Ergodic Theorem

Monday, February 13, 2017 7:18 PM

Kolmogorov's 0/1 Law

- Tail σ-algebra

$$\mathcal{T} = \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \dots)$$

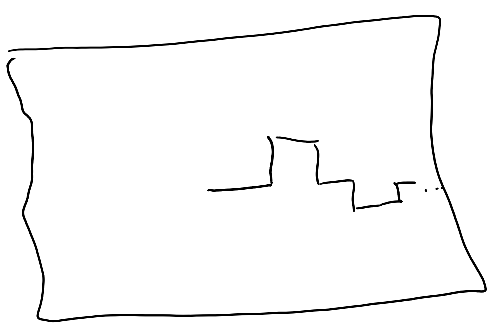
Example: $\limsup \frac{1}{n} \sum_{i=1}^{n} X_i > a$

$A \in \mathcal{T}$ if changing finitely many X_i does not affect the occurrence.

Theorem: If $A \in \mathcal{T}$, $\mathbb{P}[A] = 0$ or 1.

Percolation

$\mathbb{P}[\exists$ infinite component$] = 0$ or 1.



$\exists$ infinite path, not affected by finitely many bonds

Stochastic Process $X_0, X_1, \dots$ is stationary if $\forall m, n$

$$(X_0, \dots, X_m) \stackrel{d}{=} (X_n, \dots, X_{n+m})$$

E.G. IID, Markov Chain

On $(\Omega, \mathcal{F}, \mathbb{P})$, $\varphi : \Omega \to \Omega$ is
<u>measure preserving</u> if

$$\mathbb{P}[\varphi A] = \mathbb{P}[A].$$

- If $X(\omega)$ is a R.V.,

$X_n = X(\varphi^n \omega)$ is <u>stationary</u>

$\mathcal{I}$ is <u>invariant</u> σ-algebra.

If $\mathcal{I}$ <u>trivial</u> then X_n is <u>ergodic</u>.

<u>Example</u> $\Omega = \mathbb{R}^{\mathbb{N}}$ $\quad \varphi(x_0, x_1, \dots) = (x_1, x_2, \dots)$

left shift

Kolmogorov's 0/1 Law implies $\mathcal{I}$ is trivial.

- Also applies to $\Omega = \mathbb{R}^{\mathbb{Z}}$

- If $X(\varphi^n \omega)$ is constant a.s. then $\{X_n \in B\} \in \mathcal{I}$

and so $\mathbb{P}[X_n \in B] = 0$ or 1.

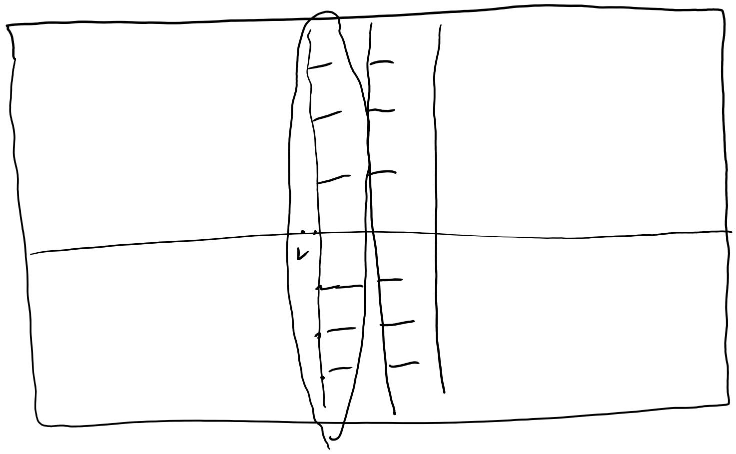
- If $X_{-1}, X_0, X_1, \dots$ IID Ber$(\frac{1}{2})$

$Y_{-1}, Y_0, Y_1, \dots$

Does $\exists\, \varphi$ increasing, Lipschitz s.t.

$Y_{\varphi(i)} = X_i$

Percolation



Let Y_e indicator of edge e.

Let $E_i = \{((x,y),(x',y')) \in E : \min\{x,x'\} = i\}$

Y_{E_i} is IID sequence.

