Combinatorial Spanning Tree Models for Knot Homologies

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Knots in Washington XXXIII

Joint work with John Baldwin (Princeton University)
Given a diagram $D$ for a knot or link $K \subset S^3$, form the Tait graph or black graph $B(D)$:

- Vertices correspond to black regions in checkerboard coloring of $D$.
- Edges between two vertices correspond to crossings incident to those regions.
Spanning tree models for knot polynomials

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A *spanning tree* is a connected, simply connected subgraph of $B(D)$ containing all the vertices.

![Diagram showing examples of spanning trees]

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Spanning Tree Models
The Alexander polynomial and Jones polynomials of $K$ can be computed as sums of monomials corresponding to spanning trees: e.g.,

$$\Delta_K(t) = \sum_{s \in \text{Trees}(B(D))} (-1)^{a(s)} t^{b(s)}$$

where $a(s)$ and $b(s)$ are integers determined by $s$. 
Knot Floer homology (Ozsváth–Szabó, Rasmussen): for a link $K \subset S^3$, bigraded, finitely generated abelian group.

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- Detects the genus of the knot (Ozsváth–Szabó):

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g(K) = \max\{a \mid \widehat{\text{HFK}}_* (K, a) \neq 0\} = -\min\{a \mid \widehat{\text{HFK}}_* (K, a) \neq 0\} \]
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- Detects the genus of the knot (Ozsváth–Szabó):

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g(K) = \max\{a \mid \hat{HF}_f(K, a) \neq 0\} = -\min\{a \mid \hat{HF}_f(K, a) \neq 0\}
\]

- Detects fiberedness: \( K \) is fibered if and only if \( \hat{HF}_f(K, g(K)) \cong \mathbb{Z} \).
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- (Ozsváth–Szabó) There is a spectral sequence whose \( E_2 \) page is \( \widetilde{Kh}(\overline{K}) \) and whose \( E_\infty \) page is \( \hat{HF}(\Sigma(K)) \), the Heegaard Floer homology of the branched double cover of \( K \). Hence \( \text{rank } \widetilde{Kh}(\overline{K}) \geq \text{rank } \hat{HF}(\Sigma(K)) \).
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(Kronheimer–Mrowka) Similar spectral sequence from \( \tilde{\text{Kh}}(K) \) to the \textbf{instanton knot Floer homology} of \( K \), which detects the unknot. Hence \( \tilde{\text{Kh}}(K) \cong \mathbb{Z} \) iff \( K \) is the unknot.
The $\delta$ grading

Often, it’s helpful to collapse the two gradings into one, called the $\delta$ grading.

$$\hat{HFK}^\delta(K) = \bigoplus_{a-m=\delta} \hat{HFK}_m(K, a) \quad \hat{Kh}_\delta(K) = \bigoplus_{i-2j=\delta} \hat{Kh}^{i,j}(K)$$
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Theorem (Manolescu–Ozsváth)

If $K$ is a (quasi-)alternating link, then $\widehat{\text{HFK}}(K; \mathbb{F})$ and $\widetilde{\text{Kh}}(K; \mathbb{F})$ are both supported in a single $\delta$ grading, namely $\delta = -\sigma(K)/2$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. 
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**Theorem (Manolescu–Ozsváth)**

*If $K$ is a (quasi-)alternating link, then $\widehat{\text{HFK}}(K; F)$ and $\widehat{\text{Kh}}(K; F)$ are both supported in a single $\delta$ grading, namely $\delta = -\sigma(K)/2$, where $F = \mathbb{Z}/2\mathbb{Z}$.***

**Conjecture**

*For any $\ell$-component link $K$,*

$$2^{\ell-1} \text{ rank } \widehat{\text{Kh}}_{\delta}(K; F) \geq \text{ rank } \widehat{\text{HFK}}^{\delta}(K; F).$$
Can we find explicit spanning tree complexes for $\widehat{\text{HFK}}(K)$ and $\widetilde{\text{Kh}}(K)$? Specifically, want to find a complex $C$ such that:

- Generators of $C$ correspond to spanning trees of $B(D)$;
- The homology of $C$ is $\widehat{\text{HFK}}(K)$ or $\widetilde{\text{Kh}}(K)$;
- The differential on $C$ can be written down explicitly.
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**Theorem (Baldwin–L., Roberts, Jaeger, Manion)**

Yes.
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Earlier results

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- Wehrli and Champarnerkar-Kofman showed that the standard Khovanov complex reduces to a complex generated by spanning trees, but they weren’t able to describe the differential explicitly.
Label the crossings $c_1, \ldots, c_n$. For $I = (i_1, \ldots, i_n) \in \{0, 1\}^n$, let $D_I$ be the diagram gotten by taking the $i_j$-resolution of $c_j$:
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Let $|I| = i_1 + \cdots + i_n$, and let $\ell_I = \ldots$ be the number of components of $D_I$. 
Resolutions correspond to spanning subgraphs of $B(D)$, and connected resolutions correspond to spanning trees.
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Let $R(D) = \{ l \in \{0, 1\}^n \mid \ell_l = 1 \}$. For $l, l' \in R(D)$, we say $l'$ is a double successor of $l$ if $l'$ is gotten by changing two 0s to 1s.
Let $\mathbb{F}(T)$ be the ring of rational functions in a formal variable $T$. 
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Label the edges of $D e_1, \ldots, e_{2n}$. For each $I \in R(D)$, we define $Y_I$ to be a vector space over $\mathbb{F}(T)$ with generators $y_1, \ldots, y_{2n}$, satisfying a single linear relation whose coefficients are powers of $T$ depending on the order in which $e_1, \ldots, e_{2n}$ occur in $D_I$. 
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Let

$$C(D) = \bigoplus_{I \in R(D)} \Lambda^*(Y_I).$$

Declare the grading of $\Lambda^*(Y_I)$ to be $\frac{1}{2}(|I| - n_-(D))$. 

