Asymptotic Solution of the Wang-Uhlenbeck Recurrence Time Problem

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A Langevin particle is initiated at the origin with positive velocity. Its trajectory is terminated when it returns to the origin. In 1945, Wang and Uhlenbeck posed the problem of finding the joint probability density function (PDF) of the recurrence time and velocity, naming it "the recurrence time problem." We show that the short-time asymptotics of the recurrence PDF is similar to that of the integrated Brownian motion, solved in 1963 by McKean. We recover the long-time $t^{-3/2}$ decay of the first arrival PDF of diffusion by solving asymptotically an appropriate variant of McKean's integral equation.

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The recurrence problem of a free Brownian particle was posed in [1] and has appeared since in different variants in such diverse physical applications as the inelastic collapse of a randomly forced particle in granular media [2–8] and the simulation of permeation of ions in protein channels of biological membranes [9,10], to name but a few. The problem can be formulated as follows. The one-dimensional Brownian motion in the gravitational field is described by the Langevin equation [11]

$$\ddot{x} + \gamma \dot{x} = -g + \sqrt{2\gamma\varepsilon}\dot{w}, \tag{1}$$

where γ is the friction (damping) parameter, ε is the noise strength (temperature), g is the gravitational acceleration, and $\dot{w}$ is a δ-correlated white (Gaussian) noise. Wang and Uhlenbeck (1945) [1] posed, among others, the *reccurrence time problem* (RT), to find the joint probability density function (PDF) of the time and velocity of first return of the Brownian motion to the origin. Another problem is that of finding the PDF of the first passage time to the origin from a point $x_0 > 0$. The latter problem was first solved in [12], whereas the former is solved here.

The PDF to find the diffusive particle at time t at position x with velocity v, given it was initiated at time $t = 0$ at location $x_0 \geq 0$, with initial velocity v_0, denoted $p(x, v, t|x_0, v_0)$, satisfies the Fokker-Planck equation [11]

$$\frac{\partial p}{\partial t} = -v\frac{\partial p}{\partial x} + \frac{\partial}{\partial v}[(\gamma v + g)p] + \varepsilon\gamma\frac{\partial^2 p}{\partial v^2}, \tag{2}$$

with the initial condition

$$p(x, v, 0|x_0, v_0) = \delta(x - x_0)\delta(v - v_0), \tag{3}$$

and the absorbing boundary condition

$$p(0, v, t) = 0, \qquad v > 0. \tag{4}$$

The absorbing boundary condition reflects the fact that the first passage time of particles with and without absorption is the same. The motion after this time is irrelevant, so we may assume that the particle is absorbed, and there are no incoming trajectories from left to right (characterized by positive velocity) at the origin. In the RT problem $x_0 = 0$, that is, the particle is initiated at the location of the absorbing boundary with a positive velocity $v_0 > 0$, whereas the first passage time to the origin problems are identified by $x_0 > 0$ and the initial velocity is not necessarily positive. In both problems the goal is to find the PDF $p(x, v, t|x_0, v_0)$.

Titulaer *et al.* ([13], and references therein) used several numerical approximation methods to solve the steady state problem. The RT and the first passage time problems were renamed the albedo and the Milne (in analogy to transport theory) problems, respectively. The mean exit time and the stationary exit distribution were calculated by Marshall and Watson [14]; Hagan, Doering, and Levermore [15]; and Kłosek [16] for the strip $-A < x < B$, $-\infty < v < \infty$ in the phase plane in the limit of large γ. A simplified calculation of the mean first-passage time (MFPT) and the asymptotic distribution of passage times for large t in the presence of an attracting field were presented in [17].

The distribution of the first passage time (the time-dependent Milne problem) was first determined in [12], where a uniform expansion for both short and long times was found. The analysis in [12] is based on constructing an initial time layer of size $t_0 = 2/\gamma$. The probability to exit before t_0 is transcendentally small in γ, as the particle is initiated away from the absorbing boundary. The above mentioned references contain the solution to Wang and Uhlenbeck's first passage time problem, yet the RT problem of Wang and Uhlenbeck remained unresolved. Marshall and Watson [14] computed the Laplace transform of the RT distribution in terms of infinite weighted sum of eigenfunctions. However, quoting ([14], p. 3542) *"The Laplace inversion . . . appears to be out of question."* The initial time layer method [12] cannot be applied in the RT problem, because, as seen below, particles are absorbed in arbitrarily short times with non-negligible probabilities.

