Component size

Wednesday, April 12, 2017 10:06 PM

Component Size $G(n, d/n)$

- $G(n, d/n)$. Components $C_1, C_2, \ldots$ ordered such that $|C_1| \geq |C_2| \geq \ldots$

  - If $d < 1$ then $|C_1| \leq O(\log n)$

  - If $d > 1$ $|C_1| = (c_d + o(1))n$

    $|C_2| \leq \log n$ $\quad c_d = \mathbb{P}[GW(d) \text{ survives}] > 0.$

  - If $d = 1$, $|C_1| \asymp n^{2/3}$

    $|C_2| \asymp n^{2/3}$

Proof: Breadth first search

- A_t - active vertices
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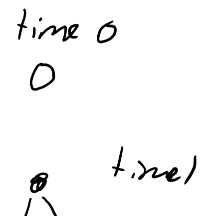
- $U_t = n - A_t - t$ unexplored vertices

$A_0 = 1,$



$A_0 = 1$

$A_1 = 2$

$A \;= A_{.} - 1 + W_{1}$

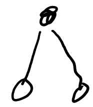
$$A_t = A_{t-1} - 1 + W_t$$

where $W_t = \mathrm{Bin}(U_t, d/n)$

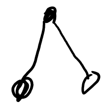
let $\tilde{W}_t = \mathrm{Bin}(n, d/n)$ IID

$W_t \le \tilde{W}_t$

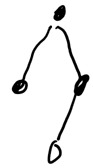
time1 $A_1 = 2$



time 2 $A_3 = 1$



time 3 $A_4 = 1$

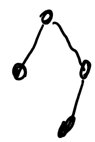


Then $A_t = 1 - t + S_t$

$\le 1 - t + \tilde{S}_t$

$S_t = \sum_{i=1}^{t} W_i$.

$A_5 = 0$.



$\mathbb{P}[\tilde{S} - t + 1 > 0] \le e^{-cn}$.

So $\mathbb{P}[|C_1| > C \log n] \to 0$.

Supercritical Case: $G(n, d/n)$ $d > 1$.

$U_t \ge n - \tilde{S}_t \underset{\text{w.h.p.}}{\ge} n(1-\delta)$ for $t < \delta' n$.

such that $(1-\delta)d > 1 + \varepsilon$

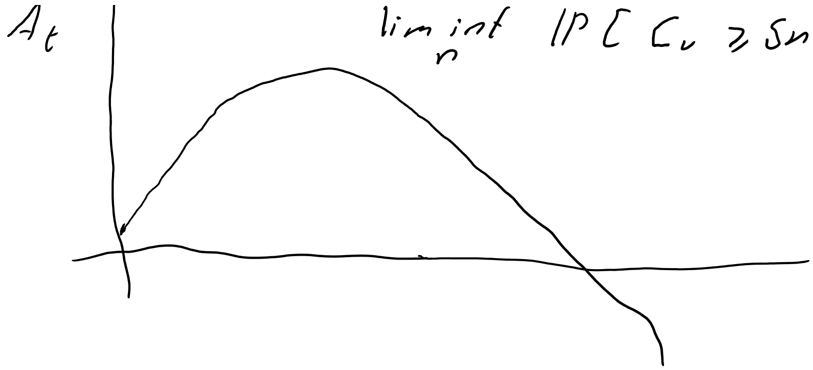
Set $\hat{W}_t \sim \mathrm{Bin}(n(1-\delta), \frac{d}{n}) \preceq W_t$ for $t < \delta' n$.

$A_t \ge 1 - t + \hat{S}_t$ a random walk with positive drift, so

$\mathbb{P}(\forall t \;\; \hat{S}_t - t \ge 0] > 0$

So $\mathbb{P}[\forall t \le \delta n, \; A_t \ge 1] > 0$.

