# FKG

Monday, March 27, 2017 9:50 PM

Basic FKG 1-dimension

$f, g$ increasing $\mathbb{E} f(X) g(X) \geq \mathbb{E} f(X) \mathbb{E} g(X)$

$X, X'$ independent

$\mathbb{E}\left[f(X) - f(X')\right]\left[g(X) - g(X')\right] \geq 0$ (same sign)

$= 2\left(\mathbb{E} f(X) g(X) - \mathbb{E} f(X) g(X')\right)$

We say $X$ stochastically dominates $Y$

$X \succcurlyeq Y$,

if $\mathbb{P}[X \geq t] \geq \mathbb{P}[Y \geq t]$.

Equivalent to $\exists$ coupling $X' \geq Y'$

$X = f(U)$, $Y = g(U)$, $f, g$ increasing

Also equivalent $\forall h(x)$ increasing

$\mathbb{E} h(X) \geq \mathbb{E} h(Y)$

Also true more generally

- partial order set $\mathcal{X}$, $x \geq y, y \geq x \Rightarrow x = y$
$x \geq y \geq z \Rightarrow x \geq z$

Set $\mathcal{X} = \{0,1\}^V$

$x \geq x'$ if $\forall i$ $x_i \geq x'_i$.

$A$ increasing set if $x \in A, y \geq x \Rightarrow y \in A$.

$X \succcurlyeq Y$ if $\forall A$ increasing $\mathbb{P}[X \in A] \geq \mathbb{P}[Y \in A]$

- Thm (Strassen)

$X \succcurlyeq Y \iff \exists$ monotone coupling $X' \geq Y'$

Proof: See notes of Roch

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Monotone Coupling: State space is partially ordered,

i.e. If $\forall x \leq y \Rightarrow P(x,\cdot) \preccurlyeq P(y,\cdot)$

then if $X_0 \preccurlyeq Y_0'$ $\exists$ coupling with

$X_t \preccurlyeq Y_t$.

- Proof: Induction each step.

Ex Lazy SRW on $\mathbb{Z}$.

If $X, Y$ have transition matrix $P, Q$,

$\forall x \leq y$ $P(x,\cdot) \preccurlyeq Q(y,\cdot)$

$\Rightarrow \quad X_t \preceq Y_t \qquad \mu_P \preceq \mu_Q$

Ex 2: Ising Model, Glauber Dynamics

$\mu(x) = \frac{1}{Z} \exp(\beta \sum x_i x_j)$

Suppose $x \le y$.

- update $v$: $\quad P_v(x,\cdot) \preceq P_v(y,\cdot)$

- Compare new distribution at $v$

$$\mu(x_v = +1 \mid x_{v^c}) = \frac{e^{\beta \sum_{j\sim i} x_j}}{e^{\beta \sum_{j\sim i} x_j} + e^{-\beta \sum x_j}} = \varphi\Big(\sum_{j\sim i} x_j\Big)$$

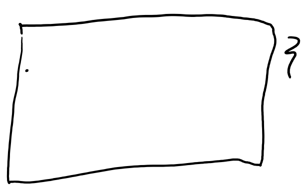
increasing function

$\le \mu(y_v = +1 \mid y_{v^c})$

$$P(x,\cdot) = \frac{1}{|V|}\sum P_v(x,\cdot) \preceq \frac{1}{|V|}\sum P_v(y,\cdot) = P(y,\cdot)$$

Let $\mu_\Lambda^\xi$ be Ising model on $\Lambda$

boundary condition $\xi$.



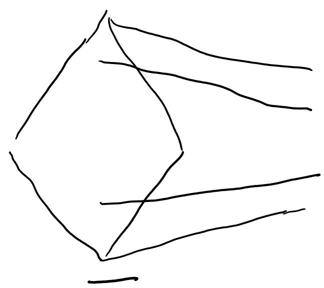
If $\xi \le \xi'$ then

$\mu_\Lambda^\xi \preceq \mu_\Lambda^{\xi'}$

Coupling: $X_t^x$ all plus minus — Grand coupling.

- Choose $v \in V$, $U \sim U[0,1]$
- Set to + if $U \le \varphi\Big(\sum_{j\sim i} x_j\Big)$

+



order preserved

If $X_t^+ = X_t^-$ then all states coupled.

