Entropy of Cohen-Lenstra measures: the u-aspect

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Abstract

Let $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ be the entropy of the Cohen-Lenstra measure on finite abelian p-groups associated to unit rank $0 \leq u \in \mathbb{N}$. In this note, we show that $0 < \mathbf{H}(\nu_{\mathrm{CL}}^u) < \infty$ for all u, $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ is a strictly decreasing function of $u \geq 0$, and $\mathbf{H}(\nu_{\mathrm{CL}}^u) \xrightarrow{u\to\infty} 0$. In particular, this shows that the groupoid measure is an entropy maximizer in the class of Cohen-Lenstra measures on finite abelian p-groups.

1 Entropy and statement

Let (X, ν) be a discrete probability space. The **Shannon entropy** of ν is defined as the expected value of $-\log \nu$

$$\mathbf{H}(\nu) := \mathbb{E}(\log \nu) = -\sum_{x\in X} \nu(x) \log \nu(x) \geq 0$$

and is a measure of the information content of the measure ν [6].

Fix a prime p and let FinAb_p denote the category of finite abelian p-groups. The Cohen-Lenstra measure on FinAb_p associated to unit rank $0 \leq u \in \mathbb{N}$, ν_{CL}^u, is the groupoid measure on FinAb_p quotiented by u randomly-chosen elements. Alternatively, ν_{CL}^u is characterized by the property

$$\nu_{\mathrm{CL}}^u(A) \propto \frac{1}{\#A^u \#\mathrm{Aut}A}$$

for all finite abelian p-groups A.

The purpose of this note is to show that $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ is finite for all u, strictly decreasing as a function of u, and converges to 0 as u approaches ∞.

Theorem 1. *Let $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ denote the entropy of the Cohen-Lenstra measure ν_{CL}^u on finite abelian p-groups associated to unit rank $0 \leq u \in \mathbb{N}$. Then*

I) $\mathbf{H}(\nu_{\mathrm{CL}}^u) < \infty$ *for all u;*

II) $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ *is a strictly decreasing function of $u \geq 0$; and,*

III) $\mathbf{H}(\nu_{\mathrm{CL}}^u) \xrightarrow{u\to\infty} 0$.

We view our statement as a first step in introducing the Principle of Maximum Entropy in the study of Cohen-Lenstra measures. Such a principle has been profitably exploited in probability, see for instance the information theoretic proofs of the Central Limit Theorem [4, 1]. In particular, Theorem 1 shows that the groupoid measure is an entropy maximizer for Cohen-Lenstra measures on finite abelian p-groups.

Remark 1. *There exists variants of the Cohen-Lenstra measures when the parameter u can be taken to any real number > -1. Our proof shows that items I) and III) of Theorem 1 hold for these variants of Cohen-Lenstra measures. We do not yet know whether item II also does.*

We also obtain an explicit formula for the relative entropy between Cohen-Lenstra measures. The precise definition will be given in §3.

Theorem 2. *Let $\nu_{\mathrm{CL}}^{u_1}$ and $\nu_{\mathrm{CL}}^{u_2}$ be Cohen-Lenstra measures on finite abelian p-groups associated to unit ranks $u_1 \geq 0$ and $u_2 \geq 0$ respectively. The relative of $\nu_{\mathrm{CL}}^{u_1}$ from $\nu_{\mathrm{CL}}^{u_2}$ is given by:*

$$D_{\mathrm{KL}}\left(\nu_{\mathrm{CL}}^{u_1} \mid\mid \nu_{\mathrm{CL}}^{u_2}\right) = \log\left(\frac{F_{u_1}}{F_{u_2}}\right) + (u_2 - u_1)\sum_{i=1}^{\infty} \frac{\log(p)}{p^{u_1+i} - 1}$$

where F_u is the normalizing constant $\prod_{i\geq 1+u}(1-p^{-i})$ for the Cohen-Lenstra measure ν_{CL}^{u}.

The proof exploits explicit formulas for the Cohen-Lenstra zeta functions $\boldsymbol{\zeta}_k^{(p)}(s)$.

