Tails and Concentration

Thursday, January 26, 2017 3:55 PM

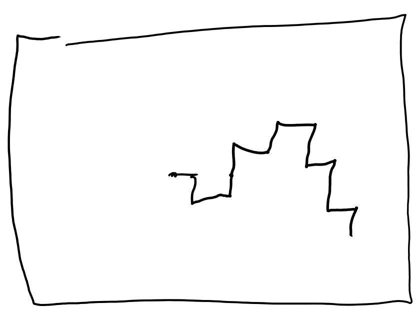
- Markov's Inequality / First moment method

$$\mathbb{P}[X > t] \leq \frac{\mathbb{E}X}{t}.$$

- Union bound

$$\mathbb{P}[\bigcup A_i] \leq \sum \mathbb{P}[A_i]$$

Ex: - 2D percolation



If $p < \frac{1}{3}$ then

$\mathbb{P}[\text{infinite component}] = 0$

Let γ be a self avoiding path from the origin

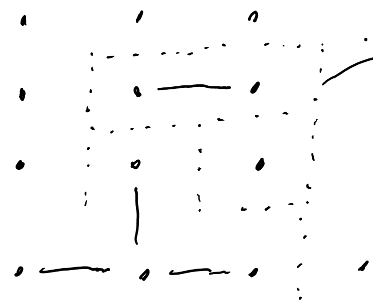
$\mathbb{P}[\gamma \text{ open}] = p^{|\gamma|}$

path S.A.W. from origin length k

$\leq 4 \cdot 3^{k-1}$

- $\mathbb{P}[\text{open } k\text{-SAW at } 0] \leq 4 \cdot 3^{k-1} \cdot p^k \to 0$

- Dual lattice



Dual edge

- percolation with $p' = 1 - p$

Phase transition at $\frac{1}{3} \le p_c \le \frac{2}{3}$, $p_c = \frac{1}{2}$.

- Random k-SAT n variables, $m = \alpha n$ clauses

Z_n = # solutions

$$\mathbb{E} Z_n = 2^n (1 - 2^{-k})^m$$

$$= \exp(n(\log 2 + \alpha \log(1 - 2^{-k})))$$

for $\alpha \geq 2^k \log 2$ $\mathbb{E} Z_n \to 0$, $\mathbb{P}[Z_n = 0] \to 1$.

Can show $\alpha_c \approx 2^k - c$.

Second moment method:

Apply Markov to $|X - \mathbb{E}X|^2$

$$\mathbb{P}[|X - \mathbb{E}X| > t] = \mathbb{P}[|X - \mathbb{E}X|^2 > t^2]$$

$$\leq \frac{\mathbb{E}(X - \mathbb{E}X)^2}{t^2} = \frac{\mathrm{Var}(X)}{t^2}.$$

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Basic example: WLLN

X_i IID, mean μ, $\mathbb{E}X_i^2 < \infty$

$\operatorname{Var}\sum X_i = n \operatorname{Var} X_i,$

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i > \mu + \varepsilon\right] \leq \frac{\frac{1}{n^2}\cdot n \operatorname{Var} X_i}{\varepsilon^2} \to 0.$$

Paley - Zygmund Inequality

For $X \geq 0$, $\theta \in [0,1]$

$$\mathbb{P}[X \geq \theta \mathbb{E}X] \geq (1-\theta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}$$

Proof:

$$\mathbb{E}[X] = \mathbb{E}[X\, I(X < \theta \mathbb{E}X)] + \mathbb{E}[X\, I(X > \theta\mathbb{E}X)]$$

$$\leq \theta\, \mathbb{E}[X] + \left(\mathbb{E}[X^2]\cdot \mathbb{P}[X > \theta \mathbb{E}[X]]\right)^{1/2}$$

Cauch - Schwartz

(2) Connectivity of random graphs $G(n, p_n)$

Let X_i indicator i is degree 0.

$\mathbb{E}X_i = \mathbb{P}[\operatorname{Bin}(n-1, p) = 0] = (1-p)^{n-1}$

$X = \sum_{i \in V} X_i$

$\mathbb{E}X = n(1-p)^{n-1}$

If $p_n \geq \frac{(1+\varepsilon)\log n}{n}$ then

$\mathbb{E}X \leq n e^{-(1+\varepsilon)\log n} = n^{-\varepsilon} \to 0$

If $p_n \leq (1-\varepsilon)\frac{\log n}{n}$

$\mathbb{E}X \geq n^{\varepsilon} \to \infty$

Not enough to say $\mathbb{P}[X > 0] \to 0$.

Second moment method,

$$\mathbb{E} X_i X_j = \mathbb{P}(\text{Bin}(2n-1, p) = 0)$$

$$= (1-p)^{2n-1}$$

$$\text{Cov}(X_i, X_j) = (1-p)^{2n-1} - ((1-p)^{(n-1)})^2$$

$$= (\mathbb{E} X_i)^2 \cdot p$$

$$\text{Var}[X] = \sum_{i,j} \text{Cov}(X_i, X_j) = n\, \mathbb{E}X_i (1 - \mathbb{E}X_i)$$

$$+ n(n-1)(\mathbb{E}X_i)^2 \cdot p$$

$$\frac{\text{Var}[X]}{(\mathbb{E}(X))^2} \to 0$$

$\mathbb{P}[X > 0] \to 1$.

