r'} - \log\frac{Z_r}{Z_r'}$

Let $u, u' \in U$, $v, v' \in V$ be random unmatched half edges

$$\mathbb{E}\log\frac{Z_r}{Z_r'} = \mathbb{E}\log\mathbb{E}\left[(1-(1-\delta)\sigma_u\sigma_v) \mid G_{r'}, u, v\right]$$

$$= -\mathbb{E}\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\mathbb{E}\left[\sigma_u\sigma_v \mid G_{r'}, u, v\right]^{\ell}$$

Let $\{\sigma^\ell\}_{\ell\geq 1}$ be IID copies of σ given $G_{r'}$.
and $I_u^p = \prod_{\ell\geq 1}^{p}\sigma_u^\ell$.

$$= -\mathbb{E}\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\,\mathbb{E}\Big[\Big(\prod_{\ell=1}^{p}\sigma_u^{\ell}\Big)\cdot\Big(\prod_{\ell=1}^{p}\sigma_v^{\ell}\Big) \,\Big|\, G_r';u,v\Big]$$

$$= -\mathbb{E}\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\,\mathbb{E}\big[\tau_u^p\,\tau_v^p \,\big|\, G_r';u,v\big]$$

Now let $A_p = \mathbb{E}[\tau_u^p \mid G_r', \tau]$, $B_p = \mathbb{E}[\tau_v^p \mid G_r', \tau]$

$$= -\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\,\mathbb{E}\big[\mathbb{E}[A_p B_p \mid G_r', \tau]\big]$$

$$\mathbb{E}\log\frac{Z_{r+1}}{Z_r'} = \frac{1}{2}\mathbb{E}\log\mathbb{E}\big[(1-(1-\delta)\sigma_u\sigma_{u'}) \mid G_r';u,u'\big]$$
$$+\frac{1}{2}\mathbb{E}\log\mathbb{E}\big[(1-(1-\delta)\sigma_v\sigma_{v'}) \mid G_r';v,v'\big]$$

$$= -\mathbb{E}\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\Big(\mathbb{E}[\sigma_u\sigma_{u'} \mid G_r';u,u']^p + \mathbb{E}[\sigma_v\sigma_{v'} \mid G_r';v,v']^p\Big)\Big/2$$

$$= -\sum_{p\geq 1}\mathbb{E}\,\mathbb{E}\Big[\frac{\tau_u^{(p)}\tau_{u'}^{(p)}}{2} + \frac{\tau_v^{(p)}\tau_{v'}^{(p)}}{2} \,\Big|\, G_r';u,u',v,v'\Big]$$

$$= -\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\,\mathbb{E}\big(\mathbb{E}[(A_p^2+B_p^2)/2 \mid G_r';\tau]\big]$$

So

$$\mathbb{E}\Delta_r = -\sum_{p\geq 1}\frac{(1-\delta)^p}{p}\,\mathbb{E}\Big[\mathbb{E}[(A_p^2+B_p^2)-A_pB_p \mid G_r', \tau]\Big]$$
$$\leq 0.$$

So $\mathbb{E}\log G_n \leq \mathbb{E}\log G^* + O(n^{2/3})$.

- G_n is a random d-regular graph plus $\frac{d}{n}n$ disconnected edges with external field m.
- G^* is n depth 1 trees with depth 1.

Let $\Phi_V = \frac{1}{n}\log Z_{G^*}$ $\quad \Phi_E = \frac{2}{dn}\log Z_{G_n \wedge u}$

$$\Phi_{Res_n} = \frac{1}{n}\log Z_{G_n \wedge V}.$$

We have shown that for any choice of μ,

$$\mathbb{E}\,\Phi_V \geq \mathbb{E}\,\frac{d}{2}\Phi_E + \Phi_{Res_n} + O(n^{-1/3})$$

and so

$$\mathbb{E}\,\Phi_{Res_n} \leq \mathbb{E}\left(\Phi_V - \frac{d}{2}\Phi_E\right)$$

$$\Leftrightarrow \mathbb{E}\log Z_n \leq \mathbb{E}\log Z_{N+n} - \log Z_n$$

$$\Leftrightarrow \mathbb{E}\log Z_n + \log Z_N \leq \mathbb{E}\log Z_{N+n} + O(n^{-1/3})$$

Superadditivity $\Rightarrow$ $\Phi = \lim \frac{1}{n}\log Z_n$ exists!

<u>Plugging in μ</u>

"Replica Symmetric Bound"

If μ is an IID measure with marginals m

$\mathbb{E}\,\Phi_{Res_n}$

$$\leq \inf_m \mathbb{E} \log \Big(\sum_{x_v \in \{0,1\}} \lambda^{x_v} \prod_{u \sim v} \sum_{x_u} m(x_u) \cdot \log(1-(1-\delta) x_v x_u) \Big)$$

$$- \frac{d}{2} \mathbb{E} \log \Big[\sum_{x_u, x_{u'}} m(x_u)\, m(x_{u'}) \log(1-(1-\delta) x_u x_{u'}) \Big]$$

1-Step Replica Symmetry Breaking (1RSB) Bound

A Poisson Dirichlet Process with parameter $\gamma \in (0,1)$ are the points of a Poisson process with intensity $x^{-1-\gamma}$, normalized to have sum 1.