$\liminf_n \mathbb{P}[C_v \ge \delta n] > 0$



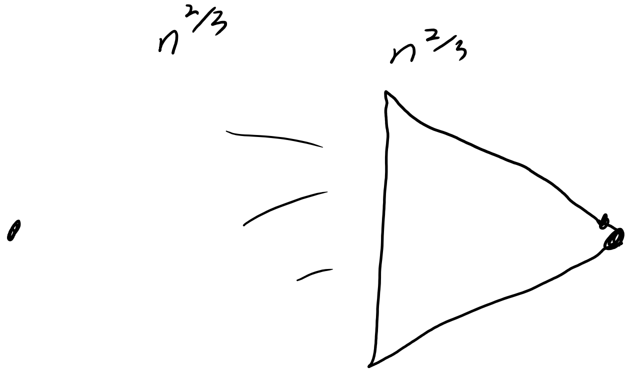
$\liminf_n \mathbb{P}[C_v \geq \delta n] > 0$

$\forall t \quad \mathbb{P}[\hat{S}_t - t \leq ((1-\delta)d - \frac{\varepsilon}{2})t] \leq e^{-ct}$

so

$\exists D$ s.t. $\mathbb{P}[D \log n \leq |C_v| \leq \delta n] \leq n^{-2}$.

— At most 1 giant component



connections $Bin(n^{4/3}, d/n)$

Concentration

Let A_v be the event

$A_v = \{|C_v| \geq D \log n\}$

If $\{v_1, \ldots, v_m\}$ random vertices, $m \approx n^{\varepsilon}$

then $\{A_{v_i}\}$ close in TV distance to a product measure since # revealed vertices is $O(n^{\varepsilon} \log n)$.

$$\mathbb{P}\left[\left|\frac{1}{m}\sum_{i=1}^{m} I_{A_{v_i}} - \mathbb{P}[A_{v_i}]\right| > \varepsilon\right] \to 0.$$

$$\Rightarrow \mathbb{P}\left[\Big||C_1| - \mathbb{P}[A_{v_i}]\Big| > \varepsilon\right] \to 0.$$

$$\mathbb{P}[A_{v_i}] = O\left(\tfrac{1}{n^2}\right) + \mathbb{P}[|C_v| \geq L]$$

$$- \mathbb{P}[L \leq |C_v| \leq \delta n]$$

$$\lim_{L\to\infty} \lim_{n\to\infty} \mathbb{P}[|C_v| < L] = 1 - \mathbb{P}[\text{Survives}]$$

$$\lim_{L\to\infty} \mathbb{P}[L \leq |C_v| \leq \delta n] \leq \lim_{L\to\infty} e^{-cL} = 0$$

So $\mathbb{P}[A_{v_i}] \xrightarrow[n\to\infty]{} \mathbb{P}[\text{Survives}]$

$$\frac{1}{n}|C_1| \xrightarrow[n\to\infty]{P} \mathbb{P}[\text{Survives}]$$

Critical Case

- Each time a component runs

out add a new vertex.

- N_t = # components visited by time t,
  $N_0 = 1$.

$$A_t = N_t - t + S_t \geq 1.$$

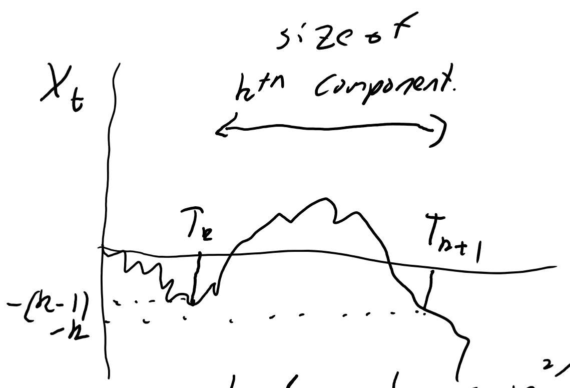
Recall $S_{t+1} - S_t \sim \mathrm{Bin}(U_t, \frac{1}{n})$
$$= \mathrm{Bin}(n - S_t - N_t, \tfrac{1}{n})$$

$$X_t = S_t - t$$

$$N_t = \min_{0 \le u \le t} X_u \quad +1$$

Let $T_k = \min\{t : X_t = -k+1\}$

Component size k is $T_{k+1} - T_k$



Let $t \le n^{2/3+\varepsilon}$.

