Decay of Correlation

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<u>Spin system:</u> $x \in \mathcal{X}^V$

$$\mu(x) = \frac{1}{Z} \exp\Big(\sum_{i \sim j} \psi_{ij}(x_i, x_j) + \sum_i h_i(x_i) \Big)$$

<u>E.G.:</u> $\psi_{ij} = x_i x_j$ Ising $\mathcal{X} = \{-1, 1\}$

<u>Hardcore:</u> $\mathcal{X} = \{0,1\}$ $\psi_{ij}(x, x') = -\infty \, I(x = x' = 1)$

$h_i(x) = x_i \log \lambda$

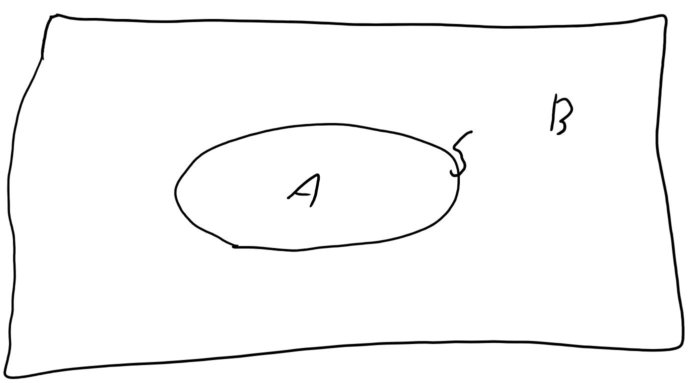
k- <u>Colouring</u> $\mathcal{X} = [k]$, $\psi_{ij}(x, x') = -\infty \, I(x = x')$

$B_n(v) = \{u : d(u,v) \le n\}$

<u>Markov random Field property</u>

If S separates A & B

X_A & X_B conditionally independent given X_S

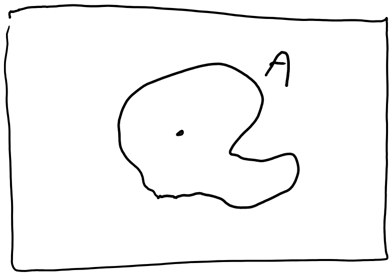


- μ factorizes

$$* \quad \mathbb{P}[X_A = x_A \mid X_{A^c} = y_{A^c}] = \mathbb{P}[X_A = x_A \mid X_{\partial A} = y_{\partial A}]$$

<u>Definition of Infinite Graph DLR</u>

μ satisfies DLR if (*) holds for all finite A.



Existence of Gibbs measures

- free measure (or any) on B_n, take a weak limit along a subsequence

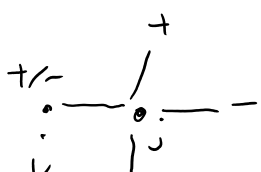
Is it unique?

If

$$\max_{\tau,\tau'} \left| P[x_v = a \mid x_{\partial B_n} = \tau] - P[x_v = a \mid x_{\partial B_n} = \tau'] \right| \to 0$$

Influence of i on j

$$C_{ij} = \max_{\substack{\tau,\tau' \in \mathcal{X}^{N_j} \\ \tau_{i^c} = \tau'_{i^c}}} d_{TV}\left(\mu(x_j \in \cdot \mid x_{N_j} = \tau), \mu(x_j \in \cdot \mid x_{N_j} = \tau')\right)$$



Dobrushin – Shlosman

~~If~~ $\max_i \sum_{j \in N_i} C_{ij} = \alpha < 1$

$i \qquad v \in N_i$

then

$$\left| \mathbb{P}[x_v = a \mid x_{\partial B_n} = \tau] - \mathbb{P}[x_v = a' \mid x_{\partial B_n} = \tau'] \right| \le e^{-cn}$$

Proof: D.S. $\Rightarrow$ path coupling contraction

$\rho_t = \#$ disagreements

$\mathbb{E}\rho_t \le \rho_0 \left(1 - \frac{1-\alpha}{n}\right)^t$

In cts time $\rho_t \le e^{-(1-\alpha)t}$

Let X_t, Y_t G.D. with B.C. τ

$X_0 \sim \mu_{B_n}^{\tau}, \quad Y_0 \sim \mu_{B_n}^{\tau'}$

- $\left| \mu_{B_n}^{\tau}(x_0 = a) - \mathbb{P}[Y_t = a] \right| \le \mathbb{E}\rho_t \le n^2 e^{-(1-\alpha)t}$

- Z_t G.D. with B.C. τ'.