Denote it $(a_i)_{i \geq 1}$. Let ξ_i be IID, then

$$\mathbb{E} \log \sum a_i \xi_i = \log\left(\mathbb{E}(\xi_i^{\gamma})^{1/\gamma}\right) + \log(\sum a_i)$$

$$= \frac{1}{\gamma} \log \mathbb{E}(\xi_i^{\gamma}).$$

P.f.: If Π is Poisson Point Process with density $x^{-1-\gamma}$, $\tilde{\Pi} = \{ x \xi_x : x \in \Pi \}$, also a P.P.P. with intensity at x,

$$\int f_{\xi}(u)\, \mathbb{E}[\#\text{pts in } [\tfrac{x}{u}, \tfrac{x}{u} + \tfrac{dx}{u}]]\, du$$

$$\int f_\xi(u)\, \mathbb{E}[\#\text{pts in } [\tfrac{x}{u}, \tfrac{x}{u} + \tfrac{dx}{u}]]\ du$$

$$= \int f_\xi(u) \left(\frac{x}{u}\right)^{-1-\gamma} \frac{dx}{u} \cdot du$$

$$= x^{-1-\gamma}\, dx \cdot \mathbb{E}[\xi^\gamma].$$

Let $\xi'_x = \xi_x / \mathbb{E}(\xi^\gamma)^{1/\gamma}$, then

$$\tilde{\Pi} = \{x \xi'_x : x \in \Pi\} \stackrel{d}{=} \Pi.$$

$$\mathbb{E} \log \sum_{x \in \Pi} x \xi'_x = \mathbb{E} \log \sum_{x \in \Pi} x \cdot \xi_x / \mathbb{E}[\xi^\gamma]^{1/\gamma}$$

$$= \mathbb{E} \log \sum_{x \in \Pi} x \xi_x - \log \mathbb{E}[\xi^\gamma]^{1/\gamma}$$

$$= \mathbb{E} \log \sum_{x \in \Pi} x$$

$$(a_i)_{i \geq 1} = \left(\frac{x_i}{\sum x_j}\right)_{i \geq 1}$$

$$\mathbb{E}\left(\log \sum_i a_i \xi_i\right) = \frac{1}{\gamma} \log \mathbb{E}\left[[\xi^\gamma]^{1/\gamma}\right].$$

Suppose μ is as follows:

- a_j – P.D. parameter γ.
- $m_{u,j}$ IID copies of $S \in \mathcal{P}(\mathcal{P}(\mathcal{X}))$

$\sim D(\Gamma$ [illegible]

- $m_{u,j}$ IID copies of $S \in P(P(\mathcal{X}))$
  $\cong P([0,1])$

$$\mu(x_U) = \sum_j a_j \prod_{u \in U} m_{u,j}(x_u)$$

Then

$$\Phi \leq \mathbb{E} \log\left[\sum_j a_j \left(1 + \lambda \prod_{u \sim v} (m_{u,j}(0) + \delta m_{u,j}(1))\right)^\gamma\right]$$

$$- \frac{d}{2} \mathbb{E} \log\left[\left(\sum_j a_j \left(1 - (1-\delta) m_{u,j}(1) m_{u',j}(1)\right)\right)^\gamma\right]$$

$$= \gamma^{-1} \log \mathbb{E}\left[\left(1 + \lambda \prod (1 - (1-\delta) m_u(1))\right)^\gamma\right]$$

$$- \gamma^{-1} \frac{d}{2} \log \mathbb{E}\left[\left(1 - (1-\delta) m_u(1) m_{u'}(1)\right)^\gamma\right]$$

Recall: Sum over WP configurations r,

$$Z = \sum w(r),$$

reweighted to those that exist,

$$Z_\gamma = \sum w(r)^\gamma.$$

When $\gamma = 0$ this is counting unweighted clusters.

$$SP_\gamma[s](A) = \frac{1}{Z_\gamma} \int I\left(\frac{\lambda \prod (1 - (1-\delta) r_i)}{1 + \lambda \prod (1 - (1-\delta) r_i)} \in A\right)$$

$$SP_\gamma[s](A) = \frac{1}{z_\gamma}\int I\left(\frac{\lambda\prod(1-(1-\delta)r_i)}{1+\lambda\prod(1-(1-\delta)r_i)} \in A\right)$$

$$\cdot\left(1+\lambda\prod(1-(1-\delta)r_i)\right)^\gamma \prod_{i=1}^{d-1} s(dr_i)$$

Choose γ, s_γ according to this prediction.

- For MAX I.S. $\delta \to 0$, $\lambda \to \infty$, $\lambda^\gamma \to \theta$

  then $S_{\delta,\lambda,\theta} \xrightarrow{w} s_\theta$

  our W.P. prediction with

  - $s_\theta(1) = q$, $s_\theta(0) = 1-q$
  - $$q = \frac{\theta(1-q)^{d-1}}{1+(\theta-1)(1-q)^{d-1}}.$$