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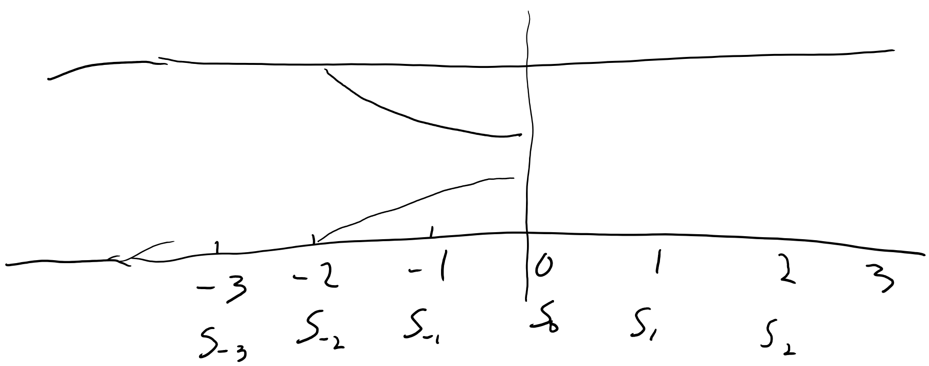
Coupling from the past.

Updates $S_i = (v_i, U_i)$




$f(x, S_1, S_2, \dots, S_t)$ - start at $x$ do updates $S_1, \dots, S_t$

$\xrightarrow{d} \pi$

$$f(x, S_{-t}, S_{-(t-1)}, \dots, S_{-1}) \xrightarrow{d} \pi$$

But

$$f(x, S_{-t}, \dots, S_{-1}) \xrightarrow{a.s.} Y_0 \text{ not depending on } x.$$

So $Y_0 \sim \pi$ exactly! Enough to find $t$ such that

$$f(+, S_{-t}, \dots, S_{-1}) = f(-, S_{-t}, \dots, S_{-1})$$

We say

$\mu$ has positive associations if

$$\mu(fg) \geq \mu(f)\mu(g) \qquad \mu(f) = \int f \, d\mu$$

Special case, if $A, B$ increasing events $1_A, 1_B$ increasing

$\mu(A \cap B) \geq \mu(A)\mu(B)$

$\mu(A | B) \geq \mu(A)$.

<u>Thm FKG</u> If $\mu$ measure on $\{0,1\}^V$,

$\forall \omega, \omega' \in \{0,1\}^V$

If

$$\mu(\omega \wedge \omega')\mu(\omega \vee \omega') \geq \mu(\omega)\mu(\omega').$$

then $\mu$ has positive associations.

<u>Example</u>: Percolation, product measure

Let $S = \{i : \omega_i \neq \omega_i'\}$

$$\mu(\omega \wedge \omega')\mu(\omega \vee \omega')$$

$$= \prod_{i \in S^c} \mu(\omega_i)\mu(\omega_i') \prod_{i \in S} \mu(\omega_i \wedge \omega_i')\mu(\omega_i \vee \omega_i')$$

$$= \prod_{i \in S^c} \quad '' \quad '' \quad \prod_{i \in S} \mu(\omega_i)\mu(\omega_i')$$

$$= \mu(\omega)\mu(\omega')$$

<u>Ex Ising model:</u> $\beta \geq 0$ $\omega \in \{-1,1\}^V$

$$\mu(\omega) = \frac{1}{Z} \exp\left(\beta \sum \omega_i \omega_j\right)$$

$\mu(\omega)\mu(\omega')$

$$= \frac{1}{Z^2} \exp\left(\beta \sum_{i\sim j} \omega_i \omega_j + \omega_i' \omega_j'\right)$$

$$(\omega_i \wedge \omega_i')(\omega_j \wedge \omega_j') + (\omega_i \vee \omega_i')(\omega_j \vee \omega_j') \quad (*)$$
$$- \omega_i \omega_j - \omega_i' \omega_j' \geq 0$$

If $\omega_i = \omega_i'$ or $\omega_j = \omega_j'$ then LHS $= 0$.

O.w. $(*) = 2$.