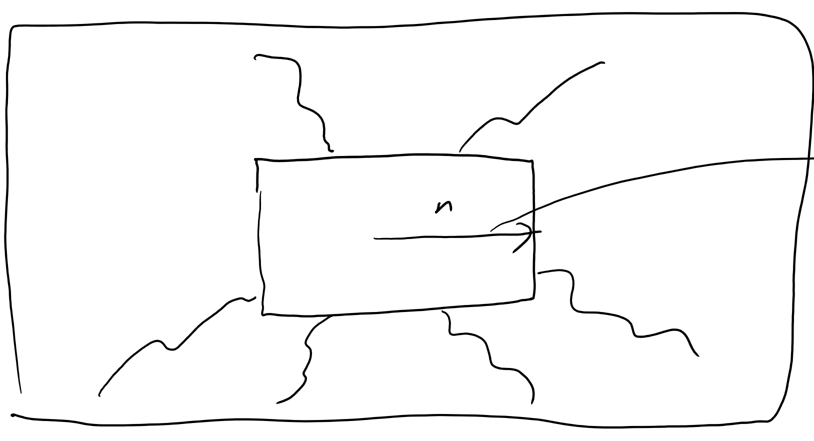
Let X = # infinite components.

$X = X(\dots, Y_{E_0}, Y_{E_1}, \dots)$

is shift invariant.

$\Rightarrow$ # infinite components is constant a.s.

Is it $\exists$? Suppose # components $= k < \infty$.



Reveal everything except B_n

$\mathbb{P}[\,k$ - infinite components touching ∂B_n in $\mathbb{Z} \setminus B_n\,] > 0$.

in $\mathcal{E} \setminus B_n$

Edges in B_n independent. If B_n is fully connected then # components $=1$

$$\mathbb{P}[\#\text{components} = 1] > 0.$$

$\Rightarrow$ # components $= 1$ or ∞.

<u>Birkhoff's Ergodic Theorem</u>

If $\mathbb{E}|X| < \infty$ then

$$\frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m \omega) \to \mathbb{E}[X \mid \mathcal{I}] \text{ a.s. \& in } L^1.$$

If X is ergodic then $\mathbb{E}[X \mid \mathcal{I}]$ constant a.s.

E.G. X_i IID $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s} \mathbb{E}X$ SLLN

<u>Percolation</u>: Let Z_x be a translation invariant function of x.

E.G. $Z_x = Z(\tau_x^{-1} Y)$.

- $Z = I(0 \in \text{infinite component})$ $\mathbb{E}Z = \theta_p$.

Then $\frac{1}{n}\sum_{i=1}^{n} Z_{i \cdot e_1} \to \theta_p$

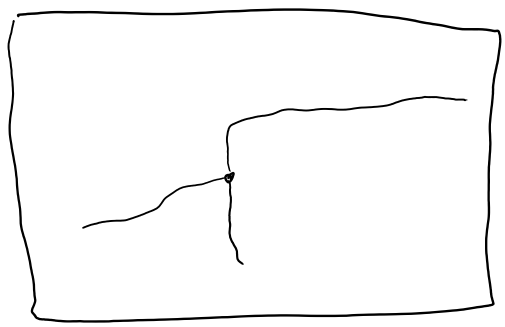
Now average over rows

$$\frac{1}{n^2}\sum_{x\in[0,n-1]^2} Z_x \xrightarrow{a.s.} \theta_p .$$

Average over boxes converge

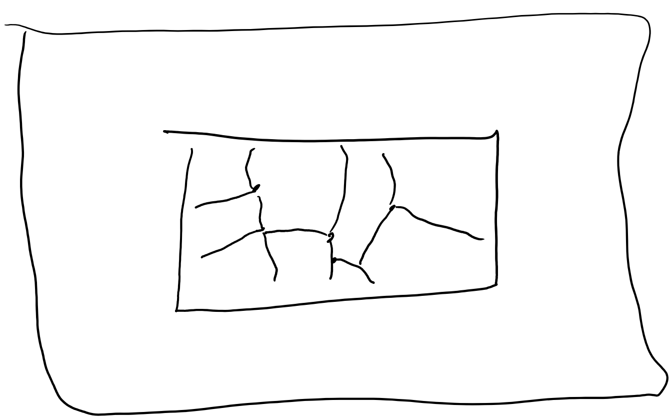
Burton - Keane

x is a trifurcation point if



removing x splits component into 3 - infinite components

Let M_n = # tri-points in B_n.