- For $p_n \geq \frac{(1+\varepsilon)\log n}{n}$.

Let $S \subset V$, $1 \leq |S| \leq \frac{n}{2}$

$$\mathbb{P}[E(S, S^c) = 0]$$

$$= (1-p)^{|S|(n-|S|)}$$

$$\mathbb{P}[\text{Connected}] \leq \sum_{s=1}^{n/2} \sum_{S : |S| = s} (1-p)^{s(n-s)}$$

$$= \sum^{n/2} \binom{n}{\cdot} e^{-(1+\varepsilon)s(n-s)/n}$$

$$= \sum_{s=1}^{\cdot\cdot} \binom{n}{s} n^{-(1+\varepsilon)s(n-s)/n}$$

$$= \sum_{s=1}^{\delta n} \frac{n^s}{s!} n^{-(1+\varepsilon)(1-\delta)s} \to 0$$

$$\sum_{s=\delta n+1}^{n/2} 2^n n^{-\delta n/2} \longrightarrow 0$$

A more careful analysis says that if we add edges one by one w.h.p. connectivity same as no more degree 0 vertices.

- E.G. If $p_n = \frac{\log n + s}{n}$

then $\mathbb{E} X = e^{-s}$

Moreover $X \xrightarrow{d} Pois(e^{-s}) = Z$

So $\mathbb{P}[X=0] = e^{-e^{-s}}$

- Poisson approximation $X \xrightarrow{d} Poisson(Z)$

Show that $\mathbb{E} X^k \to \mathbb{E}(Poisson(Z))^k$

- easier $\mathbb{E}\, X(X-1)\dots(X-(k-1)) \to Z^k$

enough to show $\mathbb{P}[X_i = 1 \mid X_{j_1}, X_{j_2}, \dots X_{j_{n-1}}]$
$\approx \mathbb{P}[X_i = 1]$

Largest IS / Clique in $G(n, \frac{1}{2})$

– Same by symmetry

Let $X = X(m)$ = # cliques of size m.

$$\mathbb{E} X = \binom{n}{m} 2^{-\binom{m}{2}}$$

$$\leq \frac{n^m}{m!} 2^{-\binom{m}{2}}$$

$$\leq 2^{m \log_2 n - m(m-1)/2} \to 0$$

if $m > 2 \log_2 n + 4$

$$\mathbb{E} X \geq (1 + o(1)) 2^{m \log_2 (n/m) - m(m-1)/2}$$

$\to \infty$

if $n < 2\log_2 n - 2 \log \log n$

Set $m_* = \max_{m > 0} \binom{n}{m} 2^{-\binom{m}{2}} > \log n$

$$\frac{\binom{n}{m+1} 2^{-\binom{m+1}{2}}}{\binom{n}{m} 2^{-\binom{m}{2}}} = \frac{n-m}{m+1} \cdot 2^{-m} \leq n^{-1+o(1)}$$

So $\mathbb{E} X(m_*) \to \infty$ $\quad \mathbb{E} X(m_* + 2) \to 0$

– Maximum clique at most m_* w.h.p.

$$\mathbb{E} X(m_*)^2 = \mathbb{E}\Big[\sum_S I(S \in IS)\Big]^2$$

$$= \mathbb{E} \sum_{S, S'} I(S, S' \in IS)$$

$$= \sum \sum$$

$$= \sum_{S} \sum_{k} \sum_{S' : |S \cap S'| = k} \mathbb{P}[S, S' \in IS]$$

$$= \binom{n}{m_*} \cdot \sum_{k=0}^{m_*} \binom{m_*}{k} \cdot \binom{n - m_*}{m_* - k} 2^{-\binom{m_*}{2}} 2^{-\binom{m_* - k}{2}} 2^{-k(m_* - k)}$$

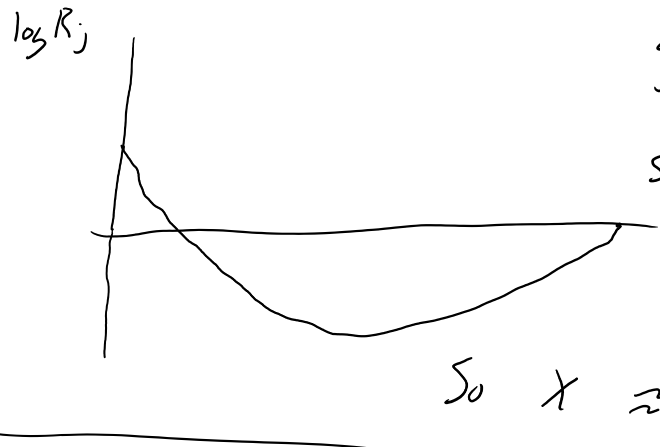
$$= \mathbb{E}X \underbrace{\sum_{k=0}^{m_*-1} \binom{n - m_*}{m_* - k} 2^{-\binom{m_* - k}{2}} 2^{-k(m_* - k)}}_{R_k}$$

$$R_0 \approx \mathbb{E}X, \quad R_{m_*} = 1$$

$$R_j / R_{j-1} = \frac{m_* - j + 1}{n - 2m_* + j} \cdot 2^{j-1}$$

For j small $\frac{R_j}{R_{j-1}} \ll 1$,

For $j \approx m_*$ $\frac{R_j}{R_{j-1}} \gg 1$



$$\sum_j R_j \leq (1 + o(1)) (\underline{R_0} + \underline{R_{m_*}})$$

Since $\mathbb{E}X \to \infty$,

$$\mathbb{E}X^2 / (\mathbb{E}[X])^2 \to 1.$$

So $X \approx \mathbb{E}X$ w.h.p.