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More general <u>Holley's Thm</u>

If $\mu_1(\omega \wedge \omega')\mu_2(\omega \vee \omega') \geq \mu_1(\omega)\mu_2(\omega')$

Then $\mu_1 \preceq \mu_2$ (will assume $\mu_i(\omega) > 0$)

Holley $\Rightarrow$ FKG

- Let $f, g$ be increasing, WLOG positive

$$\mu_1 = \mu, \qquad \mu_2(\omega) = \frac{\mu(\omega)\, g(\omega)}{\mu(g)}$$

$$\mu_1(\omega)\mu_2(\omega') = \mu(\omega)\mu(\omega') \frac{g(\omega')}{\mu(g)}$$

$$= \mu(\omega \wedge \omega')\mu(\omega \vee \omega') \; g(\omega' \vee \omega)$$

$$= \mu(w \wedge w')\mu(w \vee w') \frac{g(w' \vee w)}{\mu(g)}$$

FKG Hypothesis
+ g increasing

$$= \mu_1(w \wedge w')\mu_2(w \vee w_1)$$

So $\mu_1 \preccurlyeq \mu_2$

$$\mu(f) \leq \mu_2(f) = \sum_w \frac{\mu(w) g(w) f(w)}{\mu(g)}$$

$$= \mu(fg)/\mu(g) \quad \blacksquare$$

## Proof of Holley:

Via Coupling $P, Q$ transition matricies with S.D $\mu_1, \mu_2$. Chooses $\alpha > 0$ small

Notation: $x^{i,\gamma}(j) = \begin{cases} \gamma & j = i \\ x(j) & \text{o.w.} \end{cases}$

$$P(x^{i,0}, x^{i,1}) = \frac{\alpha}{n}$$

$$P(x^{i,1}, x^{i,0}) = \frac{\alpha}{n} \frac{\mu_1(x^{i,0})}{\mu_1(x^{i,0})}$$

$$P(x, x) = 1 - \sum_y Q(x, y) \geq 0 \text{ if } \alpha \text{ small.}$$

Detailed balances $\mu_1$ is stationary

Detailed balance $\Rightarrow \mu_1$ is stationary.

$\boxed{\text{Take } x \le y.}$ If $y_i = 1$, $x_i = 0$ then

$$P(x, x^{i,1}) \ll P(y, y^{i,1})$$

for small $\alpha$.

When $x_i = y_i = 1$

enough to check

$$\frac{\mu_1(x^{i,0})}{\mu_1(x^{i,1})} \geq \frac{\mu(y^{i,0})}{\mu(y^{i,1})}$$

Take $w = x^{i,1}$ $w' = y^{i,0}$

$\Rightarrow X_t \le Y_t$ $\mu_1 \preceq \mu_2$

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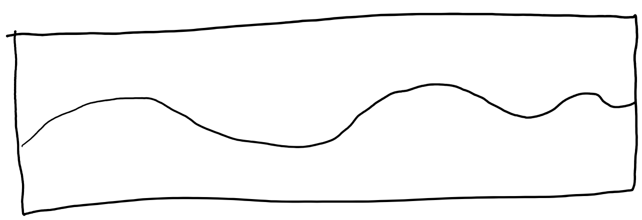
Percolation Harris' Thm $\Theta(1/2) = 0$

$n$



$\alpha n$

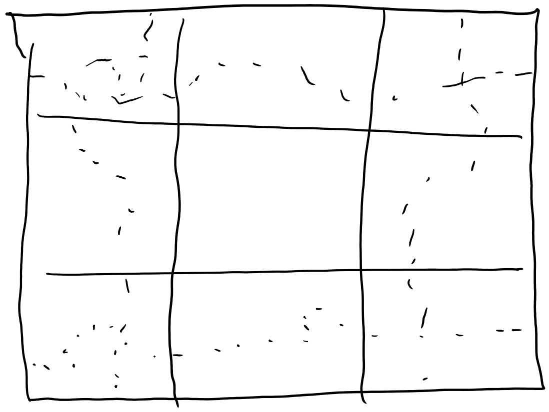
$C_{\alpha,n} = \mathbb{P}[\text{crossing } (\alpha n \times n) \text{ box}]$

RSW: Russo-Seymour-Welsh when $p = 1/2$

$$\inf_n C_{\alpha,n} = C_\alpha > 0.$$

## RSW $\Rightarrow$ Harris

$\mathbb{P}[\text{dual crossing } n \times \alpha n] = C_{\alpha n}$

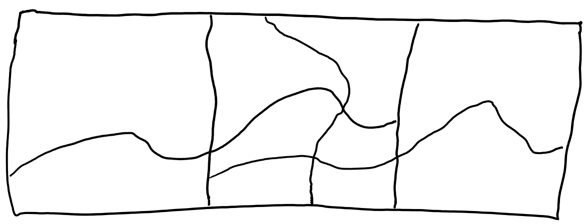


$3n \times 3n$

$\mathbb{P}[\text{Dual crossing around annulus}] \geq C_{3,n}^4$.