If we remove all bonds in B_n, at least M_n+2 components on ∂B_n

$n^{-d} M_n \to \mathbb{P}[0 \text{ tri-point}]$

$M_n \sim c\, n^d$

$\partial M_n \sim c' n^{d-1}$ Contradition

Sub-additive Ergodic Theorem

Sub-additive Ergodic Theorem

$X_{m,n} \qquad 0 \leq m \leq n$

(a) $X_{0,m} + X_{m,n} \geq X_{0,n}$ subadditive

(b) $X_{nk,(n+1)k}$ stationary sequence $k \geq 1$.

(c) $X_{m,m+k}$ does not depend on m.

(d) $\mathbb{E}\, X_{0,1} < \infty$ & $\inf \frac{1}{n} \mathbb{E}\, X_{0,n} > -\infty$

Then (i) $\lim \frac{1}{n} \mathbb{E} X_{0,n} = \inf \frac{1}{n} \mathbb{E} X_{0,n} =: M$.

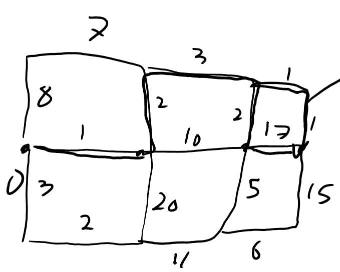
(ii) $\lim \frac{1}{n} X_{0,n}$ exists a.s.

(iii) If ergodic in (b) then $\frac{1}{n} X_{0,n} \xrightarrow{a.s.} M$.

Ex: ξ_i IID, $X_{m,n} = \sum_{i=m+1}^{n} X_i$

Ex: First passage percolation

$\{Y_e\}_{e \in \mathbb{Z}^2}$ IID. $Y_e \in [s^-, s^+]$ $\quad 0 < s^- < s^+ < \infty$.



γ For $u, v \in \mathbb{Z}^2$

$$X_{u,v} = \min_{\gamma \in \Gamma_{u,v}} \sum_{e \in \gamma} Y_e.$$

$X_{i,j} = X_{(i,0),(j,0)} \qquad X_{i,j} + X_{j,k} \geq X_{i,k}$.

Then $\frac{1}{n} X_{0,n} \to \mu$ a.s.

<u>Limit Shape</u>: Let $B_n = \{v \in \mathbb{Z}^2 : X_{0,v} \le n\}$

$\exists$ convex, symmetric shape B_* such that

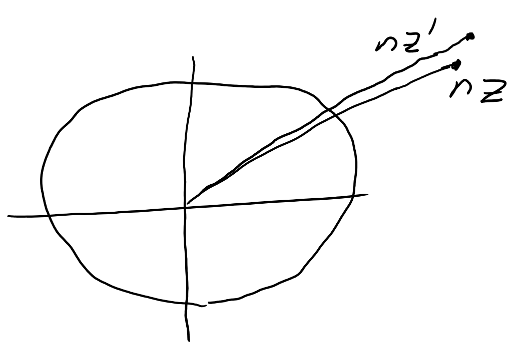
$B_n / n \to B_*$

i.e.
$$B_n \subset (1+\varepsilon) n B_*$$
$$B_n \supset (1-\varepsilon) n B_* \cap \mathbb{Z}^2$$

For $z = (x,y) \in S^1$, $\frac{y}{x} = \frac{b}{a} \in \mathbb{Q}$

$$X_{(0,0),(na,nb)} \longrightarrow \mu_z \cdot \sqrt{a^2+b^2} \quad \text{a.s}$$