Remark 2. *Unlike for the relative entropy, we do not have an explicit formula for the Shannon entropy of Cohen-Lenstra measures. The most straightforward approach would be to gain a good understanding of a variant on the Cohen-Lenstra zeta function $\boldsymbol{\zeta}_k^{(p)}(s)$ with a power of s on the $1/\#Aut$ terms instead of the $1/\#A$ terms. An explicit expression for the entropy would then fall out by taking the derivative of this new zeta function.*

2 Proof of Theorem 1

2.1 Preliminaries

Phillips Hall's Strange Formula for finite abelian p-groups states that

$$\sum_A{}' \frac{1}{\#\mathrm{Aut}A} = \sum_A{}' \frac{1}{\#A} = \sum_n \frac{\pi(n)}{p^n} = \prod_{i\geq 1}(1-p^{-i})^{-1} < \infty$$

where the sums $\sum'$ run over isomorphism classes of finite abelian p-groups and π is the partition function [3, 7]. This formula, and its variants, imply the following description of the Cohen-Lenstra measures

$$\nu_{\mathrm{CL}}^{u}(A) = \frac{1}{\#A^u\#\mathrm{Aut}A}\prod_{i\geq 1}(1-p^{-u-i}).$$

We denote by F_u the normalizing constant $\prod_{i\geq 1}(1-p^{-u-i}) = \prod_{j\geq u+1}(1-p^{-j})$ (see [2]) and note that F_u is a strictly increasing function of u.

The following lemma will be useful. The number of automorphisms of a finite abelian group is comparable, and often much larger, than the size of the group. The following is a lower bound expressing this fact.

Lemma 1. *For a finite abelian p-group A of size p^n we have*

$$\#\mathrm{Aut}(A) \geq \#A(1-p^{-1}) \geq p^{n-1}.$$

In fact, $\#\mathrm{Aut}A \geq \#A$ whenever $\mathrm{rank}_p(A) \geq 2$.

Proof. Denote by $A_{\lambda'} = \prod_i \mathbf{Z}_p/p^{\lambda'_i}\mathbf{Z}_p$ the finite abelian p-group of associated to the partition $\lambda' = (\lambda'_i)_i = (\lambda'_1 \geq \lambda'_2 \geq \ldots)$. Recall that the number of automorphisms of a finite abelian p-group of type λ' is

$$\#\mathrm{Aut}A = p^{|\lambda'|+2n(\lambda')}\prod_{j\geq 1}\prod_{k=1}^{\lambda_j-\lambda_{j+1}}(1-p^{-k})$$

where λ denotes the dual partition of λ', $|\lambda'| = \sum_{j\geq 1}\lambda_j = n$, and $n(\lambda') = \sum_i (i-1)\lambda'_i = \sum\binom{\lambda_j}{2}$ [5]. To extract the lower bound for $\#\mathrm{Aut}(A)$ when $\#A = p^n$, we rewrite this expression as follows

$$\#\mathrm{Aut}A = p^n p^{\sum_{j\geq 1}(\lambda_j^2-\lambda_j) - \frac{(\lambda_j-\lambda_{j+1})^2}{2} - \frac{(\lambda_j-\lambda_{j+1})}{2}}\prod_{j\geq 1}\prod_{k=1}^{\lambda_j-\lambda_{j+1}}(p^k-1) \tag{1}$$

Let's analyze the exponent. We use the notation λ^2 to denote the partition where each component of λ is squared and we denote $m_j := \lambda_j - \lambda_{j+1}$. We have:

$$
\begin{aligned}
&\sum_{j\geq 1}(\lambda_j^2 - \lambda_j) - \frac{(\lambda_j - \lambda_{j+1})^2}{2} - \frac{(\lambda_j - \lambda_{j+1})}{2} \\
&= |\lambda^2| - |\lambda| - \frac{|\lambda^2|}{2} - \left(\frac{|\lambda^2|}{2} - \frac{\lambda_1^2}{2}\right) - \frac{|\lambda|}{2} + \left(\frac{|\lambda|}{2} - \frac{\lambda_1}{2}\right) + \sum_{j\geq 1}\lambda_j\lambda_{j+1} \\
&= \frac{\lambda_1^2}{2} - \frac{\lambda_1}{2} - |\lambda| + \sum_{j\geq 1}\lambda_j\lambda_{j+1} \\
&= \frac{\lambda_1^2}{2} - \frac{\lambda_1}{2} - |\lambda| + \sum_{j\geq 1}(\lambda_{j+1} + m_j)\lambda_{j+1} \\
&= \frac{\lambda_1^2}{2} - \frac{\lambda_1}{2} - |\lambda| + |\lambda^2| - \lambda_1^2 + \sum_{j\geq 1} m_j\lambda_{j+1} \\
&= \frac{\lambda_1^2}{2} - \frac{3\lambda_1}{2} + (|\lambda^2| - \lambda_1^2) - (|\lambda| - \lambda_1) + \sum_{j\geq 1} m_j\lambda_{j+1}.
\end{aligned}
$$

The only term which can be negative is $\frac{\lambda_1^2}{2} - \frac{3\lambda_1}{2}$. It is in fact non-negative, except for $\lambda_1 = 1, 2$ in which case it is equal to -1.

Now to prove that $\#\mathrm{Aut}A \geq \#A$ whenever $\mathrm{rank}_p(A) \geq 2$, we simply note that when $\lambda_1 \geq 2$, then either one of the $\lambda_j - \lambda_{j+1} \geq 2$ or at least two of them are ≥ 1. In either case p^{-1} times the corresponding terms $\prod_{k=1}^{\lambda_j - \lambda_{j+1}}(p^k - 1)$ in Equation (1) is ≥ 1.

If $\lambda_1 = 1$, then A is cyclic p-group, whence $\#\mathrm{Aut}A = (p^n - p^{n-1}) = p^n(1 - p^{-1}) = \#A(1 - p^{-1})$ □

2.2 The proof

With these preliminaries in hand, we turn to the proof.

I) To show that $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ is finite for all $u \geq 0$, it will suffice to show that $\mathbf{H}(\nu_{\mathrm{CL}}^0)$ is finite as we will show in II) that $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ is strictly decreasing in u. Now, using $\log(x) \ll_\varepsilon x^\varepsilon$, we have the following estimates

$$
\begin{aligned}
\mathbf{H}(\nu_{\mathrm{CL}}^0) &= -{\sum}' \frac{F_0}{\#\mathrm{Aut}A}\log\left(\frac{F_0}{\#\mathrm{Aut}A}\right) \\
&\ll_\varepsilon -\log(F_0) + \sum_{A\neq 1}{}' \frac{1}{(\#\mathrm{Aut}A)^{1-\varepsilon}} \\
&= -\log(F_0) + \sum_{n\geq 1}\sum_{\#A = p^n}{}' \frac{1}{(\#\mathrm{Aut}A)^{1-\varepsilon}} \\
&\leq -\log(F_0) + \sum_{n\geq 1}\frac{\pi(n)}{(p^{n-1})^{1-\varepsilon}} < \infty
\end{aligned}
$$

where $\pi(\cdot)$ is the partition function. The sum on the last line is finite by the root test since

$$
\frac{\pi(n)}{(p^{n-1})^{1-\varepsilon}} \sim \frac{\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}}{(p^{n-1})^{1-\varepsilon}}
$$

as $n \to \infty$, implying that $\left(\frac{\pi(n)}{(p^{n-1})^{1-\varepsilon}}\right)^{\frac{1}{n}} \xrightarrow{n\to\infty} \frac{1}{p^{1-\epsilon}} < 1$.