The RT of a trajectory $x(t)$ is the first time it returns to the origin (or to any other point),

$$\tau = \inf_{t>0}\{t: x(t) = 0\}.$$

The probability density of returning to the origin for the first time with a given velocity,

$$f(v, t|v_0) = \Pr\{\tau = t, v(\tau) = v|x_0 = 0, v_0\}, \tag{5}$$

is called the joint recurrence density. Obviously, for $v > 0$, $v_0 > 0$, and for $v < 0$, $v_0 < 0$

$$f(v, t|v_0) = 0. \tag{6}$$

The Wang-Uhlenbeck RT problem is to determine $f(v, t|v_0)$ for $v_0 > 0$ and $v < 0$ for the problem (1), which we denote $f_c(v, t|v_0)$.

A closely related RT problem for the integrated Brownian motion (IBM),

$$\ddot{x} = \sqrt{2\gamma\varepsilon}\dot{w}, \tag{7}$$

was solved in [18] (see also [19–22]), where the explicit expression

$$f_{\mathrm{IBM}}(v, t|v_0) = \frac{|v|\sqrt{3}}{2\pi\varepsilon\gamma t^2} \times \exp\left\{-\frac{v^2 + vv_0 + v_0^2}{\varepsilon\gamma t}\right\}\mathrm{erf}\left(\sqrt{\frac{3|v|v_0}{\varepsilon\gamma t}}\right) \tag{8}$$

is given.

For both the free Brownian particle and the IBM the recurrence density $f(v, t|v_0)$ can be understood as follows. A trajectory of a particle that starts at the origin with $v_0 > 0$, and that diffuses on the entire line, returns to the origin with alternating negative and positive velocities any number of times. This means that for $t > 0$, $v_0 > 0$, and for all v the recurrence density satisfies the integral equation [18–22].

$$vp_i(0, v, t|0, v_0) = -f_i(v, t|v_0) + v\int_0^t ds \times \int_{-\infty}^0 d\eta f_i(\eta, s|v_0)p_i(0, v, t-s|0, \eta), \tag{9}$$

where $i = c$, IBM. Indeed, consider the unidirectional flux density of particles that cross the origin $x = 0$ with velocity v at time t. On the one hand, this unidirectional flux is $vp_i(0, v, t|0, v_0)$ [9]; on the other hand, it has two different contributions. The contribution $-f_i(v, t|v_0)$ is due to the trajectories that return to the origin for the first time exactly at time t with velocity v, and the second contribution in (9) is due to trajectories that return to the origin at time t with velocity v after previous returns to the origin (at any previous times with any velocities). More specifically, due to the Markov property of the pair $(x(t), v(t))$, the unidirectional flux density of trajectories that cross the origin at time t with velocity v, given that it started with a positive velocity v_0, is the probability density $f_i(\eta, s|v_0)$ that it returns to the origin for the first time at $\tau_i = s < t$ with some negative velocity η, and then it returns to the origin with velocity v (with probability density $p_i(0, v, t-s|0, \eta)$).

Next, we construct an approximate solution by truncating the Neumann series for (9). We write the equation as

$$(I - \mathcal{L})f_c = -vp_c, \tag{10}$$

where the linear integral operator $\mathcal{L}$ is defined as

$$\mathcal{L}f_c = v\int_0^t ds \int_{-\infty}^0 d\eta f_c(\eta, s|v_0)p_c(0, v, t-s|0, \eta). \tag{11}$$

Marching in sufficiently small time steps, we can assume that $\| \mathcal{L} \| <1$, so the Neumann series expansion

$$f_c = -(I + \mathcal{L} + \mathcal{L}^2 + \cdots)vp_c \tag{12}$$

converges. The terms of the series (12) are readily computable numerically, but the short- and long-time asymptotics have to be evaluated analytically.

