1-(10 points) Let \( p \) be an odd prime and
\[
f(x) = \sum_{i=0}^{p-2} (i+1)x^i.
\]
Show that if \( f(a) \equiv f(b) \pmod{p} \), then \( a \equiv b \pmod{p} \).

2-(10 points) Let \( a_1 = 1, a_2 = 2, a_3 = 24, \) and for \( n \geq 4, \)
\[
a_n = \frac{6a_{n-1}a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.
\]
Show that \( n|a_n \) for all positive integers \( n \).

3-(5 points) Let \( x \) and \( y \) be two positive integers. Show that if \( xy|x^2+y^2+1, \) then \( x^2+y^2+1 = 3xy. \)

4- Let \( \mathcal{O}_m \) be the ring of integers in \( \mathbb{Q}[\sqrt{m}] \).
   a) (5 points) If \( x, y \in \mathbb{Z} \) and \( x \) divides \( y \) in \( \mathcal{O}_m \), then \( x \) divides \( y \) in \( \mathbb{Z} \).
   b) (15 points) Let \( a \) and \( b \) be two non-zero integers such that \( a^2 + 4b \neq 0 \) and \( \gcd(a, b) = 1. \) Let \( \{x_m\}_{m=0}^{\infty} \) be a sequence of integers as follows:
\[
x_0 = 0, \ x_1 = 1, \quad x_{m+1} = ax_m + bx_{m-1}.
\]
Show that \( \gcd(x_m, x_n) = \gcd(a, a^{m+n}) \) for all positive integers \( m \) and \( n. \)
   (Hint: First show that \( x_m = \frac{a^n - \beta^m}{\alpha - \beta}, \) where \( \alpha \) and \( \beta \) are solutions of \( x^2 - ax - b = 0. \))

5-(15 points) Find all positive integers \( n \) such that \( n = x^2 + xy + y^2 \) for some integers \( x \) and \( y. \) If \( p \) is prime and \( p = x^2 + xy + y^2, \) then at most how many integer solutions does the equation \( p = x^2 + xy + y^2 \) have? (Hint: use \( \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \).)