Variance gives better concentration than first moment. Exponential moments even better

$$\mathbb{P}[X > t] = \mathbb{P}[e^{\alpha X} > e^{\alpha t}]$$

$$\leq \frac{\mathbb{E} e^{\alpha X}}{e^{\alpha t}}.$$

Binomial

$X \sim Bin(n, p)$

$$\mathbb{P}[X > n(p+\delta)]$$

$$\leq \mathbb{E} e^{\theta \Sigma X_i} e^{-\theta(p+\delta)n}$$

$$\mathbb{E} e^{\theta X_i} = e^{\theta} p + 1 - p = 1 + p(e^{\theta} - 1) \leq e^{p(e^{\theta}-1)}$$

$$\leq \exp(n(p(e^{\theta} - 1) - \theta(p+\delta)))$$

set $\theta \leq 1$ so $e^{\theta} \leq 1 + \theta + \theta^2$

$$\leq \exp(n(p(\theta + \theta^2) - \theta p - \theta\delta))$$

$$= \exp(n(p\theta^2 - \theta\delta)) \quad \text{Choose } \theta = \delta/2$$

$$\leq \exp\left(n\left(\frac{p\delta^2}{4} - \frac{\delta^2}{2}\right)\right) \leq \exp\left(-n\frac{\delta^2}{4}\right)$$

Johnson - Lindenstrauss Lemma

Let $x_1, \ldots, x_m \in \mathbb{R}^n$

If A $d \times n$ matrix IID $N(0,1)$, $L = \frac{1}{\sqrt{d}} A$

entries, $d \geq C\theta^{-2}(\log m + \log \delta)$

then

$$\mathbb{P}\left[\forall i,j: (1-\theta)\|x^{(i)} - x^{(j)}\|_2 \leq \|Lx^{(i)} - Lx^{(j)}\|_2 \leq (1+\theta)\|x^{(i)} - x^{(j)}\|_2\right]$$

$$\geq 1 - \delta.$$

<u>Proof</u>: $\displaystyle L(x^{(i)} - x^{(j)})_k = \sum_{\ell} \frac{A_{k\ell}}{\sqrt{d}}\left(x^{(i)}_{\ell} - x^{(j)}_{\ell}\right)$

$$\sim N\left(0, \tfrac{1}{d}\|x^{(i)} - x^{(j)}\|_2^2\right)$$

So $$\frac{\|L(x^{(i)} - x^{(j)})\|_2^2}{\|x^{(i)} - x^{(j)}\|_2^2} \sim \frac{1}{d}\chi^2(d-1)$$

e.g. $\displaystyle \frac{1}{d}\sum_{i=1}^{d} W_i^2$ where $W_i \sim N(0,1)$

$$\mathbb{P}\left[\sum_{i=1}^{d} W_i^2 > d(1+\theta)^2\right]$$

$$\mathbb{E}\, e^{sW_i^2} = \frac{1}{\sqrt{1-2s}}$$

$$\mathbb{E}\left[e^{s\sum_{i=1}^{d} W_i^2}\right] = (1-2s)^{-d/2}$$

Set $s = \frac{1}{2}\left(1 - \frac{1}{(1+\theta)^2}\right)$

$$= (1+\theta)^d$$

$\mathbb{P}\left[\quad s\sum W_i \quad \ldots^2\right]$

$> d(1+\theta)\}$

$$\leq \left((1+\theta)e^{-\theta-\theta^2/2}\right)^d$$

$$\leq \exp(-\tfrac{1}{2}\theta^2 d)$$

Large Deviations

Basic example X_i IID $\mathbb{E}X_i = \mu$, CDF $F(x)$

For $a > \mu$ what is $\mathbb{P}[\frac{1}{n}\sum_{i=1}^n X_i > a]$

How does it happen?

Ⓐ There is some $X_i > \delta n$.

$F_n(x) = \frac{1}{n}\#\{i : X_i \leq x\}$

$\approx F(x)$

Ⓑ $\max X_i = o(n)$

$F_n(x)$ different from $F(x)$

Answer: Depends on X_i.