We note that the short-time ($\gamma t \ll 1$) asymptotics of the exact solution of the Fokker Planck equation (2) and (3) for a free Brownian motion on the entire line [11] is given by

$$p_c(x, v, t|x_0, v_0) \sim \sum_{n=1}^{\infty} t^{n-3}Z_n(x, v) \times \exp\left\{-\frac{\psi_0(x) + t\psi_1(x, v) + t^2\psi_2(x, v)}{\varepsilon\gamma t^3}\right\}, \tag{13}$$

(the dependence of the eikonals ψ_i on x_0, v_0 is suppressed), where

$$\psi_0(x) = 3(x - x_0)^2, \qquad \psi_1(x, v) = -3(x - x_0)(v + v_0),$$
$$\psi_2(x, v) = v^2 + vv_0 + v_0^2 + \frac{3}{10}\gamma^2(x - x_0)^2, \tag{14}$$

and

$$Z_1(x, v) = \frac{\sqrt{3}}{2\pi\varepsilon\gamma}\exp\left\{g\frac{(v_0 - v) + \gamma(x_0 - x)}{2\varepsilon\gamma}\right\} \times \exp\left\{\frac{\gamma(x - x_0)(v + v_0)}{20\varepsilon} + \frac{v_0^2 - v^2}{4\varepsilon}\right\}. \tag{15}$$

Setting $x = x_0 = 0$ in Eqs. (14), we find that both eikonals ψ_0 and ψ_1 vanish. Therefore, the short-time asymptotics is governed by the first nonvanishing eikonal $\psi_2(0, v) = v^2 + vv_0 + v_0^2$, so that

$$p_c(0, v, t|0, v_0) \sim \exp\left\{\frac{v_0^2 - v^2}{4\varepsilon} + g\frac{v_0 - v}{2\varepsilon\gamma}\right\}\frac{\sqrt{3}}{2\pi\varepsilon\gamma t^2} \times \exp\left\{-\frac{v^2 + vv_0 + v_0^2}{\varepsilon\gamma t}\right\}. \tag{16}$$

We note that the corresponding density for the IBM is

$$p_{\rm IBM}(0, v, t|0, v_0) = \frac{\sqrt{3}}{2\pi\varepsilon\gamma t^2}\exp\left\{-\frac{v^2 + vv_0 + v_0^2}{\varepsilon\gamma t}\right\}, \tag{17}$$

so that

$$p_c(0, v, t|0, v_0) \sim \exp\left\{\frac{v_0^2 - v^2}{4\varepsilon} + g\frac{v_0 - v}{2\varepsilon\gamma}\right\} \times p_{\rm IBM}(0, v, t|0, v_0). \tag{18}$$

To find the short-time asymptotics of $f_c(v, t|v_0)$, we use (18) in (9) with $i = c$. The resulting equation is identical to that for IBM, but instead of $f_{\rm IBM}(v, t|v_0)$, we have $f_c(v, t|v_0)\exp\{v_0^2/4\varepsilon + gv_0/2\varepsilon\gamma\}$. We conclude that the short-time expansion of $f_c(v, t|v_0)$ is given by

$$f_c(v, t|v_0) \sim \exp\left\{\frac{v_0^2 - v^2}{4\varepsilon} + g\frac{v_0 - v}{2\varepsilon\gamma}\right\} f_{\rm IBM}(v, t|v_0). \tag{19}$$

To find the long-time asymptotics of $f_c(v, t|v_0)$, we first Laplace transform (9) with respect to t,

$$v\hat{p}_c(0, v, \theta|0, v_0) = -\hat{f}_c(v, \theta|v_0) + v\int_{-\infty}^{0} d\eta \hat{f}_c(\eta, \theta|v_0)\hat{p}_c(0, v, \theta|0, \eta), \tag{20}$$

where θ is the Laplace time coordinate. The long-time asymptotics of $p_c(0, v, t|0, v_0)$ is obtained from [11] (for $g = 0$) as

$$p_c(0, v, t|0, v_0) = \frac{\gamma}{2\pi\varepsilon\sqrt{2\gamma t}}\exp\left\{-\frac{v^2}{2\varepsilon}\right\} + O\left(\frac{1}{\gamma t}\right). \tag{21}$$

