Stability and Instability of Extremal Black Holes

Stefanos Aretakis

Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge

s.aretakis@dpmms.cam.ac.uk

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General Relativiy/Black Holes

- General relativity: classical theory for evolution of physical systems under the effect of gravity.
- Einstein equations: $(\mathcal{M}, g)$ such that $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$.
  1. Einstein-vacuum: $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$.
  2. Einstein-Maxwell: $T_{\mu\nu} = F_{\mu\alpha} F^{\alpha}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$, where F is a Maxwell field.
- Solutions include: flat Minkowski metric, black holes
- Black hole regions are defined to be the (non-empty) complement of the past of (null) infinity.
- Explicit solutions
  1. Vacuum: Rotating Kerr black holes parametrized by (M, a) with $0 \leq |a| \leq M$.
  2. Maxwell: Spherically symmetric charged Reissner-Nordström black holes parametrized by (M, e) with $0 \leq |e| \leq M$.
- Extremal case: $|e| = M$ or $|a| = M$.

The Cauchy Problem for the Wave Equation

We consider solutions of the Cauchy problem of the wave equation

$$\Box_g \psi = 0 \tag{1}$$

on the domain of outer communications of **extremal** black holes, arising from regular initial data prescribed on a Cauchy hypersurface Σ_0 crossing the event horizon $\mathcal{H}^+$ and terminating at spacelike or null infinity.

- Main two examples:
  1. Extremal Reissner-Nordström; toy model for extremality
  2. Extremal Kerr; much harder

The Wave Equation on Black Holes

- The study of the wave equation has been a very active research area in geometric analysis for the past 30 years, and especially the past decade.
- The evolution of waves on **subextremal** black holes (including subextremal Reissner-Nordström and Kerr) has been completely understood and sharp quantitative decay estimates have been shown. Contributors include:
  Andersson, Blue, Dafermos, Dyatlov, Holzegel, Kay, Laul, Luk, Metcalfe, Regge, Rodnianski, Schlue, Smulevici, Soffer, Tataru, Tohaneanu, Vasy, Wald, Wheeler, ...
- Regarding general subextremal Kerr backgrounds, Dafermos and Rodnianski (2010) have shown definitive decay results for the wave equation.
- So, what about the **extremal** black holes? Although, extremal black holes have been extensively studied in the Physics community, the analysis of waves on such backgrounds had not been adequately studied from a mathematical point of view.

The Main Theorems

1. Stability of extremal R-N: Pointwise and energy decay **away** from the event horizon $\mathcal{H}^+$ for all solutions.
2. Stability of extremal Kerr: Pointwise and energy decay **away** from the event horizon $\mathcal{H}^+$ for axisymmetric solutions.
3. Instability of extremal R-N: Pointwise and energy **non-decay** and **blow-up** along the event horizon for higher order derivatives.

Difficulties of the Analysis of Black Holes

1. Near horizon geometry: redshift effect
2. Dispersion in intermediate region: trapping effect
3. Near null infinity
4. "Low frequency" obstructions
5. Superradiance

New Difficulties of Extremal Black Holes

1. Near horizon geometry: **degeneracy** of redshift effect
2. Dispersion in intermediate region: trapping effect
3. Near null infinity
4. "Low frequency" obstructions
5. Superradiance

Preliminaries: The Vector Field Method

Given a vector field V and the energy momentum tensor

$$\mathbf{T}_{\mu\nu}[\psi] = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^a \psi \partial_a \psi$$

we can define the currents

$$\begin{aligned} J^V_\mu[\psi] &= \mathbf{T}_{\mu\nu}[\psi] V^\nu, \\ K^V[\psi] &= \mathbf{T}[\psi](\nabla V), \\ \mathcal{E}^V[\psi] &= \mathrm{Div}(\mathbf{T}) V = (\Box_g \psi) V(\psi). \end{aligned}$$

