L-spaces, Taut Foliations, Left-Orderability, and Incompressible Tori

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Heegaard Floer homology: invariants for closed 3-manifolds, defined by Ozsváth and Szabó in the early 2000s.
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  \[ Y \text{ closed, oriented 3-manifold } \implies \widehat{\text{HF}}(Y), \text{ f.d. vector space} \]
Heegaard Floer homology: invariants for closed 3-manifolds, defined by Ozsváth and Szabó in the early 2000s.

Most basic version (over $\mathbb{F} = \mathbb{Z}_2$):

Let $Y$ be a closed, oriented 3-manifold. Denote $\hat{\text{HF}}(Y)$ as the Heegaard Floer homology, a finite-dimensional vector space.

A cobordism $W: Y_1 \to Y_2$ induces a map $F_W: \hat{\text{HF}}(Y_1) \to \hat{\text{HF}}(Y_2)$. 

This framework provides a rich structure for studying 3-manifolds and their topological properties.
Heegaard Floer homology

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- Most basic version (over $\mathbb{F} = \mathbb{Z}_2$):

  $Y$ closed, oriented 3-manifold $\Rightarrow \hat{HF}(Y)$, f.d. vector space

  $W : Y_1 \to Y_2$ cobordism $\Rightarrow F_W : \hat{HF}(Y_1) \to \hat{HF}(Y_2)$

- Defined in terms of a chain complex $\hat{CF}(\mathcal{H})$ associated to a Heegaard diagram $\mathcal{H}$ for $Y$:
  - Generators correspond to tuples of intersection points between the two sets of attaching curves.
Heegaard Floer homology

\[ \hat{\text{HF}}(Y) \] decomposes as a direct sum of pieces corresponding to spin\(^c\) structures on \( Y \):

\[ \hat{\text{HF}}(Y) \cong \bigoplus_{s \in \text{Spin}^c(Y)} \hat{\text{HF}}(Y, s). \]

Spin\(^c\) structures on \( Y \) are in 1-to-1 correspondence with elements of \( H^2(Y; \mathbb{Z}) \).
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**Theorem (Ozsváth–Szabó)**

If \( Y \) is a 3-manifold with \( b_1(Y) > 0 \), the collection of spin\(^c\) structures \( s \) for which \( \hat{HF}(Y, s) \) is nontrivial detects the Thurston norm on \( H_2(Y; \mathbb{Z}) \). Specifically, for any nonzero \( x \in H_2(Y; \mathbb{Z}) \),

\[
\xi(x) = \max \{ \langle c_1(s), x \rangle \mid s \in \text{Spin}^c(Y), \hat{HF}(s) \neq 0 \}.
\]
Let $Y$ be a rational homology sphere: a closed 3-manifold with $b_1(Y) = 0$. The nontriviality theorem above doesn’t tell us anything since $H_2(Y; \mathbb{Z}) = 0$. 
L-spaces

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- For any rational homology sphere $Y$ and any $s \in \text{Spin}^c(Y)$,

$$\dim \widehat{HF}(Y, s) \geq \chi(\widehat{HF}(Y, s)) = 1.$$
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For any rational homology sphere $Y$ and any $s \in \text{Spin}^c(Y)$,

$$\dim \widehat{HF}(Y, s) \geq \chi(\widehat{HF}(Y, s)) = 1.$$  

$Y$ is called an **L-space** if equality holds for every spin$^c$ structure, i.e., if

$$\dim \widehat{HF}(Y) = \left| H^2(Y; \mathbb{Z}) \right|.$$
Examples of L-spaces:

- $S^3$
- Lens spaces (whence the name)
- All manifolds with finite fundamental group
- Branched double covers of (quasi-)alternating links in $S^3$
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**Question**

*Can we find a topological characterization (not involving Heegaard Floer homology) of which manifolds are L-spaces?*
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**Theorem (Ozsváth–Szabó)**

*If $Y$ is an L-space, then $Y$ does not admit any taut foliation.*
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**Theorem (Ozsváth–Szabó)**

If $Y$ is an L-space, then $Y$ does not admit any taut foliation.

**Conjecture**

If $Y$ is an irreducible rational homology sphere that does not admit any taut foliation, then $Y$ is an L-space.
A left-ordering on a group $G$ is a total order $<$ such that for any $g, h, k \in G$,

$$g < h \implies kg < kh.$$ 

$G$ is left-orderable if it is nontrivial and admits a left-ordering.
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If $Y$ is a 3-manifold with $b_1(Y) > 0$, then $\pi_1(Y)$ is left-orderable.
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**Conjecture (Boyer–Gordon–Watson, et al.)**

Let $Y$ be an irreducible rational homology sphere. Then $Y$ is an L-space if and only if $\pi_1(Y)$ is not left-orderable.
L-spaces and left-orderability

Theorem (L.–Lewallen, arXiv:1110.0563)

If $Y$ is a strong L-space — i.e., if it admits a Heegaard diagram $H$ such that $\dim \hat{CF}(H) = |H^2(Y; \mathbb{Z})|$ — then $\pi_1(Y)$ is not left-orderable.
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Theorem (Greene–L.)