Hence, the small θ asymptotics is

$$\hat{p}_c(0, v, \theta|0, v_0) = \frac{1}{2\sqrt{2\pi}\varepsilon}\exp\left\{-\frac{v^2}{2\varepsilon}\right\}\sqrt{\frac{\gamma}{\theta}} + O\left(\sqrt{\frac{\theta}{\gamma}}\right). \tag{22}$$

Expanding

$$\hat{f}_c(v, \theta|v_0) = f_0(v, v_0) + f_1(v, v_0)\sqrt{\frac{\theta}{\gamma}} + f_2(v, v_0)\frac{\theta}{\gamma} + \ldots, \tag{23}$$

and equating leading order terms, we find that

$$\int_{-\infty}^{0} f_0(\eta, v_0)d\eta = 1, \tag{24}$$

which means that the free particle is recurrent, and at $O(1)$, we find that

$$f_0(v, v_0) = \frac{v\exp\{-\frac{v^2}{2\varepsilon}\}}{2\sqrt{2\pi}\varepsilon}\int_{-\infty}^{0} f_1(\eta, v_0)d\eta. \tag{25}$$

Equations (24) and (25) give

$$f_0(v, v_0) = -\frac{v\exp\{-\frac{v^2}{2\varepsilon}\}}{\varepsilon}. \tag{26}$$

Similarly, $f_1(v, v_0)$ is determined by comparing terms of order $O(\sqrt{\frac{\theta}{\gamma}})$, as

$$f_1(v, v_0) = 2\sqrt{2\pi}\frac{v\exp\{-\frac{v^2}{2\varepsilon}\}}{\varepsilon}. \tag{27}$$

We conclude that the half-integer power series (23) is self consistent. Note that $\frac{d}{d\theta}\hat{f}_c \sim \frac{1}{\sqrt{\theta}}$ is singular for $\theta = 0$. Therefore, the mean time of return is infinite

$$\int_0^\infty tf_c(v, t|v_0)dt = -\frac{d}{d\theta}\hat{f}_c(v, 0|v_0) = \infty,$$

which is the Gambler's ruin paradox [23]. Moreover, $\frac{d}{d\theta}\hat{f}_c \sim \frac{1}{\sqrt{\theta}}$ means that the asymptotic behavior of $tf_c(\cdot, t|\cdot)$ for large t is the inverse Laplace transform of $\frac{1}{\sqrt{\theta}}$, which is proportional to $\frac{1}{\sqrt{t}}$, hence

$$f_c(v, t|v_0) \sim -\frac{v\exp\{-\frac{v^2}{2\varepsilon}\}}{\varepsilon\sqrt{2\gamma t^3}}, \quad \text{for} \quad \gamma t \gg 1. \tag{28}$$

Note that (28) means that at long times the return probability is to leading order independent of the initial velocity v_0, because the initial velocity is thermalized for times longer than relaxation time. The PDFs of the RT are the marginal densities