## Proof of RSW

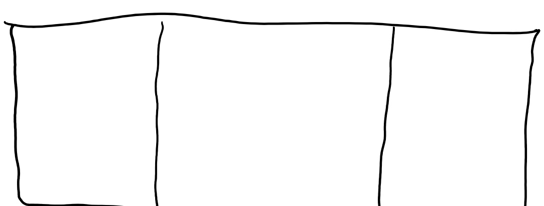
A) $C_3 \geq C_2^2 C_1$



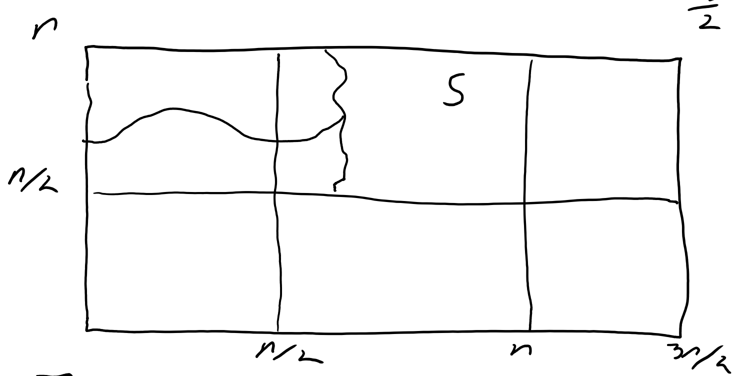


$LR(B_1 \cup B_2) \cup LR(B_2 \cup B_3) \cup TB(B_2)$

B) $C_2 \geq C_{3/2}^2 C_1$

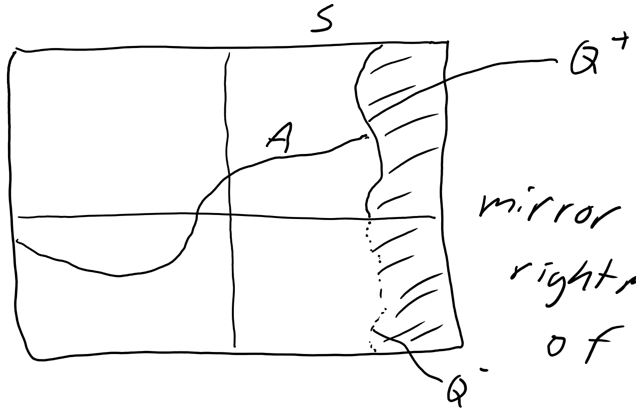


(c) $\frac{3}{2} n \times n$




$\Gamma$ = TB(S) connected to LHS of

$\mathbb{P}[\Gamma]$




mirror image of rightmost TB crossing of S

Let

$A^{\pm}$ = Left $\longleftrightarrow Q^{\pm}$

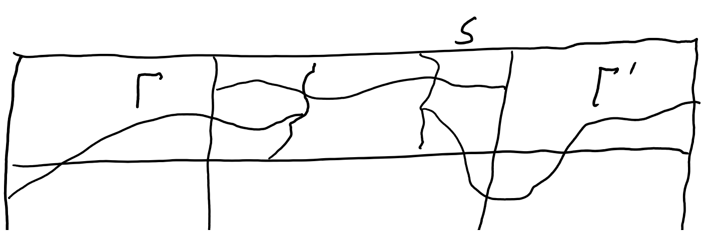
$\mathbb{P}[A^+ \mid Q] = \mathbb{P}[A^- \mid Q] \geq \frac{1}{2} \mathbb{P}[A \mid Q]$ $(A = A^+ \cup A^-)$

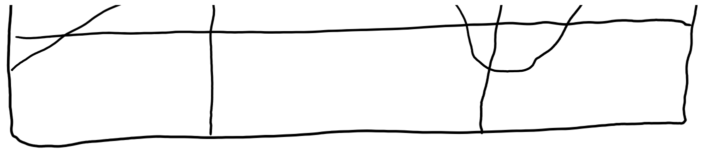
by symmetry since bonds left of Q are independent of $\{Q = q\}$.

