Midterm MAT 214: Theorems and Problem Sets.

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1-(15 points) Let $a$ and $b$ be two non-zero integers. Show that there are integers $r$ and $s$ such that $\gcd(a, b) = ar + bs$.

2-(15 points) State a version of Wolstenholme’s theorem and prove it.

3-(20 points) Define primitive root and show that there is a primitive root modulo a prime number.

4- Let $f(n) = \sum_{m \leq n, (m,n)=1} m$. Show that
   a) (5 points) $f(n) = n\phi(n)/2$.
   b) (10 points) If $f(n) = f(m)$, then $n = m$.

5-(5 points) Find all positive integers $n$ such that, for any $k$, the binomial coefficient $\binom{n}{k}$ is odd.

6-(5 points) Show that $\gcd(2^{2n} + 1, 2^{2m} + 1) = 1$.

7-a) (10 points) Let $p$ be a prime number and $k$ a positive integer. Show that
   $$\sum_{i=0}^{p-1} i^k \equiv \begin{cases} 0 \pmod{p} & \text{if } p - 1 \nmid k, \\ -1 \pmod{p} & \text{if } p - 1 | k. \end{cases}$$

    b) (15 points) Let $f \in (\mathbb{Z}/p\mathbb{Z})[x_1, \ldots, x_n]$ be a polynomial in $n$ variables with coefficients in $\mathbb{Z}/p\mathbb{Z}$. Show that if degree of $f$ is less than $n$, then $p$ divides $\#V_f(\mathbb{Z}/p\mathbb{Z})$, where
    $$V_f(\mathbb{Z}/p\mathbb{Z}) = \{ a \in (\mathbb{Z}/p\mathbb{Z})^n \mid f(a) \equiv 0 \pmod{p} \}.$$