For any $N$, there exist only finitely may strong L-spaces with $|H^2(Y; \mathbb{Z})| = n$. 
Conjecture

If $Y$ is an irreducible 3-manifold with $\dim \widehat{HF}(Y) = 1$, then $Y$ is homeomorphic to either $S^3$ or the Poincaré homology sphere.
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This is known for all Seifert fibered spaces (Rustamov), graph manifolds (Boileau–Boyer, via taut foliations), and manifolds obtained by Dehn surgery on knots in $S^3$ (Ozsváth–Szabó).
Conjecture

If $Y$ is an irreducible 3-manifold with $\dim \widehat{HF}(Y) = 1$, then $Y$ does not contain an incompressible torus.
Incompressible tori

Conjecture

If $Y$ is an irreducible $3$-manifold with $\dim \widehat{HF}(Y) = 1$, then $Y$ does not contain an incompressible torus.

By geometrization and Rustamov’s work, this would imply that it suffices to look at hyperbolic $3$-manifolds for the L-space homology sphere conjecture.
If $K_1 \subset Y_1$, $K_2 \subset Y_2$ are knots in homology spheres, let

$$M(K_1, K_2) = (Y_1 \setminus \text{nbd } K_1) \cup_{\phi} (Y_2 \setminus \text{nbd } K_2)$$

where $\phi: \partial(Y_1 \setminus \text{nbd } K_1) \to \partial(Y_2 \setminus \text{nbd } K_2)$ is an orientation-reversing diffeomorphism taking

- meridian of $K_1 \to$ 0-framed longitude of $K_2$
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- meridian of \( K_1 \rightarrow 0\)-framed longitude of \( K_2 \)
- 0-framed longitude of \( K_1 \rightarrow \) meridian of \( K_2 \).

If \( Y \) is a homology sphere and \( T \subset Y \) is a separating torus, then \( Y \cong Y(K_1, K_2) \) for some \( K_1 \subset Y_1, K_2 \subset Y_2 \), knots in homology spheres, and \( T \) is incompressible if and only if \( K_1 \) and \( K_2 \) are both nontrivial knots.

If $Y_1$ and $Y_2$ are homology sphere L-spaces, and $K_1 \subset Y_1$ and $K_2 \subset Y_2$ are nontrivial knots, then

$$\dim \hat{HF}(M(K_1, K_2)) > 1.$$

If $Y_1$ and $Y_2$ are homology sphere $L$-spaces, and $K_1 \subset Y_1$ and $K_2 \subset Y_2$ are nontrivial knots, then

$$\dim \widehat{HF}(M(K_1, K_2)) > 1.$$ 

Removing the hypothesis that $Y_1$ and $Y_2$ are $L$-spaces will complete the proof of the incompressible torus conjecture.
Lipshitz, Ozsváth, and Thurston define invariants of 3-manifolds with parametrized boundary:

Surface $F \mapsto$ DG algebra $\mathcal{A}(F)$
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3-manifold $M_1$, $\phi_1 : F \xrightarrow{\simeq} \partial M_1 \mapsto$ Right $\mathcal{A}_\infty$-module $\widehat{\text{CFA}}(M_1)$
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3-manifold $M_2$, $\phi_2 : F \xrightarrow{\cong} -\partial M_2 \mapsto$ Left DG module $\widehat{\text{CFD}}(M_2)$
Theorem (Lipshitz–Ozsváth–Thurston)

\[ \widehat{\text{CFA}}(M_1) \text{ and } \widehat{\text{CFD}}(M_2) \text{ are invariants up to chain homotopy equivalence.} \]
Theorem (Lipshitz–Ozsváth–Thurston)

1. $\hat{\text{CFA}}(M_1)$ and $\hat{\text{CFD}}(M_2)$ are invariants up to chain homotopy equivalence.

2. If $Y = M_1 \cup_{\phi_2 \circ \phi_1^{-1}} M_2$, then

$$\hat{\text{HF}}(Y) \cong H_*(\hat{\text{CFA}}(M_1) \otimes A(F) \hat{\text{CFD}}(M_2)).$$
If $K \subset Y$ is a knot in a homology sphere, the bordered invariants of $X_K = Y \setminus \text{nbd}(K)$ are related to the knot Floer homology of $K$, $\widehat{\text{HFK}}(Y, K)$, which detects the genus of $K$. 
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If \( K_1 \) and \( K_2 \) are nontrivial knots in L-space homology spheres, we can explicitly identify at least two cycles in

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\widehat{\text{CF}}A(X_{K_1}) \otimes_{A(T^2)} \widehat{\text{CFD}}(X_{K_2})
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Hope to extend this approach for knots in general homology spheres.