Note that since $\mathrm{Div}\,\mathbf{T}[\psi] = (\Box_g \psi) d\psi$ we have

$$\mathrm{Div}(J) = K^V[\psi] + \mathcal{E}^V[\psi].$$

Finally, if ψ is a solution to the wave equation, then the divergence identity takes the form

$$\int_{\Sigma_\tau} J^V_\mu[\psi] n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}^+} J^V_\mu[\psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{R}} K^V[\psi] = \int_{\Sigma_0} J^V_\mu[\psi] n^\mu_{\Sigma_0}.$$

Local Geometry of Extremal Reissner-Nordström

We will start by investigating solutions to the wave equation on extremal R-N. In view of the spherical symmetry, these spacetimes serve as a toy model for extremality.

- In the ingoing Eddington-Finkelstein coordinates $(v, r, \omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$ the metric of extremal R-N takes the form

$$g = -Ddv^2 + 2dvdr + r^2 g_{\mathbb{S}^2}, \tag{2}$$

  where

$$D = D(r) = \left(1 - \frac{M}{r}\right)^2$$

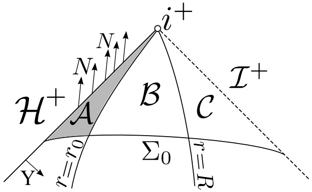
  and $g_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$. We denote $T = \partial_v, Y = \partial_r$.
- We will refer to the hypersurface $r = M$ as the event horizon (and denote it by $\mathcal{H}^+$) and the region $r \leq M$ as the black hole region. The region where $M < r$ corresponds to the domain of outer communications.

The Region $\mathcal{A}$ Near $\mathcal{H}^+$

- In view of the existence of null geodesics (with bounded r) on $\mathcal{H}^+$, one could expect trapping to take place on the event horizon. However, this is not the case in the **subextremal** case.
- Indeed, for the subextremal case, Dafermos and Rodnianski (2005) constructed a ϕ^T-invariant timelike vector field N which satisfies the following

  $$K^N[\psi] \geq J^N_\mu[\psi]n^\mu, \text{ in } \mathcal{A}.$$

  Their construction relied on the positivity of the surface gravity $\kappa > 0$ on the event horizon. Recall that κ satisfies $\nabla_T T = \kappa T$ on $\mathcal{H}^+$.
- Physically, the positivity of the surface gravity is related to the so-called *redshift effect* that is observed along (and close to) $\mathcal{H}^+$.



A New Multiplier and Trapping on $\mathcal{H}^+$

- Using the redshift vector field N, Dafermos and Rodnianski showed

$$\int_{\Sigma_\tau} J^N_\mu[\psi]n^\mu + \int_{\mathcal{A}} J^N_\mu[\psi]n^\mu \leq C \int_{\Sigma_0} J^N_\mu[\psi]n^\mu.$$

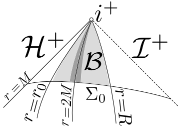
- However, in the extremal case, we have $\kappa = 0$ and no redshift takes place on $\mathcal{H}^+$.
- This can be resolved by introducing a Lagrangian term of the form $\psi\nabla_\mu\psi$. Using also appropriate Hardy inequalities and a spacetime estimate in region $\mathcal{B}$ one can then show the following

$$\int_{\Sigma_\tau} J^N_\mu[\psi]n^\mu + \int_{\mathcal{A}} \left(1 - \frac{M}{r}\right) J^N_\mu[\psi]n^\mu \leq C \int_{\Sigma_0} J^N_\mu[\psi]n^\mu.$$

- The above estimate degenerates at $\mathcal{H}^+$. In order to remove this degeneracy one needs to commute with Y (however, such commutation is not always possible!).
- This allows us to conclude that **the trapping effect takes place on degenerate horizons**.

The Intermediate Region $\mathcal{B}$ and Trapping Effect

- Existence of trapped null geodesics for general black hole backgrounds.
- On extremal R-N, all trapped null geodesics approach $r = 2M$ which is the so-called photon sphere.