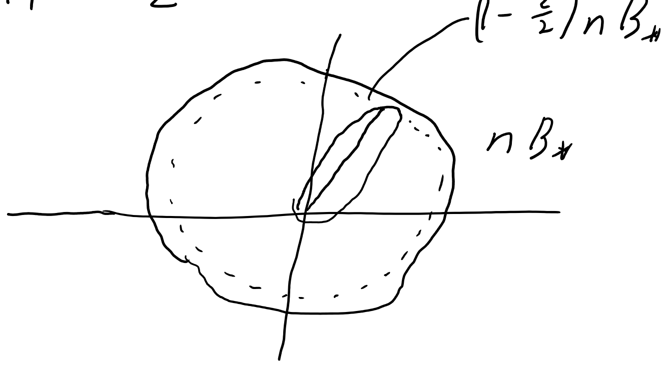
<u>Claim</u> μ_x is Lipschitz. If $\mu_{z'} - \mu_z > 5^+ |z - z'|$



$$\limsup \frac{1}{n}\left(X_{0,nz} + X_{nz,nz'}\right) \ge \lim \frac{1}{n} X_{0,nz'}$$
$$= \mu_z + 5^+ |z - z'| \qquad \overset{\shortparallel}{\mu_{z'}}$$

$B_* = \{ z : |z| \mu_{z/|z|} \le 1 \}$. Convex, Symmetric.

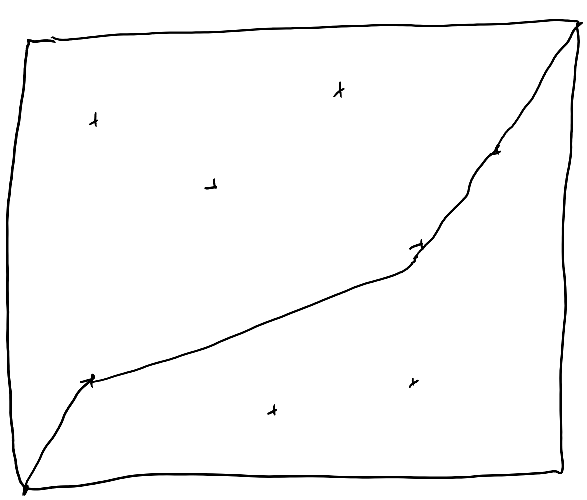
If Z



For each ray, can reach all points in time $(1-\frac{\varepsilon}{3})n$.

At time n can reach strip width

Ex:



Poisson Last Passage Percolation

Rate 1 Poisson points on $[0,n]^2$.

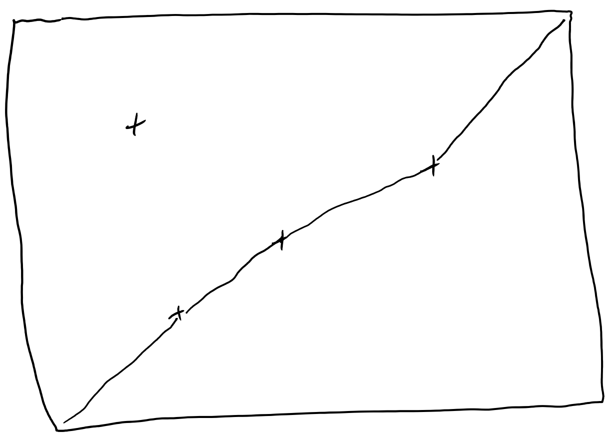
- $X_{m,n}$ = # pts on maximal oriented path from (m,m) to (n,n)

- $X_{0,n} \geq X_{0,m} + X_{m,n}$ super-additive

$\frac{1}{n} X_{0,n} \to C$ a.s.

Why is $C < \infty$?

- has been shown by other methods that $C = 2$.



n points in a box

$(4\ \dot{1}\ \dot{2}\ \dot{3})$ encodes random permutation

LIS = maximal oriented path

Take Poisson on $[0, (1-\varepsilon)\sqrt{n}]^2$

w.h.p. $\approx (1-\varepsilon)^2 n$ points

$X_{0,(1-\varepsilon)\sqrt{n}} \approx (1-\varepsilon) C \sqrt{n}$

Now add $n -$ # pts U.A.R.

$$\Rightarrow \mathbb{P}[LIS(n) \geq (1-\varepsilon) c\sqrt{n}]$$

Similarly:

$$\mathbb{P}[LIS(n) \leq (1+\varepsilon) c\sqrt{n}]$$

So $LIS(n) \xrightarrow{P} c\sqrt{n}$.

- An example of coupling, our next topic.