$\mathbb{P}[\Gamma] \geq$

$\frac{1}{2} \mathbb{P}[A \cap Q] \geq \frac{1}{2} \mathbb{P}[A] \cap \mathbb{P}[Q] = \frac{1}{2} c_1^2$

D)





$$C_{\frac{3}{2}} \geq \mathbb{P}[\Gamma \cap \Gamma' \cap LR(S)] \geq \left(\tfrac{1}{2} c_1\right)^2 \cdot c_1$$

$$\geq 1/32.$$

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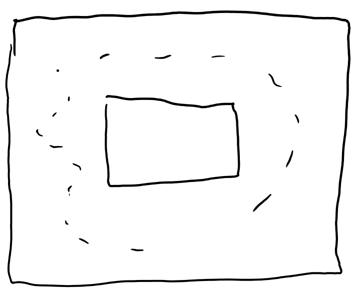
<u>Proof</u> Harris' Thm $\theta(\tfrac{1}{2}) = 0$

Let $A_n$ be event of a dual circuit in the annulus



$[-3n, 3n]^2 \setminus [-n, n]^2$

$\{0 \longleftrightarrow \infty\} \subset A_n^c$

$\mathbb{P}[A_n] \geq c_3^4 > 0$

$A_{3^k}$ $k \in \mathbb{N}$ independent

$\mathbb{P}[0 \longleftrightarrow \infty] \leq \mathbb{P}\left[\bigcap_{k \geq 1} A_{3^k}^c\right] = 0$

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Let $A$ be increasing

Edge $e$ is pivotal in $A$ for $\omega$ if

$\omega^{e,0} \notin A$, $\omega^{e,1} \in A$.

## Russo's Formula:

$$\frac{d}{dp} \mathbb{P}_p[A] = \sum_e \mathbb{P}_p[e \text{ is pivotal}]$$

$$LHS = \lim_{\delta \to 0} \frac{1}{\delta}\left(\mathbb{P}_{p+\delta}[A] - \mathbb{P}_p[A]\right)$$

$$= \lim_{\delta \to 0} \frac{1}{\delta}\left( \sum_e \mathbb{P}[e \text{ is pivotal}, w_e(p)=0, w_e(p+\delta)=1] + O(\delta^2)\right)$$

$$= \sum_e \mathbb{P}_p[e \text{ is pivotal}]$$

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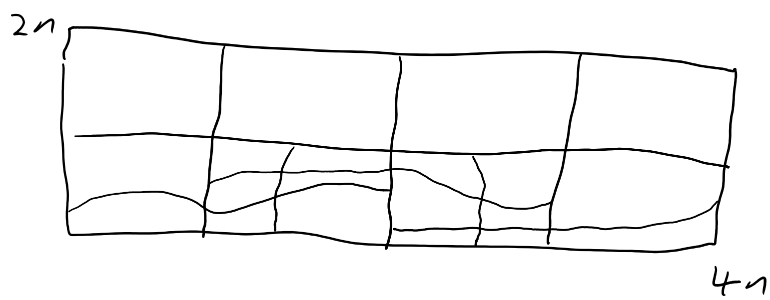
Enough to show $C_{2,n}(p) \geq 1 - 1/100$