- The significance of this hypersurface is that any L^2 spacetime estimate over $\mathcal{B}$ must degenerate exactly on the photon sphere.
- Construction of a purely physical space multiplier.
- The degeneracy may be removed by 'losing' derivatives. This reveals that (unstable) **trapping** takes place in $\mathcal{B}$.
- Dispersion away from $\mathcal{H}^+$ necessary even for boundedness of non-degenerate energy.

New Difficulties of Extremal Black Holes

1. Near horizon geometry: **degeneracy** of redshift effect
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The Far Away Region $\mathcal{C}$

- Regarding region $\mathcal{C}$, Dafermos and Rodnianski (2009) have introduced a method for obtaining a hierarchy of estimates with appropriate weights at infinity.
- In the **subextremal** case, one can obtain definitive energy decay by combining this hierarchy with the non-degenerate estimates in regions $\mathcal{A}, \mathcal{B}$.
- However, in view of the trapping on $\mathcal{H}^+$, this method cannot be directly applied in the extremal case. In fact, one is left with showing decay for the degenerate energy $\int_{\Sigma_0} J^T_\mu[\psi] n^\mu$. This would at least give us some form of decay *away* $\mathcal{H}^+$.

A Novel Vector Field P

In order to obtain the full decay for the degenerate energy we apply as a multiplier a novel vector field P.

Specifically, *there exists a translation-invariant causal vector field P such that the following hierarchy of estimates holds in $\mathcal{A}$ (S.A. 2010)*

$$\int_{\tau_1}^{\tau_2}\left(\int_{\mathcal{A}\cap\Sigma_\tau} J^T_\mu[\psi]n^\mu_{\Sigma_\tau}\right)d\tau \leq C\int_{\Sigma_{\tau_1}} J^P_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}$$

and

$$\int_{\tau_1}^{\tau_2}\left(\int_{\mathcal{A}\cap\Sigma_\tau} J^P_\mu[\psi]n^\mu_{\Sigma_\tau}\right)d\tau \leq C\int_{\Sigma_{\tau_1}} J^N_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}$$

The vector field P 'sits' in some sense between T and N. In fact, $P \sim T - \left(1-\frac{M}{r}\right)Y$.

- Decay of degenerate energy
- Pointwise decay for ψ up to and including $\mathcal{H}^+$.

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Stability Results for Extremal Reissner-Nordström, I

Let J denote the standard energy flux and N be a globally timelike (translation-invariant future directed) vector field. For all solutions ψ to the wave equation we have (S.A. 2010):

Boundedness of Non-Degenerate Energy:

$$\int_{\Sigma_\tau} J^N_\mu[\psi] n^\mu_{\Sigma_\tau} \leq C \int_{\Sigma_0} J^N_\mu[\psi] n^\mu_{\Sigma_0}$$

Integrated Local Energy Decay:

$$\int_{\mathcal{A}} \left[(T\psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] + (r-M)(Y\psi)^2 + \int_{\mathcal{B}} (r-2M)^2 J^N_\mu[\psi] n^\mu + \int_{\mathcal{C}} \frac{1}{r^2} J^N_\mu[\psi] n^\mu \leq C \int_{\Sigma_0} J^N_\mu[\psi] n^\mu,$$

where $\mathcal{A} = \{M \leq r \leq r_0 < 2M\}$, $\mathcal{B} = \{r_0 \leq r \leq R_0\}$, $\mathcal{C} = \{2M < R_0 \leq r\}$.

Stability Results for Extremal Reissner-Nordström, II

For all solutions ψ to the wave equation we have (S.A. 2010):

Decay of Degenerate Energy: $\int_{\Sigma_\tau} J^T_\mu[\psi]n^\mu \leq C\frac{1}{\tau^2}$

Decay of Non-Degenerate Energy: $\int_{\Sigma_\tau} J^N_\mu[\psi_{\geq k}]n^\mu \leq \frac{C}{\tau^k},\ k=1,2,$

where $\psi_{\geq l}$ denotes the projection of ψ on the eigenspaces of the spherical Laplacian $\triangle\!\!\!/$ with spherical number greater or equal to l.