Ⓐ Corresponds to heavy tailed R.V.'s

If $\mathbb{P}[X_i > t] \sim t^{-\alpha-1}$

then $\mathbb{P}[X_i > (a-\mu)n] \approx C n^{-\alpha}$

$$\mathbb{P}[\,|F_n(x) - F(x)| > \varepsilon\,]$$
$$= \mathbb{P}(\,|\mathrm{Bin}(n, F(x)) - F(x)n| > \varepsilon n)$$
$$\leq e^{-cn}.$$

(B) Light-tailed

Example: $X_i \sim N(\mu, \sigma^2)$

Then $\frac{1}{n}\sum X_i \sim N(\mu, \frac{1}{n}\sigma^2)$

$$\mathbb{P}[\,\tfrac{1}{n}\sum X_i > a\,] \approx e^{-(a-\mu)n/2\sigma^2 + o(n)}$$

$$X_i \mid \sum X_i = a \sim N(a, \sigma^2 \cdot \tfrac{n-1}{n})$$

$$\mathrm{Cov}(X_i, X_j) = -\frac{\sigma^2}{n}$$

$$F_n(x) \to \text{CDF of } N(a, \sigma^2)$$

M_n Satisfies a Large Deviation Principle with rate I if

$$-\inf_{x \in \Gamma^o} I(x) \leq \liminf_n \frac{1}{n}\log M_n(\Gamma)$$
$$\leq \limsup_n \frac{1}{n}\log M_n(\Gamma) \leq -\inf_{x \in \overline{\Gamma}} I(x)$$

- Theory of LD's tells us how F_n shifts under conditioning.

$$S_n = \sum_{i=1}^n X_i$$

Set $\varphi(\theta) = \mathbb{E}\, e^{\theta X_i}$ $\quad K = \log \varphi$

Assume $\varphi(\theta) < \infty$ for some $\theta > 0$.

Let $\gamma(a) = \lim_{n} \frac{1}{n} \log \mathbb{P}[S_n \geq an]$

- limit exist because

$$\pi_{n+m}(a) = \mathbb{P}[S_{n+m} \geq a(n+m)]$$
$$\geq \mathbb{P}[S_n \geq an]\,\mathbb{P}[S_m \geq am]$$
$$= \pi_n(a)\,\pi_m(a)$$

π superadditive so $\frac{\pi_n}{n} \to c$.

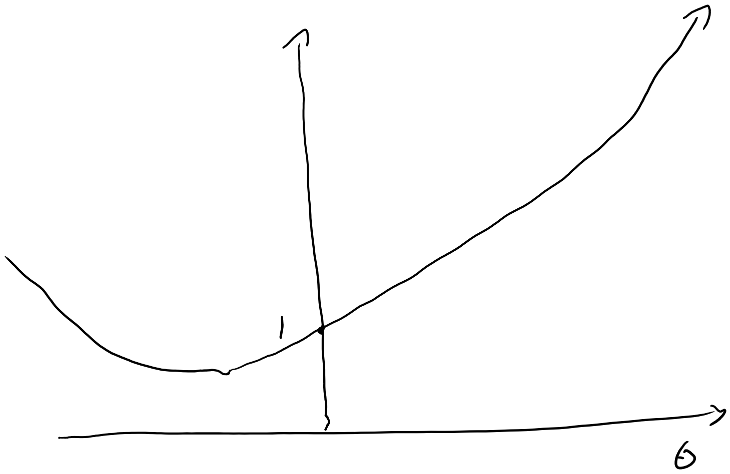
$$\mathbb{P}[S_n \geq an] = \mathbb{P}[e^{\theta S_n} \geq e^{\theta an}]$$
$$\leq \frac{\mathbb{E}e^{\theta S_n}}{e^{\theta an}} = \left(\frac{\mathbb{E}e^{\theta X_i}}{e^{\theta a}}\right)^n \leq \left(e^{K(\theta) - a\theta}\right)^n$$

So

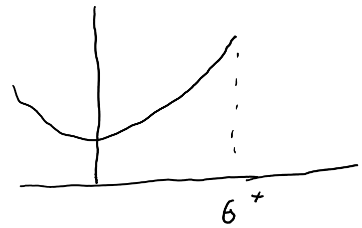
$$\gamma(a) \leq K(\theta) - a\theta$$
$$\leq \inf_{\theta} K(\theta) - a\theta$$

Legendre Transform of K.

Theorem: $\gamma(a) = \inf K(\theta) - a\theta$



Also



- φ is smooth.

$$\varphi'(0) = E\, X e^{\theta X}\big|_{\theta=0} = \mu, \quad K' = \frac{\varphi'(\theta)}{\varphi(\theta)}$$

- Result is meaningful for $a > \mu$

so $\frac{d}{d\theta} K(\theta) - a\theta \big|_{\theta=0} = \mu - a < 0$

$\Rightarrow \gamma(a) < 0.$

Change of measure Explain tilting better

$$F_\theta(x) = \frac{1}{\varphi(\theta)} \int_{-\infty}^{x} e^{\theta y}\, dF(y), \qquad \frac{dF_\theta}{dF}(x) = \frac{e^{\theta x}}{\varphi(\theta)}$$

$$K''(\theta) = \frac{\varphi''(\theta)}{\varphi(\theta)} - \left(\frac{\varphi'(\theta)}{\varphi(\theta)}\right)^2$$

$$= \int x^2 \frac{e^{\theta x}}{\varphi(x)}\, dF(x) - \left(\int x \frac{e^{\theta x}}{\varphi(x)}\, dF(x)\right)^2$$

$$= \mathbb{E}_\theta X^2 - (\mathbb{E}_\theta[X])^2 = \mathrm{Var}_\theta[X] \geq 0$$

So K is strictly convex unless F is a point mass.