If $C_{2,n}(p) \geq 1-\varepsilon$ then

$C_{2,2n}(p) \geq 1 - 25\varepsilon^2$

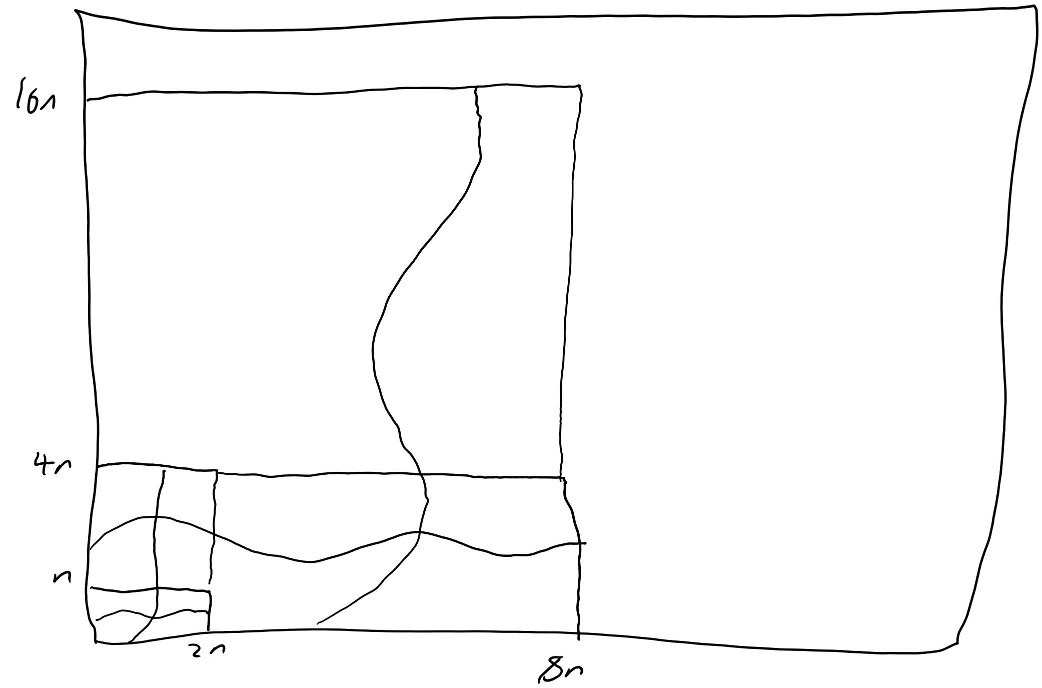
(for $\varepsilon \leq \frac{1}{100}$) $\geq 1 - \varepsilon/2$

Pf:




$$C_{2,2n} \geq 1 - (1 - C_{4,n})^2$$

$$C_{4,n} \geq C_{2,n}^5 \geq 1 - 5\varepsilon$$




$\mathbb{P}_p[\exists \text{ infinite component}]$

$$\geq \mathbb{P}_p\Big[\bigcap_{k\geq 0} LR[2\cdot 4^k n, 4^k n] \cap TB[2\cdot 4^k n, 4^{k+1} n]\Big]$$

$$\geq \prod \Big(1 - \frac{1}{2^k\cdot 100}\Big) > 0.$$

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If $p_c > \frac{1}{2}$ $\qquad J_n = LR[2n, 4n]$

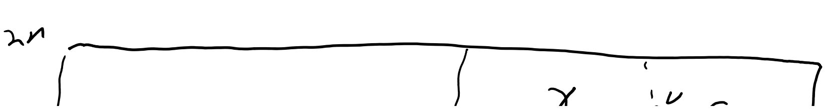
$V_n = \#$ pivitols of $J_n$

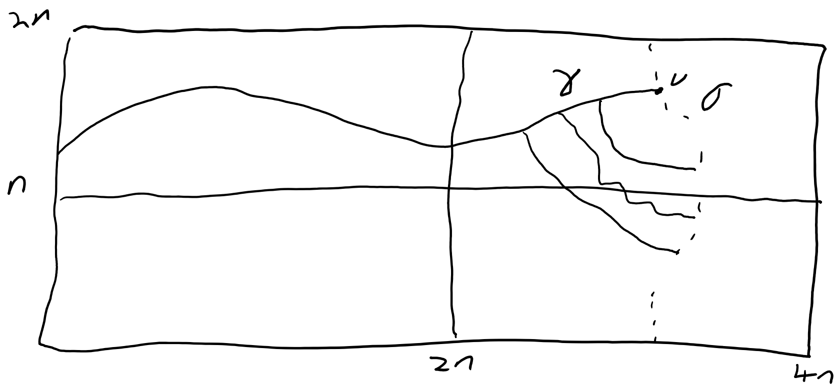
Let $p' = \frac{1}{2}(\frac{1}{2} + p_c)$

$\mathbb{P}_{p'}[J_n] \leq 99/100$

$$\mathbb{P}_{p'}[J_n] \geq \int_{1/2}^{p'} \frac{d}{dp}[J_n]\, dp$$

$$\geq (p' - \tfrac{1}{2}) \inf_{\frac{1}{2}\leq p \leq p'} \mathbb{E}_p[V_n]$$






$U_n$ = event TB dual path $[2n, 4n] \times [0, 2n]$

$\sigma$ - rightmost such path

$W_n$ = LR crossing to $\sigma$ in $[0, 4n] \times [n, 2n]$

$\mathbb{P}_p[U_n, W_n] \geq (1/100)^2$

$\gamma$ - top most crossing

$v$ - intersection point.