Pointwise Decay: $\begin{cases} |\psi(\tau,\cdot)| \leq C\frac{1}{\tau^{\frac{3}{5}}}, \\ |\psi_{\geq 1}(\tau,\cdot)| \leq C\frac{1}{\tau^{\frac{3}{4}}}, \\ |\psi_{\geq 2}(\tau,\cdot)| \leq C\frac{1}{\tau}, \end{cases}$

for all $r \geq M$.

Local Geometry of Extremal Kerr

In the ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ^*) the metric takes the form

$$g = g_{vv} dv^2 + g_{rr} dr^2 + g_{\phi^*\phi^*} (d\phi^*)^2 + g_{\theta\theta} d\theta^2 \\ + 2g_{vr} dv dr + 2g_{v\phi^*} dv d\phi^* + 2g_{r\phi^*} dr d\phi^*,$$

where

$$g_{vv} = -\left(1 - \frac{2Mr}{\rho^2}\right), \quad g_{rr} = 0, \quad g_{\phi^*\phi^*} = g_{\phi\phi}, \quad g_{\theta\theta} = \rho^2$$

$$g_{vr} = 1, \quad g_{v\phi^*} = -\frac{2Mar\sin^2\theta}{\rho^2}, \quad g_{r\phi^*} = -a\sin^2\theta,$$

where $\rho^2 = r^2 + a^2 \sin^2\theta$. One can immediately read off that the vector fields $T = \partial_v$ and $\Phi = \partial_{\phi^*}$ are Killing. Of course, the metric g is not spherically symmetric.

We will refer to the hypersurface $r = M$ as the event horizon (and denote it by $\mathcal{H}^+$) and the region $r < M$ as the black hole region. The exterior region corresponds to $r > M$.

Main Difficulties of (general) Kerr

1. Superradiance
   - The Killing field T is spacelike in a region $\mathcal{E}$ outside $\mathcal{H}^+$ known as *ergoregion* ($\mathcal{H}^+ \subset \mathcal{E}$ and $\sup_{\mathcal{E}} r = 2M$).
   - In the context of $\Box_g \psi = 0$, this means that the flux $J^T_\mu[\psi] n^\mu_{\Sigma_\tau}$ fails to be non-negative definite.
   - Hence, we can have $\int_{\mathcal{I}^+} J^T[\psi] n^\mu_{\mathcal{I}^+} > \int_{\Sigma_0} J^T_\mu[\psi] n^\mu_{\Sigma_0}$.
2. More complicated trapping in $\mathcal{B}$.
   - There are trapped null geodesics with constant r for an open range of Boyer-Lindquist r values.
   - Hence, dispersive estimate in intermediate region $\mathcal{B}$ must degenerate in for an open range of r.

Regarding extremal Kerr, we additionally need to overcome the degeneracy of redshift.

To simplify our analysis, we consider **axisymmetric** solutions to the wave equation.

Why Axisymmetry?

1. Superradiance vs Axisymmetry: superradiance is absent if ψ is assumed to be axisymmetric. In fact, if $\Phi\psi = 0$ then

$$J_\mu^T[\psi]n^\mu_{\Sigma_\tau} \sim (T\psi)^2 + \left(1 - \frac{M}{r}\right)^2 (Y\psi)^2 + |\nabla\!\!\!/\psi|^2$$

for $r \leq R$ where the constants in $\sim$ depend on M and R. Hence, the axisymmetry allows us to concentrate on the effects of the degenerate redshift and the lack of spherical symmetry.

2. Trapping vs Axisymmetry: All trapped null geodesics orthogonal to the axial Killing vector field Φ approach $r = (1 + \sqrt{2})M$. Therefore, one expects to derive an estimate which degenerates only on this hypersurface.