  $Z_0 = Y_0 \sim \mu_{B_n}^{\tau'}$

  $\left| \mu_{B_n}^{\tau'}(x_0 = a) - \mathbb{P}[Y_t = a] \right|$

  $\le \mathbb{P}[Y_t(0) \ne Z_t(0)]$

  $\le \mathbb{P}[$ path of updates from ∂B_n to 0 by time $t]$

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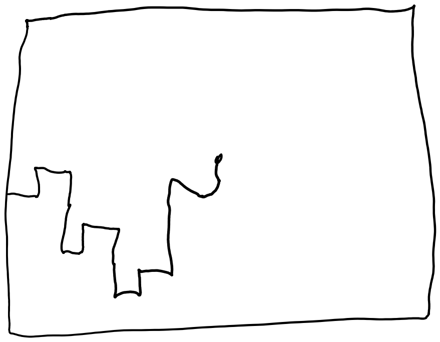
Set $t = \delta n$



paths length n
$\leq 4n \cdot 4^{\delta n}$

$\mathbb{P}[\mathrm{Pois}(\delta t) \geq n]$
$\leq 5^{-n}$ for δ small

$$\left| \mu_{B_n}^{\tau'}(x_0 = a) - \mathbb{P}[Y_t = a] \right| \leq e^{-cn}$$

So $\left| \mu^{\tau}(x_0 = a) - \mu^{\tau'}(x_0 = a) \right| \leq 2e^{-cn}$.

Non-uniqueness

E.G. 2-colouring

E.G. With soft constraints.

Ising $\mu(x) \sim \exp(\beta \sum x_i x_j)$

- If $\tau \geq \tau'$ then

$\mu_\Lambda^{\tau} \geq \mu_\Lambda^{\tau'}$ - by Glauber Dynamics Coupling

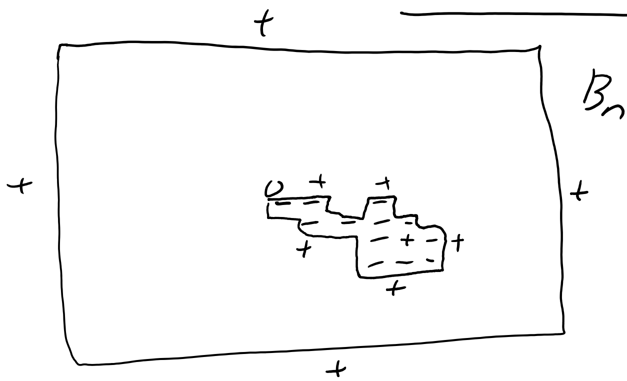
Plus measure

- weak limit of $\mu_{B_n}^{+}$, decreasing in n.
- limit μ^{+} exist.

If $\beta > \beta_c$ then $\mu^+ \neq \mu^-$ and
$\mu^+(x_0 = +) > \frac{1}{2}$.

Assume $\beta > 10$.

Peierls Argument



Let γ be a dual circuit, A_γ is the event the inner boundary is $-$,
outer boundary is $+$.

Let $\varphi_\gamma : \{+,-\}^V \to \{+,-\}^V$
flip sign interior of γ.