$\exists\, \theta_a$ such that $a = K'(\theta_a)$ - the minimizer

$$\mathbb{E}_{\theta_a} X = \frac{1}{\varphi(\theta_a)} \int x e^{\theta_a x} dx = \frac{\varphi'(\theta_a)}{\varphi(a)} = K'(\theta_a) = a.$$

$$\frac{S_n - an}{\sqrt{n}\; K''(\theta_a)} \to N(0,1) \qquad \text{CLT}$$

$$\mathbb{P}[an \leq S_n \leq an + \sqrt{n}]$$

$$= \int_{an}^{an+\sqrt{n}} dF^n(x)$$

$$\geq \int_{an}^{an+\sqrt{n}} \frac{e^{\theta_a x}}{\varphi(\theta_a)^n} dF^n(x) \cdot \left(e^{-\theta_a(an+\sqrt{n})} \varphi(\theta_a)^n\right)$$

$$= \mathbb{P}_{\theta_a}[an \leq S_n \leq an + \sqrt{n}] \; e^{-(\theta_a \cdot a + o(1))n} \cdot \varphi(\theta_a)^n$$

$$\frac{1}{n} \log \mathbb{P}[S_n \geq an] \geq -\theta_a a + \log \varphi(\theta_a) + o(1)$$

$$= K(\theta_a) - \theta_a \cdot a.$$

Example

<u>Normal</u>: $K(\theta) = \theta^2/2$, $K'(\theta) = \theta$, $\theta_a = a$

$$\gamma(a) = a^2/2 - a^2 = -a^2/2$$

- Easy since $\sum X_i \sim N(0, n)$

Exponential:

$$\mathbb{E}\, e^{\theta X_i} = \int_0^\infty e^{\theta x - x}\, dx = \frac{1}{1-\theta}$$

$$K(\theta) = -\log(1-\theta) \quad K'(\theta) = \frac{1}{1-\theta} \quad \theta_a = \frac{a-1}{a}$$

$$\gamma(a) = -\log\left(1 - \frac{a-1}{a}\right) - \frac{a-1}{a}\cdot a$$

$$= -\log\left(\frac{1}{a}\right) - (a-1) = 1 = \log(a) - (a-1)$$

$$X_i \mid \sum X_i = an \stackrel{d}{=} an\,\mathrm{Beta}(1, n-1) \to \mathrm{Exp}\left(\frac{1}{a}\right)$$

Sanov's Theorem: Discrete Case

$\Sigma = \{a_1, \dots, a_n\}$ discrete space

$$L_n(y)(a_i) = \frac{1}{n}\#\{y_j : y_j = a_i\} \in M_1(\Sigma)$$

$$T_n(\nu) = \{y \in \Sigma^n : L_n(y) = \nu\}$$

Let μ be a measure on Σ a discrete space.

$$H(\mu) = -\sum_i \mu(a_i)\log \mu(a_i)$$

$$H(\nu \mid \mu) = \sum_i \nu(a_i)\log\frac{\nu(a_i)}{\mu(a_i)}$$

- convex
- non-negative
- minimized at $\nu = \mu$.

For $y \in T_n(\nu)$

$$\mathbb{P}_\mu[(X_1, \dots, X_n) = y] = \prod_i \mu(a_i)^{n\nu(a_i)}$$

$$= \exp\left(\sum_i n\nu(a_i)\log\mu(a_i)\right)$$

$$= \exp(-n\,(H(\nu) + H(\nu\mid\mu)))$$

Also

$$(n+1)^{-\Sigma} e^{nH(r)} \leq |T_n(r)| \leq e^{nH(r)}$$

- $1 \geq \mathbb{P}_r [L(Y) = r] \geq (n+1)^{-|\Sigma|}$ (mode)

$$= e^{-nH(r)} \cdot |T_n(r)|$$

- Mode under $\mathbb{P}_r^n$ is r.

$M_1(\Sigma)$ set of prob meas on Σ

$$-\inf_{r \in \Gamma_0} H(r|\mu) \leq \liminf_n \frac{1}{n} \log \mathbb{P}_\mu [L_n(Y) \in \Gamma]$$

$$\leq \limsup \frac{1}{n} \log \mathbb{P}_\mu [L_n(Y) \in \Gamma]$$

$$\leq -\inf_{r \in \Gamma} H(r|\mu)$$

Let $\Gamma_n = \{r \in \Gamma : |T_n(r)| \geq 1\}$

for $r \in \Gamma_n$

$$\mathbb{P}_\mu [L_n(Y) = r] = |T_n(r)| e^{-n(H(r) + H(r|\mu))}$$

$$\leq e^{-nH(r|\mu)}$$

$$\mathbb{P}_\mu [L_n(Y) = r] \geq (n+1)^{-|\Sigma|} e^{-nH(r|\mu)}$$

$$- \quad \mathbb{P}_\mu [L_n(Y) \in \Gamma] = \sum_{r \in \Gamma_n} \mathbb{P} [L_n(Y) = r]$$

$$\leq (n+1) \exp\left(-n \inf_{r\in\Gamma} H(r|\mu)\right)$$

If $r^* \in \Gamma^\circ$ then $\exists\, r_n \in \Gamma_n$ such that $H(r_n|\mu) \longrightarrow H(r^*|\mu)$