More about the general (non-axisymmetric) case later.

Integrated Local Energy Decay

- Main goal: Dispersive estimate in intermediate region $\mathcal{B}$. Such an estimate is needed even for the boundedness of the non-degenerate energy.
- In view of the lack of spherical symmetry, the multiplier for extremal R-N cannot be applied for obtaining integrated decay in extremal Kerr.
- Insight of Dafermos and Rodnianski: The separability of the wave equation can be viewed as a geometric microlocalization which allows us to frequency-localize energy currents
- Then, it suffices to construct a microlocal multiplier for each frequency separately.
- This microlocalization, however, requires taking Fourier transform in time. For this reason, we cut off in time.

A Geometric Microlocalization

If $\psi_{\text{✂}} = \xi_\tau \psi$ is compactly supported in t and satisfies the inhomogeneous wave equation $\Box_g \psi_{\text{✂}} = F$, then we have the following frequency decomposition of $\psi_{\text{✂}}$:

$$\psi_{\text{✂}}(t, r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{\omega \in \mathbb{R}} \overbrace{\sum_{m,\ell} \underbrace{R_{\omega,m,\ell}(r) \cdot S^{(a\omega)}_{m\ell}(\theta, \phi)}_{\text{Oblate Spheroidal Expansion}} \cdot e^{-i\omega t}}^{\text{Fourier Expansion}} \, d\omega.$$

If we define $u^{(a\omega)}_{m\ell}(r) = \sqrt{(r^2 + a^2)} \cdot R_{\omega,m,\ell}(r)$ then by suppressing the indices, the Carter's equation reads

$$u'' + (\omega^2 - V(r))u = H,$$

where V is a potential function and H depends on the cut-off ξ_τ. In view of Parseval's identity, for $M < r_0 < R_0$ and b a uniform constant in frequencies and in cut-off time, we ideally want to construct microlocal currents $J^{(a\omega)}_{m\ell}[u]$ such that

$$b \int_{r_0}^{R_0} \left[|u|^2 + |u'|^2 + \left[\Lambda + \omega^2\right] |u|^2 \right] dr^* \leq J[u](r = R_0) - J[u](r = r_0)$$

for all frequencies (ω, Λ), where Λ is an angular frequency.

Geometric Microlocalization vs Redshift

- Dafermos and Rodnianski have constructed such currents $J^{(a\omega)}_{m\ell}[u]$ for the general subextremal Kerr family, which, however, require coupling with the redshift effect on $\mathcal{H}^+$.
- Therefore, the above currents cannot be readily adapted to extremal Kerr.
- However, we still have the following

Theorem (S.A. 2011)

For all frequencies (ω, m, ℓ), there exist currents $J^{(a\omega)}_{m\ell}[u]$ which are completely decoupled from the redshift effect. Hence, the local L^2 spacetime integral controlling the 1-jet of ψ in region $\mathcal{B}$ is bounded by the initial (conserved) T-flux.

Microlocal Energy Currents

- We construct appropriate multipliers $y(r^*), h(r^*), f(r^*)$ such that combinations of the currents

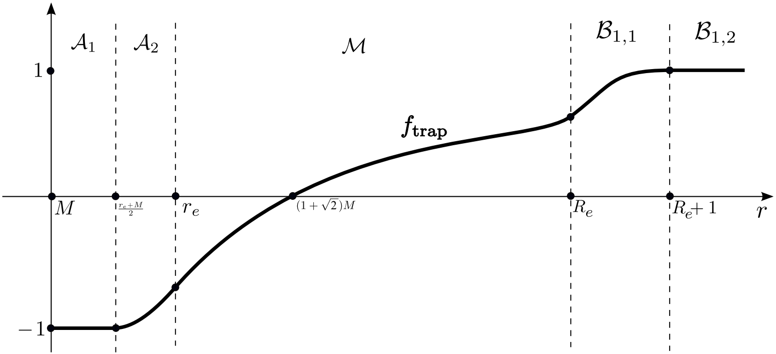
$$\mathcal{J}_1^y[u] = y\left[|u'|^2 + (\omega^2 - V)|u|^2\right],$$
$$\mathcal{J}_2^h[u] = h\text{Re}(u'\overline{u}) - \frac{1}{2}h'|u|^2,$$
$$\mathcal{J}_3^f[u] = f\left[|u'|^2 + (\omega^2 - V)|u|^2\right] + f'\text{Re}(u'\overline{u}) - \frac{1}{2}f''|u|^2.$$