II) We have:

$$\begin{aligned}
\mathbf{H}(\nu_{\mathrm{CL}}^u) &= -\sum{}' \nu_{\mathrm{CL}}^u \\
&= -\sum{}' \frac{F_u}{\#A^u\#\mathrm{Aut}A}\log\left(\frac{F_u}{\#A^u\#\mathrm{Aut}A}\right) \\
&= -\sum{}' \frac{F_u}{\#A^u\#\mathrm{Aut}A}\log(F_u) + \sum{}' \frac{F_u}{\#A^u\#\mathrm{Aut}A}\log(\#A^u\#\mathrm{Aut}A) \\
&= -\log(F_u) + F_u \sum_{A\neq 1}{}' \frac{\log(\#A^u\#\mathrm{Aut}A)}{\#A^u\#\mathrm{Aut}A}
\end{aligned}$$

Since F_u is a strictly increasing function of u, the term $-\log(F_u)$ is strictly decreasing. It would then be sufficient to show that the terms $F_u\frac{\log(\#A^u\#\mathrm{Aut}A)}{\#A^u\#\mathrm{Aut}A}$ are individually decreasing (note the tension since F_u is strictly increasing). This reduces to proving:

$$\#A^{u+1}\#\mathrm{Aut}A \leq (\#A^u\#\mathrm{Aut}A)^{(1-p^{-(u+1)})\#A} \tag{2}$$

for $A \neq 1$ and $u \geq 0$.

When $p \geq 3$ and $u \geq 1$ we have:

$$\#A^{u+1}\#\mathrm{Aut}A \leq (\#A^u\#\mathrm{Aut}A)^{p-1} \leq (\#A^u\#\mathrm{Aut}A)^{(1-p^{-(u+1)})\#A}.$$

When $p = 2$ and $u \geq 1$, we have:

$$\#A^{u+1}\#\mathrm{Aut}A \leq (\#A^u\#\mathrm{Aut}A)^{\frac{3}{4}\#A} \leq (\#A^u\#\mathrm{Aut}A)^{(1-p^{-(u+1)})\#A}.$$

except when $A = \mathbf{Z}/2$ and $u = 1$.

Now, when $u = 0$, we need to show that:

$$\#A\#\mathrm{Aut}A \leq (\#\mathrm{Aut}A)^{(1-p^{-1})\#A}.$$

The reduces to

$$\#A \leq (\#\mathrm{Aut}A)^{\#A-1-\#A/p}$$

Using Lemma 1, we see that this is true except: if $A = \mathbf{Z}/2$, for which $(\#\mathrm{Aut}A)^{\#A-1-\#A/p} = 1$, if $A = \mathbf{Z}/4$, for which $(\#\mathrm{Aut}A)^{\#A-1-\#A/p} = 2$, or if $A = \mathbf{Z}/3$, for which $(\#\mathrm{Aut}A)^{\#A-1-\#A/p} = 2$.

Thus, we have that each term is individually decreasing with the following four exception:

(a) when $A = \mathbf{Z}/2$ and $u = 0$;
(b) when $A = \mathbf{Z}/4$ and $u = 0$;
(c) when $A = \mathbf{Z}/2$ and $u = 1$;
(d) when $A = \mathbf{Z}/3$ and $u = 0$.

For these, one checks directly that:

$$\log(F_0) + F_0\frac{\log(\#\mathrm{Aut}\mathbf{Z}/2)}{\#\mathrm{Aut}\mathbf{Z}/2} + F_0\frac{\log(\#\mathrm{Aut}\mathbf{Z}/4)}{\#\mathrm{Aut}\mathbf{Z}/4} - \log(F_1) - F_1\frac{\log(\#\mathbf{Z}/2\#\mathrm{Aut}\mathbf{Z}/2)}{\#\mathbf{Z}/2\#\mathrm{Aut}\mathbf{Z}/2} - F_1\frac{\log(\mathbf{Z}/4\#\mathrm{Aut}\mathbf{Z}/4)}{\mathbf{Z}/4\#\mathrm{Aut}\mathbf{Z}/4} \geq 0.44$$

$$\log(F_1) + F_1\frac{\log(\#\mathbf{Z}/2\#\mathrm{Aut}\mathbf{Z}/2)}{\#\mathbf{Z}/2\#\mathrm{Aut}\mathbf{Z}/2} - \log(F_2) - F_2\frac{\log((\#\mathbf{Z}/2)^2\#\mathrm{Aut}\mathbf{Z}/2)}{(\#\mathbf{Z}/2)^2\#\mathrm{Aut}\mathbf{Z}/2} \geq 0.21$$