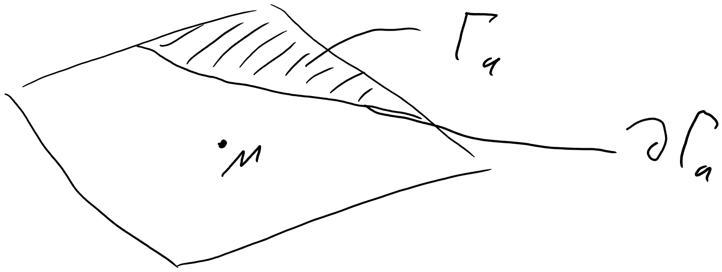
$$\liminf \tfrac{1}{n} \log \mathbb{P}_\mu[L_n(Y)\in\Gamma]$$
$$\geq \liminf \tfrac{1}{n} \log \mathbb{P}_\mu[L_n(Y) = r_n]$$
$$\geq \liminf \tfrac{1}{n} \log (n+1)^{-|\Sigma|} e^{-n H(r_n|\mu)}$$
$$= -H(r^*|\mu)$$

– Most of probability may be from one particular empirical distribution.

Re-visit

Let $\Gamma_a = \{r : \sum_i r_i a_i \geq a\}$

$$\mathbb{P}[S_n \geq an] = \mathbb{P}[L_n(Y) \in \Gamma_a]$$



Finding $\inf_{r\in\Gamma} H(r|\mu)$ Also Kullback-Leibler Divergence

Lagrange multipliers

$$\sum r_i \log\left(\frac{r_i}{\mu_i}\right) - \theta\left(\sum r_i a_i - a\right)$$

$$= \sum r_i \log(r_i/\mu_i) - \left(\sum r_i \log(e^{\theta a_i}/\varphi(\theta)) + K(\theta) - a\theta \right)$$

$$= \sum r_i \log\left(\frac{r_i}{\mu_i e^{\theta}/\varphi(\theta)}\right) - (K(\theta) - a\theta)$$

$$= H(r \mid \mu_\theta) - (K(\theta) - a\theta)$$

maximized at $r_i \equiv \dfrac{\mu_i e^{\theta}}{\varphi(\theta)}$

Large deviations - Binomial

$$\mathbb{P}[\,\mathrm{Bin}(n,p) \geq nq\,] = \exp(-n H(r \mid \mu) + o(n))$$

$$\mu = \mathrm{Ber}(p) \qquad r = \mathrm{Ber}(q)$$

$$= \exp\left(-n\left(q \log\left(\frac{q}{p}\right) + (1-q)\log\frac{1-q}{1-p}\right) + o(n)\right)$$

Configuration Model

... regular graph is hard to use

- random regular graph is hard to use directly

- perfect matching of dn - half edges
- might have
  - self loops 
  - multiple edges 
- Simple graph has no self loops / multiple edges
- $G \mid$ simple $\sim$ random d-regular

$$\#\text{ self loops} \to \mathrm{Pois}\Big(\underbrace{\tfrac{dn}{2}}_{\#\text{edges}} \cdot \underbrace{\tfrac{d-1}{dn}}_{\#\text{ choices for other end}}\Big) = \mathrm{Pois}\left(\tfrac{d-1}{2}\right)$$

$\#$ (pair) multiple edges

$$\to \mathrm{Pois}\left(\binom{n}{2} d^2 (d-1)^2 \cdot \frac{1}{(dn)^2}\right) = \mathrm{Pois}\left(\frac{(d-1)^2}{4}\right)$$

$$\#\text{selfloops} + \#\text{multedges} \to \mathrm{Pois}\left(\frac{d(d+1)}{2}\right)$$

$$\mathbb{P}[\text{ simple }] = \exp(-d(d+1)/2) > 0.$$

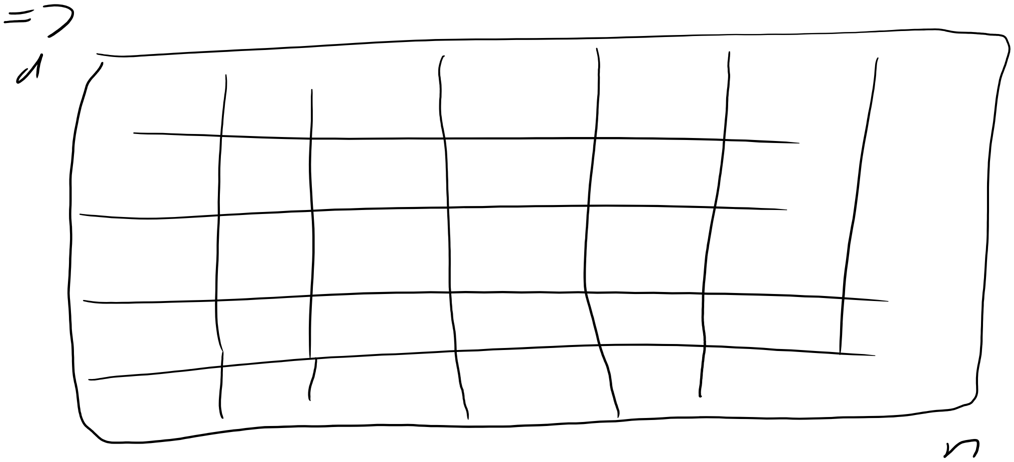
So $\mathbb{P}_{\text{Config}}[A] \to 0 \implies \mathbb{P}_{d\text{-reg}}[A] \to 0$

-

Ex: Random d-regular graph with $\frac{1}{2} z d n$ edges selected, $zd > 1$.