have positive-definite derivatives.

- Appropriate multipliers for several frequency ranges.
- Main difficulty: Low frequencies

Degenerate Estimate for the Trapped Frequencies $\Lambda \sim \omega^2$

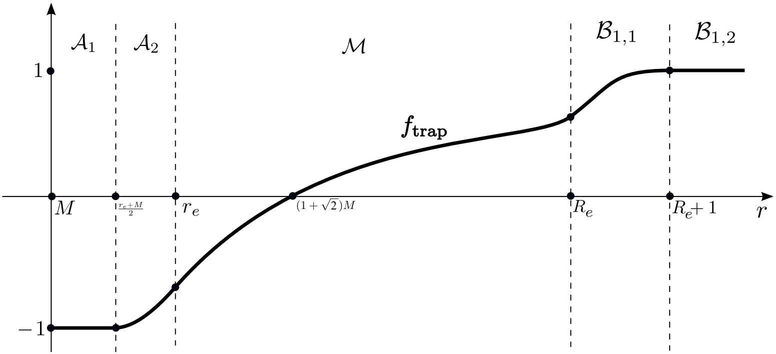
$$\left(\mathcal{J}_3^f[u]\right)' = 2f'|u'|^2 + \left(-fV' - \frac{1}{2}f'''\right)|u|^2 + E(H).$$



Time Dominated High Frequencies $\Lambda \precsim \omega^2$

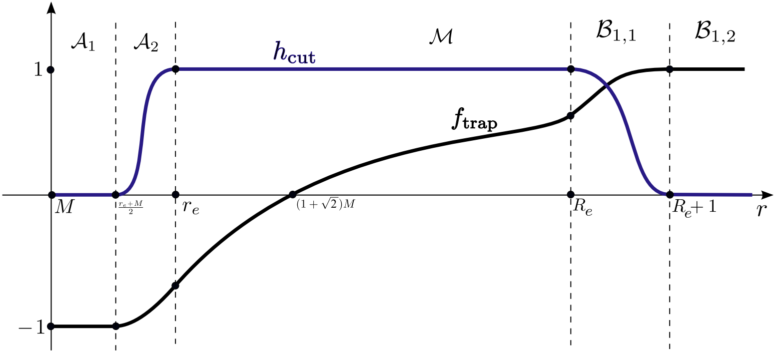
$$(\mathcal{J}_1^y[u])' = y'|u|^2 + y' \cdot (\omega^2 - V)|u|^2 - yV'|u|^2 + E(H).$$

We choose $y = f_{\text{trap}}$:



Angular Dominated High Frequencies $\omega^2 \precsim \Lambda$

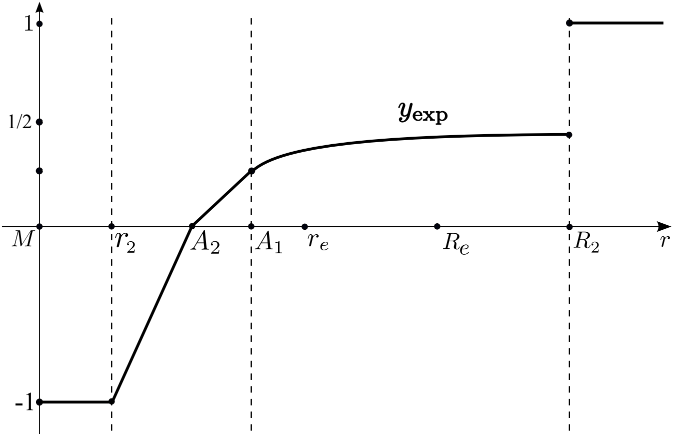
$$\left(\mathcal{J}_2^h[u] + \mathcal{J}_3^f[u]\right)' = (2f' + h)|u'|^2 + \left[h \cdot (V - \omega^2) - \frac{1}{2}h'' - fV' - \frac{1}{2}f'''\right]|u|^2 + E(H),$$