$$\log(F_0) + F_0\frac{\log(\#\mathrm{Aut}\mathbf{Z}/3)}{\#\mathrm{Aut}\mathbf{Z}/3} - \log(F_1) - F_1\frac{\log(\#\mathbf{Z}/3\#\mathrm{Aut}\mathbf{Z}/3)}{\#\mathbf{Z}/3\#\mathrm{Aut}\mathbf{Z}/3} \geq 0.34$$

by using the bound $(1-p^{-k})^{p/(p-1)} \leq \prod_{j=k}^{\infty}(1-p^{-j}) \leq 1$ (obtained using the concavity of $\log(1-t)$) to estimate the F_u terms.

We conclude that $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ is a strictly decreasing function of $u \in \mathbb{N}$.

III) By the expression for $\mathbf{H}(\nu_{\mathrm{CL}}^u)$ obtained in the proof of item II) above, and the fact that $\log(x) \leq x$, we have the bound

$$\mathbf{H}(\nu_{\mathrm{CL}}^u) = -\log(F_u) + F_u \sum_{A\neq 1}{}' \frac{\log(\#A^u \#\mathrm{Aut}A)}{\#A^u \#\mathrm{Aut}A} \leq -\log(F_u) + F_u \sum_{A\neq 1}{}' \frac{u}{\#A^{u-1} \#\mathrm{Aut}A} + F_u \sum_{A\neq 1}{}' \frac{1}{\#A^u}$$

for $u \geq 2$. Now, since $\#A \geq p$ for $A \neq 1$, we obtain

$$\begin{aligned}\mathbf{H}(\nu_{\mathrm{CL}}^u) &\leq -\log(F_u) + \frac{uF_u}{p^{u-1}} \sum_{A\neq 1}{}' \frac{1}{\#\mathrm{Aut}A} + \frac{F_u}{p^{u-1}} \sum_{A\neq 1}{}' \frac{1}{\#A} \\ &= \sum_{k\geq 1} \frac{1}{k} \frac{1}{(p^k-1)p^{ku}} + \frac{uF_u}{p^{u-1}} \sum_{A\neq 1}{}' \frac{1}{\#\mathrm{Aut}A} + \frac{F_u}{p^{u-1}} \sum_{A\neq 1}{}' \frac{1}{\#A} \xrightarrow{u\to\infty} 0.\end{aligned}$$

The entropy always being non-negative, we get $\mathbf{H}(\nu_{\mathrm{CL}}^u) \xrightarrow{u\to\infty} 0$.

3 Relative entropy and the proof of Theorem 2

Let ν and μ be two discrete probability measures on X with the property that μ is absolutely continuous with respect to μ, $\mu \ll \nu$. The **relative entropy**, also called the Kullback–Leibler divergence, of ν from μ is defined as the expected value of $\log(\nu/\mu)$

$$D_{\mathrm{KL}}(\mu \mid\mid \nu) := \mathbb{E}_\mu\big(\log(\mu/\nu)\big) = \sum_{x\in X} \mu(x) \log\left(\frac{\mu(x)}{\nu(x)}\right) \geq 0$$

where we interpret contributions of terms with $\mu(x) = 0$ as 0. The relative entropy is non-negative by Gibbs' inequality. The relative entropy measures the informational content of ν from the point of view of μ.

Theorem 3. *Let $\nu_{\mathrm{CL}}^{u_1}$ and $\nu_{\mathrm{CL}}^{u_2}$ be Cohen-Lenstra measures on finite abelian p-groups associated to unit ranks $u_1 \geq 0$ and $u_2 \geq 0$ respectively. The relative of $\nu_{\mathrm{CL}}^{u_1}$ from $\nu_{\mathrm{CL}}^{u_2}$ is given by:*

$$D_{\mathrm{KL}}\left(\nu_{\mathrm{CL}}^{u_1} \mid\mid \nu_{\mathrm{CL}}^{u_2}\right) = \log\left(\frac{F_{u_1}}{F_{u_2}}\right) + (u_2 - u_1) \sum_{i=1}^{\infty} \frac{\log(p)}{p^{u_1+i} - 1}$$

where F_u denotes the normalizing constant $\prod_{i\geq 1+u}(1-p^{-i})$.