Find $\mathbb{P}\left[\begin{array}{l}\text{each vertex adjacent to a} \\ \text{selected edge}\end{array}\right]$

$\Longleftrightarrow$



Choose zn from $d \times n$ grid

- $W_i = \#$ in column i.

- $W_1, \dots, W_n$ IID $\mathrm{Bin}(d, z)$ given $\frac{1}{n} \sum W_i = zd$.

$$\mathbb{P}\left[\forall i,\ W_i \neq 0 \mid \tfrac{1}{n} \sum W_i = zd\right]$$

$$= \frac{\mathbb{P}\left[\forall i\ \ W_i \neq 0,\ \frac{1}{n} \sum W_i = zd\right]}{\mathbb{P}\left[\frac{1}{n} \sum W_i = zd\right]}$$

Denominator $\approx \dfrac{1}{\sqrt{2\pi\, dn\, z(1-z)}}$ LCLT

Numerator: Find

$$\inf_{\substack{r:\, r(0)=0 \\ \mathbb{E}_r[Y]=zd}} H(r|\mu)$$

Same as before but with the additional constraint, $r(0)=0$. Let $\mu_0(x) = \frac{\mu(x)\, I(x>0)}{1-\mu(0)}$

$$e^{\theta i}\mu_0(i) = \binom{d}{i} z^i (1-z)^{d-i} e^{\theta i}$$

$$\propto \binom{d}{i}\left(\frac{ze^{\theta i}}{ze^{\theta i}+1-z}\right)^i \left(\frac{1-z}{ze^{\theta i}+1-z}\right)^{d-i} \quad \text{Binomial.}$$

so r is $\mathrm{Bin}(d,p)$ conditioned to be positive

$$zd = \mathbb{E}_r Y = \frac{pd}{1-(1-p)^d}.$$

$$\mathbb{P}[\forall_i\ W_i \neq 0,\ \tfrac{1}{n}\textstyle\sum W_i = zd]$$

$$= \mathbb{P}[\tfrac{1}{n}\textstyle\sum W_i = zd \mid \forall_i W_i \neq 0]\cdot (1-(1-z)^d)^n$$

$$= \frac{(\varphi(\theta))^n}{e^{\theta z d n}}(1-(1-z)^d)^n\, \mathbb{P}_r[\tfrac{1}{n}\textstyle\sum W_i = zd]$$

$$\to \frac{1}{\sqrt{2\pi \mathrm{Var}_r(W)n}}$$

So $\mathbb{P}$[all vertices next to an edge]

$$= \frac{[\varphi_{\mu_0}(\theta)]^n}{e^{\theta z d n}}(1-(1-z)^d)^n \frac{\sqrt{\mathrm{Var}_\mu(W_i)}}{\sqrt{\mathrm{Var}_r(W_i)}}$$

Azuma - Hoeffding

If M_n is a martingale, $|M_i - M_{i-1}|_\infty \leq k_i$

$$\mathbb{P}[M_n - M_0 \geq t] \leq \exp\left(-t^2 / 2\sum_{i=1}^n k_i^2\right)$$

$$\mathbb{E}[e^{\theta M_i} \mid M_{i-1}] \leq \frac{e^{\theta k_i} + e^{-\theta k_i}}{2} e^{\theta M_{i-1}}$$

$$\cosh x = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \leq \sum_{j=0}^{\infty} \frac{x^{2j}}{2^j j!} = \exp(x^2/2)$$

$$\mathbb{E}\, e^{\theta(M_n - M_0)} \leq \exp\left(\tfrac{1}{2}\theta^2 \sum_{i=1}^n k_i^2\right)$$

$$\mathbb{P}[M_n - M_0 \geq t] \leq \exp\left(\tfrac{1}{2}\theta^2 \sum k_i^2 - \theta t\right)$$

Optimize over θ, $\theta = t / \sum k_i^2$.

If $g : \mathbb{R}^n \to \mathbb{R}^n$ is 1-Lipschitz,

i.e. $|g(x_1, \ldots, x_i, \ldots, x_n) - g(x_1, \ldots, x_i', \ldots, x_n)| \leq 1$

$w_1, \ldots, w_n$ independent $0 \leq w_i \leq 1$ then

$X = g(w_1, \ldots, w_n)$

Then $\mathbb{P}[|X - \mathbb{E}X| \geq t\sqrt{n}] \leq 2\exp(-t^2/2)$

Let $X_i = \mathbb{E}[X \mid W_1, \dots, W_i]$ martingale

Suppose W_i^* independent copy of W_i

$$X_i = \mathbb{E}[g(W_1, \dots, W_i, W_{i+1}, \dots, W_n) \mid W_1, \dots, W_i]$$

$$X_{i-1} = \mathbb{E}[g(W_1, \dots, W_i, W_{i+1}, \dots, W_n) \mid W_1, \dots, W_{i-1}]$$

$$= \mathbb{E}[g(W_1, \dots, W_i^*, W_{i+1}, \dots, W_n) \mid W_1, \dots, W_{i-1}, W_i]$$

$$|g(\dots, W_i, \dots) - g(\dots, W_i^*, \dots)| \le 1$$

$\Rightarrow |X_i - X_{i-1}| \le 1.$

$\Rightarrow \mathbb{P}[X - \mathbb{E}X \ge t\sqrt{n}] \le e^{-t^2/2}$ (A-H).