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Spanning Tree Models
For each double successor pair, we define a linear map

$$f_{I, I'} : \Lambda^*(Y_I) \rightarrow \Lambda^*(Y_{I'})$$

which is (almost always) a vector space isomorphism. Let

$$\partial_D : C(D) \rightarrow C(D)$$

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\[ \Lambda^*(Y_{000}) \xrightarrow{f_{000,101}} \Lambda^*(Y_{101}) \]

\[ \Lambda^*(Y_{000}) \xrightarrow{f_{000,110}} \Lambda^*(Y_{110}) \]

\[ \text{gr} = -1 \]

\[ \text{gr} = 0 \]
Spanning tree model for $\hat{HFK}$

Theorem (Baldwin–L. 2011)

For any diagram $D$ of an $\ell$-component link $K$, $(C(D), \partial_D)$ is a chain complex, and

$$H_*(C(D), \partial_D) \cong \hat{HFK}(K; F) \otimes F(T)^{2n-\ell}$$

where $\hat{HFK}(K)$ is equipped with its $\delta$ grading.
Roberts defined a complex consisting of a copy of $F(X_1, \ldots, X_{2n})$ for each $I \in R(D)$, and a nonzero differential for each double successor pair $I, I'$, which is multiplication by some element of the field determined by the two two-component resolutions in between $I$ and $I'$. The grading is the same as in our complex.
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Jaeger proved that when \( K \) is a knot, the homology of this complex is \( \tilde{\text{Kh}}(K; \mathbb{F}) \otimes \mathbb{F}(X_1, \ldots, X_{2n}) \), with its \( \delta \) grading.
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Manion showed how to do this with coefficients in $\mathbb{Z}$ rather than $\mathbb{F}$. The resulting homology theory is odd Khovanov homology.
Khovanov associates a vector space $V_I$ of dimension $2^{\ell_I-1}$ to each resolution, and a map $d_{I,I'} : V_I \to V'_I$ whenever $I'$ is an immediate successor of $I$. Let $\partial_{Kh}$ be the differential of this complex.
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$\tilde{\text{Kh}}(K)$ is defined to be $H_*(\partial_{Kh})$. 
Roberts: Let $\mathcal{F} = \mathbb{F}(X_1, \ldots, X_{2n})$, and let $\mathcal{V}_l = V_l \otimes \mathcal{F}$. Define an internal differential $\partial_l$ on $\mathcal{V}_l$ such that

$$H_*(\mathcal{V}_l, \partial_l) = \begin{cases} V_l & l_l = 1 \\ 0 & l_l > 1. \end{cases}$$

Let $\partial_V = \sum_l \partial_l$. By choosing $\partial_l$ carefully, we can arrange that $\partial_V \partial_{\text{Kh}} = \partial_{\text{Kh}} \partial_V$, so that $(\partial_V + \partial_{\text{Kh}})^2 = 0$. 
Twisted Khovanov homology
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- The $d_1$ differential is zero, since no two connected resolutions are connected by an edge, so $E_2 = E_1$. 
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- All higher differentials vanish for grading reasons, so $H_*(E_2, d_2) \cong E_\infty$. 

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Roberts showed that the resulting homology is a link invariant. Jaeger showed that if $K$ is a knot, this homology is isomorphic to $\tilde{\text{Kh}}(K) \otimes \mathcal{F}$.
Twisted Khovanov homology
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Let $V$ be a $\mathbb{F}$-vector space of rank 2. Manolescu showed that there is an unoriented skein sequence for $\widehat{\text{HFK}}$:

$$
\begin{align*}
\widehat{\text{HFK}}(K) \otimes V^\otimes m - \ell &\longrightarrow \widehat{\text{HFK}}(K_0) \otimes V^\otimes m - \ell_0 \\
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Essentially, we need these extra powers of $V$ because $\widehat{\text{HFK}}$ of a link is “too big.” For example, $\widehat{\text{HFK}}$ of the Hopf link has rank 4, while both resolutions at a crossing are unknots, for which $\widehat{\text{HFK}}$ has rank 1. This is the big difference between $\widehat{\text{HFK}}$ and other invariants ($\widetilde{\text{Kh}}(K)$, $\widehat{\text{HF}}(\Sigma(K))$, instanton knot Floer homology, etc.)
Iterating this (à la Ozsváth–Szabó), we get a cube of resolutions for $\widehat{\text{HFK}}$: a differential on

$$\bigoplus_{l \in \{0,1\}^n} \widehat{\text{HFK}}(K_l) \otimes V^{m-\ell_l}$$

consisting of a sum of maps

$$f_l: \widehat{\text{HFK}}(K_l) \otimes V^{\otimes m-\ell_l} \rightarrow \widehat{\text{HFK}}(K_{l'}) \otimes V^{\otimes m-\ell_{l'}}$$

for every pair $l, l'$, whose homology is $\widehat{\text{HFK}}(K) \otimes V^{\otimes m-\ell}$.
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If we use twisted coefficients instead, with coefficients in $\mathbb{F}(T)$, we can arrange that $\widehat{\text{HFK}}(K_I) = 0$ whenever $\ell_I > 0$. And then a similar analysis goes through as with Khovanov homology.
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Can also do something similar for the spectral sequence from $\tilde{Kh}(K)$ to $\hat{HF}(\Sigma(-K))$. The only problem is that we don’t have the grading argument that would imply the spectral sequence collapses after $E_2$. But $E_3$ is an invariant (Kriz–Kriz).