Proof. The definition of $D_{\mathrm{KL}}\left(\nu_{\mathrm{CL}}^{u_1} \mid\mid \nu_{\mathrm{CL}}^{u_2}\right)$ gives

$$\begin{aligned}D_{\mathrm{KL}}\left(\nu_{\mathrm{CL}}^{u_1} \mid\mid \nu_{\mathrm{CL}}^{u_2}\right) &= \sum_A{}' \nu_{\mathrm{CL}}^{u_1}(A) \log\left(\frac{\nu_{\mathrm{CL}}^{u_1}(A)}{\nu_{\mathrm{CL}}^{u_2}(A)}\right) &(3)\\ &= \sum_A{}' \frac{F_{u_1}}{\#A^{u_1}\#\mathrm{Aut}A} \log\left(\frac{F_{u_1}\#A^{u_2}}{F_{u_2}\#A^{u_1}}\right) &(4)\\ &= \log\left(\frac{F_{u_1}}{F_{u_2}}\right) + F_{u_1}(u_2-u_1) \sum_A{}' \frac{\log(\#A)}{\#A^{u_1}\#\mathrm{Aut}A} &(5)\\ &= \log\left(\frac{F_{u_1}}{F_{u_2}}\right) + (u_2-u_1)F_{u_1}\left(-\lim_{k\to\infty} \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\zeta}_k^{(p)}(s)\Big|_{s=u_1}\right) &(6)\end{aligned}$$

where $\boldsymbol{\zeta}_k^{(p)}(s)$ denotes the Cohen-Lenstra zeta function. Recall that $\boldsymbol{\zeta}_k^{(p)}(s)$ is defined as

$$\boldsymbol{\zeta}_k^{(p)}(s) := \sum_A{}' \frac{w_k(A)}{\#A^s}$$

with $w(G) := \frac{1}{\#\mathrm{Aut}(G)}$ and

$$w_k(G) = \begin{cases} w(G)\prod_{i=k-r+1}^{k}(1-p^{-i}) & \text{if } k \geq r := \mathrm{rank}(G), \\ 0 & \textit{else} \end{cases}.$$

Note that $w_k(G)$ is increasing in k, $w_k(G)$ is bounded by $w(G)$, and $w_k(G) \xrightarrow{k\to\infty} w(G) := \frac{1}{\#\mathrm{Aut}G}$.The Cohen-Lenstra zeta function converges for $\Re(s) > -1$ and satisfies the following explicit formula

$$\zeta_k^{(p)}(s) = \prod_{i\geq 1}^{k}(1-p^{-s-i})^{-1}. \tag{7}$$

We compute its derivative in two ways. Using the definition, we first find

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^{(p)}(s) = -\sum_A{}' \frac{w_k(A)\log\#A}{\#A^s}$$

since the series $\sum_A{}' \frac{w_k(A)\log\#A}{\#A^s}$ is absolutely uniformly convergent for $s \in [t,\infty)$ for any $t > -1$. This follows by comparing it to

$$\sum_A{}' \frac{1}{\#A^{s-\varepsilon}\#\mathrm{Aut}A}$$

and using the proof of item I) of Theorem 1. The limit interchange giving line (6) follows from Lebesgue's Dominated Convergence Theorem and the same comparison.

On the other hand, by formula (7), we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^{(p)}(s) = \zeta_k^{(p)}(s)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(\log\zeta_k^{(p)}(s)\right) = \left(\prod_{i=1}^{k}(1-p^{-s-i})^{-1}\right)\left(-\sum_{i=1}^{k}\frac{\log(p)}{p^{s+i}-1}\right).$$

It follows that

$$\lim_{k\to\infty} -\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^{(p)}(s)\bigg|_{s=u_1} = \frac{1}{F_{u_1}}\sum_{i=1}^{\infty}\frac{\log(p)}{p^{u_1+i}-1},$$

which completes the proof. □

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