If $x \in A_\gamma$ $\quad \mu_\Lambda^+(x) = e^{-\beta|\gamma|} \mu_\Lambda^+(\varphi_\gamma(x))$

$$\mu_\Lambda^+[A_\gamma] \leq \frac{\sum_{x \in A_\gamma} \mu_\Lambda^+[x]}{\sum_{x \in A_\gamma} \mu_\Lambda^+(\varphi_\gamma(x))} \leq e^{-|\gamma|\beta}$$

$$\mu_\Lambda^+(x_0 = -1) \leq \sum_{\gamma : 0 \in \gamma} \mu_\Lambda^+(A_\gamma)$$
$$\leq \sum^{\infty} \ell \, 4^\ell e^{-\beta \ell}$$

$\leq 1/10 \quad$ if $\beta \geq 10$.

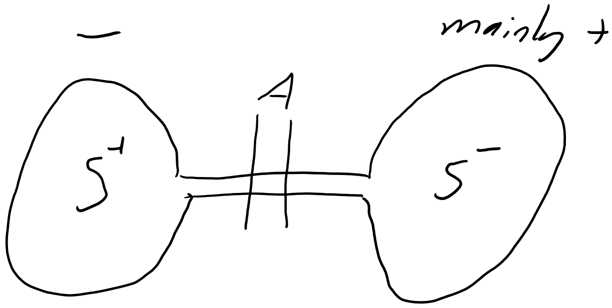
Slow Mixing: β large

on torus $\mathbb{Z}_n^2$

Bottleneck: Intuitively balanced configurations $=$
$A = \{x : \sum x_i \approx 0\}$ (hard to analyze)

- Let S^+ be event LR crossing of $+$

  S^- be event TB crossing of $-$.



Claim: $\mu(\partial S^\pm) \leq e^{-cn}$, $\mu(S^+ \cup S^-) \geq 1 - e^{-cn}$.

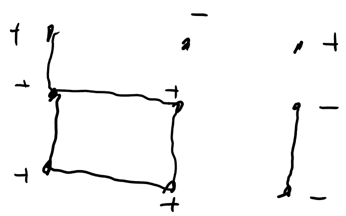
$\Phi_* \leq e^{-cn}$ so $t_{mix} \geq e^{cn}$

FK - model

$\nu(\xi) = \frac{1}{Z} p^{\sum_e \xi_e} (1-p)^{|E| - \sum_e \xi_e} \cdot 2^{\# \text{components}}$

FK to Ising

Set each component of ξ to $+/-$



ξ to $+/-$

$$\mu(\xi, \sigma) = \frac{1}{Z} p^{\Sigma \xi} (1-p)^{|E| - \xi}$$

$$\mu(\sigma) = \frac{1}{Z} \sum_{\substack{\xi \text{ compatible} \\ \text{with } \sigma}} p^{\Sigma \xi} (1-p)^{|E| - \xi}$$

$$= \frac{1}{Z} \prod_{i \sim j} I(\sigma_i \sigma_j = +) + (1-p) I(\sigma_i \sigma_j = -)$$

$$= \frac{1}{Z'} \exp\left(\beta \sum \sigma_i \sigma_j\right)$$

such that $e^{-2\beta} = 1 - p$, $p = 1 - e^{-2\beta}$.

<u>Ising to FK:</u>

$$\mathbb{P}[\xi_{(i,j)} = 1] = \begin{cases} p & \text{if } \sigma_i = \sigma_j \\ 0 & \text{o.w.} \end{cases}$$

<u>Swendsen Wang Dynamics</u>

Ising $\to$ FK $\to$ Ising.

<u>Theorem</u> Mixing time $O(n^c)$ for <u>any</u> β.

<u>Properties:</u>

(a) If $\beta < \beta'$ then $\nu_\beta \leq \nu_{\beta'}$.

(b) FKG.

(c) $\mathbb{E}\, \sigma_u \sigma_v = \mathbb{P}[u \longleftrightarrow v]$

Define $\beta_c = \inf\{\beta : \mathbb{P}_{\text{wired}}[0 \longleftrightarrow \infty] > 0\}$

For $\beta > \beta_c$ Ising has non-uniqueness