$$\mathbb{E}[X_{t+1} - X_t \mid \mathcal{F}_t] = \mathbb{E}[\mathrm{Bin}(U_t, \tfrac{1}{n}) - 1 \mid \mathcal{F}_t]$$
$$= \frac{U_t}{n} - 1 = -\frac{1}{n}(S_t + N_t)$$

$$Var[S_{t+1} - (t+1) \mid \mathcal{F}_t] = Var(Bin(U_t, \tfrac{1}{n}) \mid \mathcal{F}_t)$$
$$= U_t \tfrac{1}{n}(1 - \tfrac{1}{n})$$

$Z_t = X_t + \frac{1}{n}\sum_{i=0}^{t-1} S_i + N_i$ is a martingale

and $Var(Z_t) = \sum_{i=0}^{t-1} \mathbb{E}\, U_i \frac{1}{n}(1-\frac{1}{n}) \leq t$

$S_t \approx t$ for $t = o(n)$.

$$N_{n^{3/4}} = \min_{0 \leq t \leq n^{3/4}} X_t = \min_t Z_t - \frac{1}{n}\sum_{i=0}^{t-1}(S_i + N_i)$$
$$\geqslant \min_t Z_t - \frac{n^{3/4}}{n}(S_{n^{3/4}} + n^{3/4})$$

$$\mathbb{P}[N_{n^{3/4}} \geqslant -n^{4/7}]$$
$$= \mathbb{P}[\min_{t \leq n^{3/4}} X_t < -n^{4/7}]$$
$$\leqslant \mathbb{P}[\max_{t \leq n^{3/4}} -Z_t \geqslant \tfrac{1}{2}n^{4/7}] \longrightarrow 0$$ Doobs maximal inequality

$$+ \mathbb{P}[\tfrac{n^{3/4}}{n}(S_{n^{3/4}} + n^{3/4}) \geqslant \tfrac{1}{2}n^{4/7}]$$
$\longrightarrow 0$ L.D. for S_t.

So for $t \simeq n^{2/3}$, $S_t + N_t \approx t$.

Rescaling :

$$Y_t = n^{-1/3} \left(S_{n^{2/3}t} - n^{2/3} t \right)$$

$\boxed{\text{Martingale CLT.}}$ $n^{-1/3} Z_{n^{2/3}t} \longrightarrow B_t$ (std B.M.)

since $\sum_{i=1}^{t} \mathbb{E}(M_{i+1} - M_i)^2 \approx t.$

(and no big jumps)

- $Y_t - n^{-1/3} Z_{n^{2/3}t}$

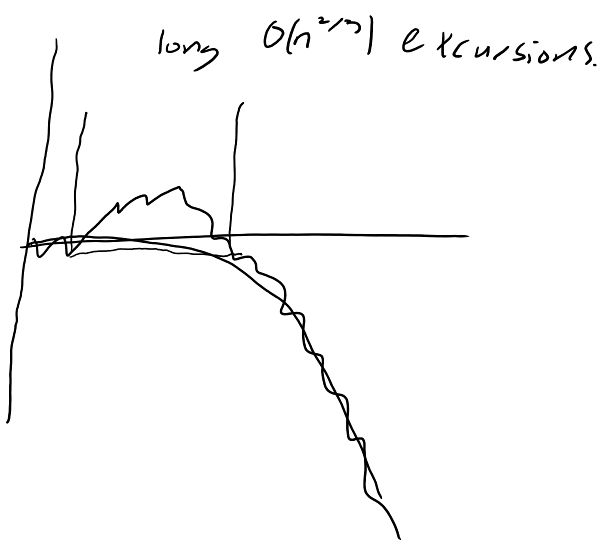
$$= - n^{-1/3} \frac{1}{n} \sum_{i=0}^{n^{2/3}t-1} S_i + N_i$$

$$\approx - n^{-4/3} \sum_{i=0}^{n^{2/3}t-1} i$$

$$\approx - n^{-4/3} \cdot \frac{1}{2} \left(n^{2/3} t\right)^2$$

$$= -\frac{1}{2} t^2$$

$$Y_t \xrightarrow{d} B_t - t^2/2$$



Cluster sizes

$$Z_t = Y_t - \min_{s \le t} Y_s$$

$n^{-2/3}$(Cluster sizes) $\Longleftrightarrow$ excursion lengths