Example:

- Concentration of chromatic number of $G(n, d/n)$

X = chromatic number

$X_i = \mathbb{E}[X \mid \mathcal{F}_i]$ - Doob martingale

$\mathcal{F}_i$ - edges connected to vertices $1, \dots, i$.

X monotone in edge set.

Let $G^{i,+}$ be graph with all edge from i to $\{i+1, \dots, n\}$ present.

$G^{i,-}$ – no edges i to $\{i+1, \dots, n\}$

$\mathcal{F}_{i-1}$ measurable

$$\mathbb{E}[X^{i,+} \mid \mathcal{F}_i] \leq X_{i-1} \leq \mathbb{E}[X^{i,-} \mid \mathcal{F}_i]$$

$$0 \leq X^{i,-} - X^{i,+} \leq 1$$

So $|X_i - X_{i-1}| \leq 1$.

$$\Rightarrow \mathbb{P}[X_n - X_0 > t\sqrt{n}] \leq \exp\left(\frac{-t^2 n}{2n}\right)$$

$$\mathbb{P}[|X - \mathbb{E}X| > t\sqrt{n}] \leq 2e^{-t^2/2}$$

If $d_n = n^{-\alpha}$ $\alpha > 5/6$,

then $\exists\, u_n$ s.t.

$$\mathbb{P}[u_n \leq X \leq u_n + 3] \to 1.$$

Set $\varepsilon > 0$, choose

$$u_n = \min\{u : \mathbb{P}[X(n) \leq u] > \varepsilon/3\}$$

So $\mathbb{P}[X(n) < u_n] \leq \varepsilon/3$.

Let U be size of minimal set of vertices such that $G \setminus U$ is u_n colourable.

- $\mathbb{P}[U = 0] \geq \frac{\varepsilon}{3}$.

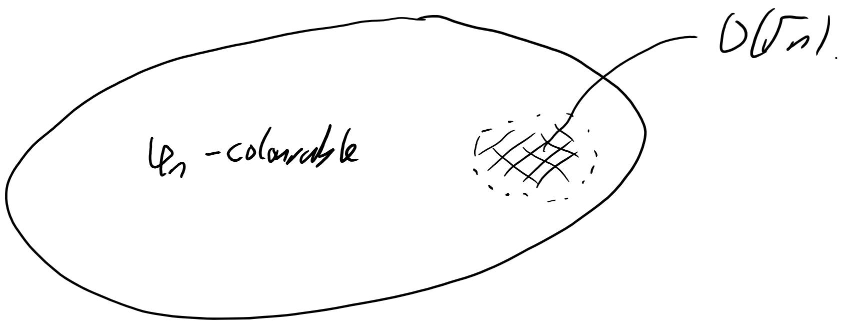
Let $U_i = \mathbb{E}[U \mid \mathcal{F}_i]$.

$|U_i - U_{i-1}| \leq 1$ so

$$\mathbb{P}[|U_i - \mathbb{E}U_i| > t\sqrt{n}] \leq 2e^{-t^2/2}.$$

$$\Rightarrow \mathbb{E}U_i \leq 2\sqrt{-\log(\varepsilon/6)}\,\sqrt{n}$$

$$\mathbb{P}[U_i > c(\varepsilon)\sqrt{n}] \leq \frac{\varepsilon}{3}.$$



Colour remaining vertices with 3 colours

- Let S be smallest non-3 colourable subset.

$\Rightarrow$ all vertices of S degree ≥ 3.

$\Rightarrow$ at least $\frac{3|S|}{2}$ edges

$$\mathbb{P}[\exists S, |S| \leq c\sqrt{n}, E(S) \geq \tfrac{3s}{2}]$$

$$\leq \sum_{s=4}^{c\sqrt{n}} \binom{n}{s} \cdot \binom{\binom{s}{2}}{\frac{3s}{2}} \cdot p^{3s/2}$$

$$\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$$

$$\leq \sum_s \left(\frac{en}{s}\right)^s \left(\frac{e(s-1)}{3}\right)^{3s/2} n^{-\alpha \cdot \frac{3s}{2}}$$

$$\leq \sum_s \left(\frac{en}{s}\right)^s \left(\frac{e(s-1)}{3}\right)^{\cdots} n^{-\alpha \cdot \frac{\cdots}{2}}$$

$$\leq \sum_s \left(\frac{n^{1-\frac{3}{2}\alpha}\, s^{1/2}\, e^{5/2}}{3^{3/2}}\right)^s$$

$$\leq \sum_s \left(C\, n^{\frac{5}{4}-\frac{3}{2}\alpha}\right)^s \to 0$$