$A_e$ - open path from $\gamma$ to $\sigma$ in

Annuls radius, $3^e, 3^{e+1}$ centered at $v$.

$\mathbb{P}_p[A_e] \geq \varepsilon$.

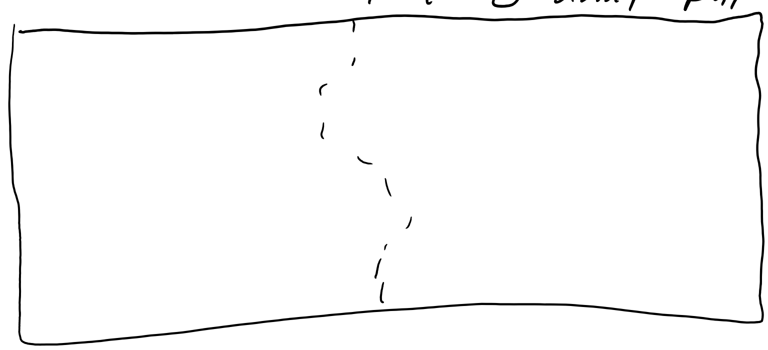
$\mathbb{E}\, V \geq \mathbb{P}[U \wedge W] \sum \mathbb{P}_p[A_e] \to \infty$.

<u>Contradiction</u>

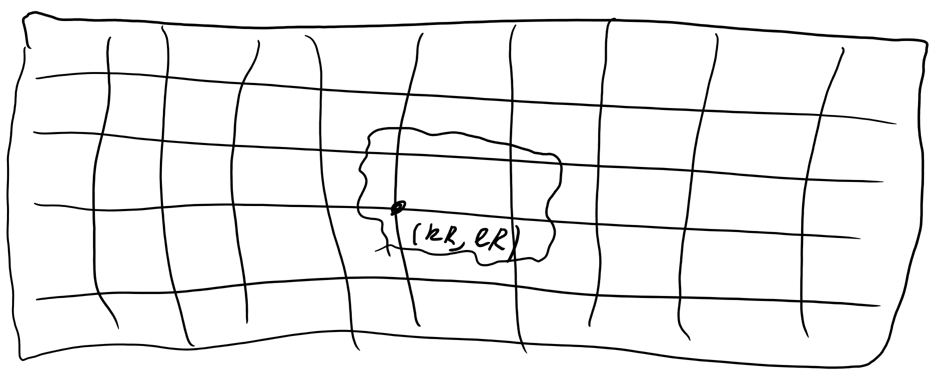
<u>Example</u>: If $p > p_c$ then

$A$ = crossing a $n \times n$ Box

$\mathbb{P}_p[A] \geq 1 - e^{-cn}$



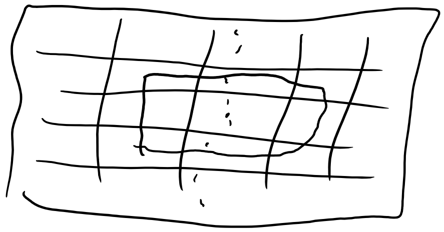

Renormalize




Let $R$ be a large constant.

Let $Y_{k,\ell}$ event that exists circuit in annuls of width $3n \times 3n$ surrounding box $(k,\ell)$. $\mathbb{P}[Y_{k,\ell}] \geq 1 - \varepsilon(R)$.

If the dual circuit passes through box $(k,\ell)$ then $Y_{k,\ell}^c$ must occur



If $\|(k,\ell)-(k',\ell')\|_\infty \geq 3$ then

$Y_{k,\ell}$ & $Y_{k',\ell'}$ independent.

- Let $S$ be the set of boxes that dual crossing touches.

- For any $S$ such that