The Non-Stationary Low Frequencies

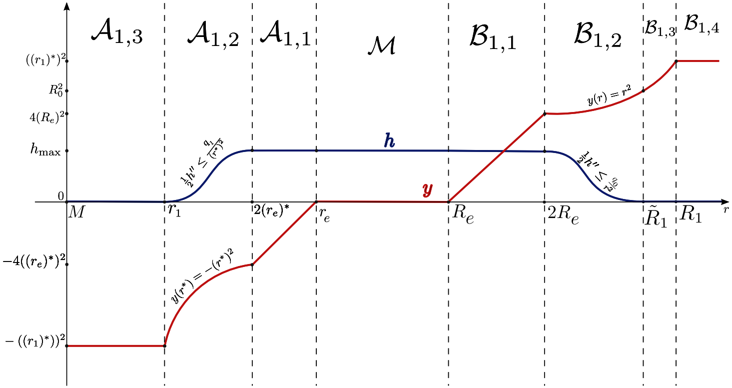
- Low frequency multipliers exploit the geometric properties of the two ends.

$$(\mathcal{J}_1^y[u])' = y'|u'|^2 + \omega^2 y'|u|^2 - (yV)'|u|^2 + E(H).$$



The Near-Stationary Low Frequencies

- Extremal end exhibits similar properties to the asymptotically flat end!



Holzegel and Smulevici have very recently adapted this current for use in Kerr-AdS.

New Difficulties of Extremal Black Holes

1. Near horizon geometry: **degeneracy** of redshift effect
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Stability Results on Extremal Kerr under Axisymmetry

The current capturing the degenerate redshift can be adapted for use in extremal Kerr. Moreover, one can also construct the analogue of the vector field P. Hence, we can still get the following stability results:

1. Dispersive estimates in region $\mathcal{B}$ (ILED)
2. Uniform boundedness of non-degenerate energy
3. Trapping effect on the event horizon
4. Decay of Degenerate Energy
5. Pointwise Decay; commutation with the second order (hidden) symmetry operator of extremal Kerr.

What about eliminating the degeneracies?

Instability Results for Extremal Reissner-Nordström

Theorem (S.A. 2010)

For generic solutions ψ to the wave equation we have:

Non-Decay:

The transversal to $\mathcal{H}^+$ derivative $Y\psi$ does not decay along $\mathcal{H}^+$.

Pointwise Blow-up:

$$\left|Y^k\psi\right| \to +\infty,$$

along $\mathcal{H}^+$ for all $k \geq 2$.

Energy Blow-up:

$$\int_{\Sigma_\tau} J_\mu^N[Y^k\psi]n^\mu \to +\infty$$

as $\tau \to +\infty$ for all $k \geq 1$.

Conservation Laws

The “source” of all instability results in extremal R-N is the following:

Proposition (S.A. 2010)

For all $\ell \in \mathbb{N}$ there exist constants $\beta_i, i = 0, 1, ..., \ell$, which depend on M and ℓ such that for all solutions ψ which are supported on the (fixed) angular frequency ℓ the quantity

$$H_l[\psi] = Y^{\ell+1}\psi + \sum_{i=0}^{\ell} \beta_i Y^i \psi$$

is conserved along the null geodesics of $\mathcal{H}^+$. Moreover, $H_\ell[\psi](\theta, \phi) \neq 0$ almost everywhere on the sphere for generic solutions ψ.

