PROBLEM SET 3

Do as many of the problems as you can. Some of the problems are not so easy, feel free to come and see me if you want to discuss any of them or some hints - I'm mainly interested in you really thinking about them.

Problem 1: Let G be and Erdos-Renyi random graph $G(n, d/n)$ and let $\mathcal{C}_1$ be the largest component of G (if there is a tie with equally large components, one is chosen uniformly at random). Let D_1 be the diameter of $\mathcal{C}_1$, show that with high probability the order of D_1 is

- $\log n$ when $d > 1$.
- $\sqrt{\log n}$ when $d < 1$.
- $n^{1/3}$ when $d = 1$.

Problem 2: With the same setup as Problem 1, when $d = 1$, find an explicit function $\epsilon(\delta) > 0$ such that

$$\lim_{n} \mathbb{P}[|\mathcal{C}_1| \geq \delta n^{2/3}] \geq 1 - \epsilon(\delta).$$

and satisfying $\epsilon(\delta) \to 0$ as $\delta \to 0$.

Problem 3: Let G be and Erdos-Renyi random graph $G(n, d/n)$ with $d > 1$. Let u and v be two vertices chosen uniformly at random from the giant component. Show that the graph distance $d(u, v)$ satisfies

$$\frac{d(u,v)}{\log_d n} \xrightarrow{p} 1$$

in probability as $n \to \infty$.

What can you say about the order of $|d(u, v) - \log_d n|$?

Problem 4: A *real tree* $(\mathcal{T}, d)$ is a compact metric space such that:

(1) For all $a, b \in \mathcal{T}$ there exists an isometry $f_{a,b} : [0, d(a, b)] \to \mathcal{T}$ such that $f_{a,b}(0) = a$, $f_{a,b}(d(a, b)) = b$.
(2) For any injective continuous function $q : [0, 1] \to \mathcal{T}$ with $q(0) = a, q(1) = b$ we have that

$$q([0, 1]) = f_{a,b}([0, d(a, b)]).$$

Let $x(t) : [0,1] \to \mathbb{R}_+$ be a continuous non-negative function with $x(0) = x(1) = 0$. We identify points i and j via the relation $i \sim j$ if

$$x(i) = x(j) = \min_{i \leq k \leq j} x(k).$$

Let $\mathcal{T}_x = [0,1]/\sim$ be the space under this identification and we give it the metric

$$d_x(i,j) = x(i) + x(j) - 2 \min_{i \leq k \leq j} x(k).$$

Verify that $(\mathcal{T}_x, d_x)$ is a real tree.