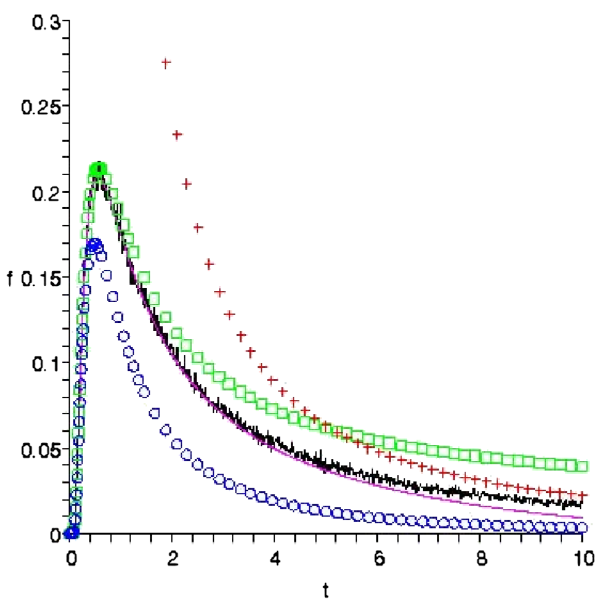

FIG. 1 (color online). Numerical and asymptotics of $\Pr\{\tau_c = t|v_0\}$, with $\gamma = 1$, $\varepsilon = 1$, $v_0 = 1$, $g = 0$. Black/noisy, simulation of 10^6 Langevin trajectories with $\Delta t = 10^{-3}$. Blue/circles, short-time asymptotics of the marginal density from (19). Red/crosses, long-time asymptotics (29). Green/boxes, first term of the Neumann series (12). Magenta/solid curve, numerical evaluation of two terms of the Neumann series.

$$\Pr\{\tau_i = t|v_0\} = \int_{-\infty}^{0} f_i(v, t|v_0)dv, \qquad i = c, \text{IBM}$$

The long-time PDF of the RT is independent of v_0 and is simply the marginal of (28),

$$\Pr\{\tau_c = t\} \approx \frac{1}{\sqrt{2\gamma t^3}}, \quad \text{for} \quad \gamma t \gg 1. \tag{29}$$

Note that the decay rate of the RT density is faster than that of the IBM, which is $O(t^{-5/4})$ [21].

We have three different approximations to $\Pr\{\tau_c = t\}$: the marginal of the truncated Neumann series (12), the marginal of the short-time approximation (19), and the long-time approximation (29). Figure 1 shows the three approximations together with results of simulations of Langevin trajectories. The Neumann series is truncated at one term (green, boxes) and at two terms (magenta, solid curve).

The approximate solutions of the RT problem can be used in the above mentioned applications, e.g., for the computation of the critical restitution coefficient and the collapse time for the case of the damped Brownian motion [2].

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[1] M. C. Wang and G. E. Uhlenbeck,Rev. Mod. Phys. **17**, 323 (1945); *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).
[2] S. J. Cornell, M. R. Swift, and A. J. Bray, Phys. Rev. Lett. **81**, 1142 (1998).
[3] T. W. Burkhardt, Phys. Rev. E **63**, 011111 (2001).
[4] D. J. Bicout and T. W. Burkhardt, J. Phys. A: Math. Gen. **33**, 6835 (2000).
[5] G. De Smedt, C. Godréche, and J. M. Luck, Europhys. Lett. **53**, 438443 (2001).
[6] L. Anton, Phys. Rev. E **65**, 047102 (2002).
[7] T. W. Burkhardt and S. N. Kotsev, Phys. Rev. E **70**, 026105 (2004).
[8] S. N. Kotsev and T. W. Burkhardt, Phys. Rev. E **71**, 046115 (2005).
[9] A. Singer and Z. Schuss, Phys. Rev. E **71**, 026115 (2005).
[10] B. Nadler, Z. Schuss, and A. Singer, Phys. Rev. Lett. **94**, 218101 (2005).
[11] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943); *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).
[12] P. S. Hagan, C. R. Doering, and C. D. Levermore, J. Stat. Phys. **54**, 1321 (1989).
[13] A. J. Kainz and U. M. Titulaer, J. Phys. A: Math. Gen. **24**, 4677 (1991).
[14] T. W. Marshall and E. J. Watson, J. Phys. A **18**, 3531 (1985).
[15] P. S. Hagan, C. R. Doering, and C. D. Levermore, SIAM J. Appl. Math. **49**, 1480 (1989).
[16] M. M. Kłosek, J. Stat. Phys. **79**, 313 (1995).
[17] T. W. Marshall and E. J. Watson, J. Phys. A: Math. Gen. **20**, 1345 (1987).
[18] H. P. McKean, Jr., Journal of Mathematics of Kyoto University **2**, 227 (1963).
[19] M. Goldman, Annals of Mathematical Statistics **42**, 2150 (1971).
[20] A. Lachal, Ann. Inst. Henri Poincaré **27**, 386 (1991).
[21] Y. Sinai, Theor. Math. Phys. (Engl. Transl.) **90**, 219 (1992).
[22] P. Groeneboom, G. Jongbloed, and J. A. Wellner, Ann. Probab. **27**, 323 (1999).
[23] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1957), 2nd ed.