Conservation Laws along Degenerate Horizons

Conservation laws hold for more general spherically symmetric extremal black hole spacetimes.

Proposition (S.A. 2010)

Let the metric with respect to the coordinate system (v, r, θ, ϕ) *take the form*

$$g = -Ddv^2 + 2dvdr + r^2 g_{\mathbb{S}^2},$$

for a general $D = D(r)$*. If this spacetime has a black hole region and the event horizon is located at* $r = r_{\mathcal{H}^+}$ *where* $D(r_{\mathcal{H}^+}) = 0$*, then the previous conservation laws still hold if*

$$D'(r_{\mathcal{H}^+}) = 0, \qquad (3)$$

$$D''(r_{\mathcal{H}^+}) = \frac{2}{r_{\mathcal{H}^+}^2}. \qquad (4)$$

The equation (3) expresses the extremality of the black hole and (4) is required for frequencies $\ell \geq 1$.

Proof of Conservation Laws

Recall that $D = \left(1 - \frac{M}{r}\right)^2$ and let $R = D' + \frac{2D}{r}$. The wave equation in local coordinates

$$DYY\psi + 2TY\psi + \frac{2}{r}T\psi + RY\psi + \not\Delta\psi = 0.$$

- For $l = 0$, the quantity $Y\psi + \frac{1}{M}\psi$ is conserved along $\mathcal{H}^+$.
- What about higher spherical numbers ℓ? If, for example, $\ell = 1$ then

$$2TY\psi + \frac{2}{M}T\psi = \frac{2}{M^2}\psi \tag{5}$$

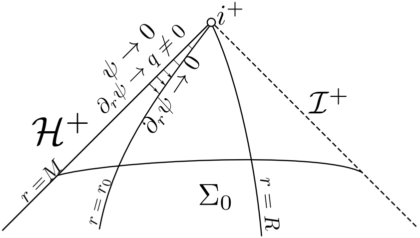
on $\mathcal{H}^+$. Moreover,

$$\begin{aligned} Y\left(\Box_g\psi\right) =& DYYY\psi + 2TYY\psi + \frac{2}{r}TY\psi + RYY\psi \\ & + Y\not\Delta\psi + D'YY\psi - \frac{2}{r^2}T\psi + R'\partial_r\psi = 0 \end{aligned}$$

Therefore, for $\ell = 1$, the quantity $YY\psi + \frac{3}{M}Y\psi + \frac{1}{M^2}\psi$ is conserved along the null geodesics of $\mathcal{H}^+$.

Energy and Pointwise Blow-up

- The decay of the degenerate energy and the fact that T is timelike in the exterior region imply that the waves fully disperse away from $\mathcal{H}^+$.
- Therefore, for any $r_0 > M$ we have $Y\psi \to 0$ along $r = r_0$. Moreover, recall that $\psi \to 0$ along $\mathcal{H}^+$.
- Non-decay and blow-up results follow from the conservation laws and the dispersion of the waves away from the horizon.



Ongoing Work

1. Stability and instability of extremal Reissner-Nordström in the context of the Cauchy problem of the E-M-S field system under spherical symmetry.
2. Instabilities on extremal Kerr: Conservation laws do not hold (at least in physical space).
3. General solutions on extremal Kerr: New major difficulty since the upper limit of superradiant frequencies is trapped. This affects even the boundedness of energy and may be related to the existence of "essentially undamped" quasinormal modes located in this frequency regime. The latter phenomenon has been numerically investigated by Andersson and Glampedakis.

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1. Near horizon geometry: **degeneracy** of redshift effect

2. Dispersion in intermediate region: trapping effect

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Conclusion

- Our results suggest that the redshift is a necessary ingredient to decay and not just a helpful mechanism.
- Instability of Degenerate Horizons?
- Extremality has opened a new area in this problem, with novel phenomena and fascinating results.

THANK YOU!