Interlacing Families and Kadison–Singer

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Notations and Simplifications

Throughout this talk, the following will (hopefully) hold:
Notations:

- d is the dimension of the vector space
- m is the number of vectors
- v is a (deterministic) vector
- $\widehat{v}$ is a random variable that takes deterministic vectors as its possible values
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Simplifications:

- All vectors will be real vectors (in $\mathbb{R}^d$) [though all proofs will hold for complex vectors by replacing transposes with adjoints]
- All random vectors will choose from 2 possible values [though proofs can be extended to any type of random vector]

Goals

In this talk I plan to

1. Give a brief history of Kadison–Singer and its relatives
2. Convey how we went about attacking the problem
3. Introduce a technique for showing the existence of combinatorial objects we call "the method of interlacing polynomials"
4. Introduce a class of polynomials we call "mixed characteristic polynomials".
5. Use these to prove two known equivalents of Kadison–Singer
6. Discuss some related open questions

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And *please* interrupt if there are any questions.

Outline

The beginning of quantum mechanics

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This predates the formalization of quantum mechanics using C^* algebras.

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Precisely:

Question

Let $\mathcal{A}$ be a discrete maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a (separable, complex) Hilbert space. Let $\rho : \mathcal{A} \to \mathbb{C}$ be a pure state on that subalgebra. Is the (pure) extension $\rho' : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ of ρ to all of $\mathcal{B}(\mathcal{H})$ unique?

Note: Pure states are the rank 1 operators *plus* a bunch more guaranteed by Axiom of Choice (these are the ones Dirac ignored).

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Showed it is *not* true in the continuous case (their counterexample was $\mathcal{H} = L^2([0, 1])$).

A World of Equivalences

Kadison–Singer
(1959)

A World of Equivalences

Conjecture (Anderson Paving Conjecture (1979))

For every $\epsilon > 0$, there is a universal positive integer $k = k(\epsilon)$ so that for every zero-diagonal finite matrix A with n rows (and columns), there exists a partition $\{S_1, \ldots, S_k\}$ of $[n]$ so that

$$\|A[S_i, S_i]\| \leq \epsilon \|A\|$$

for all $i \in [k]$.

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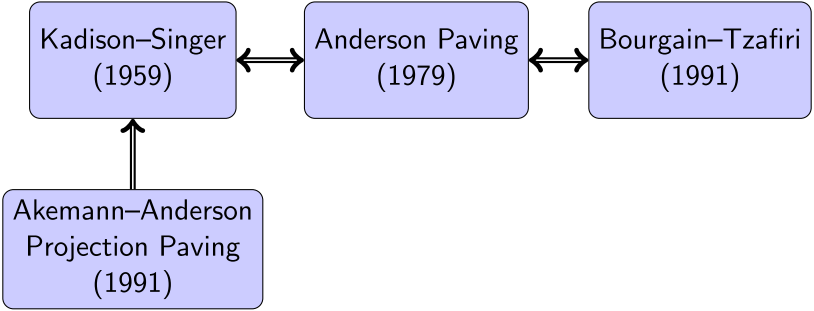

A World of Equivalences

Conjecture (Weaver's KS_r (2004))

There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds: if $w_1, \ldots, w_m \in \mathbb{C}^d$ satisfy $\|w_i\| \leq 1$ for all i and

$$\sum_i |\langle u, w_i \rangle|^2 = \eta$$

for all unit vectors $u \in \mathbb{C}^d$. Then there exists a partition of the vectors into parts $\{S_1, \ldots, S_r\}$ so that

$$\sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq \eta - \theta$$

for all unit vectors $u \in \mathbb{C}^d$ and each $j \in [r]$.

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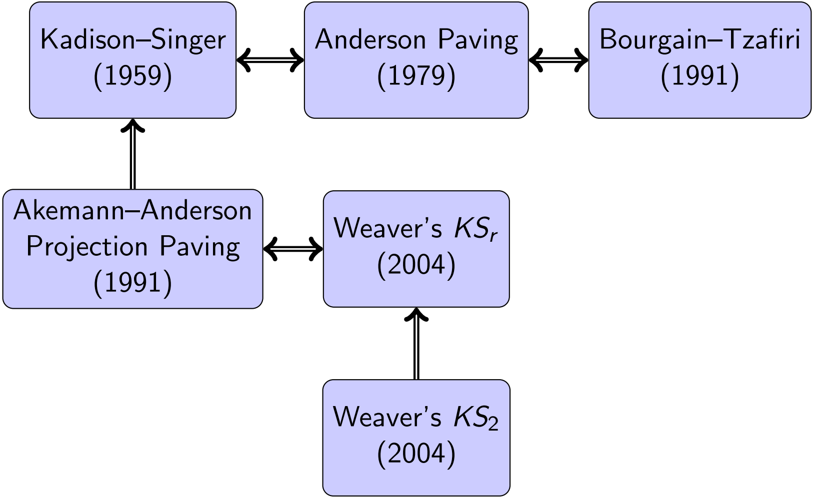

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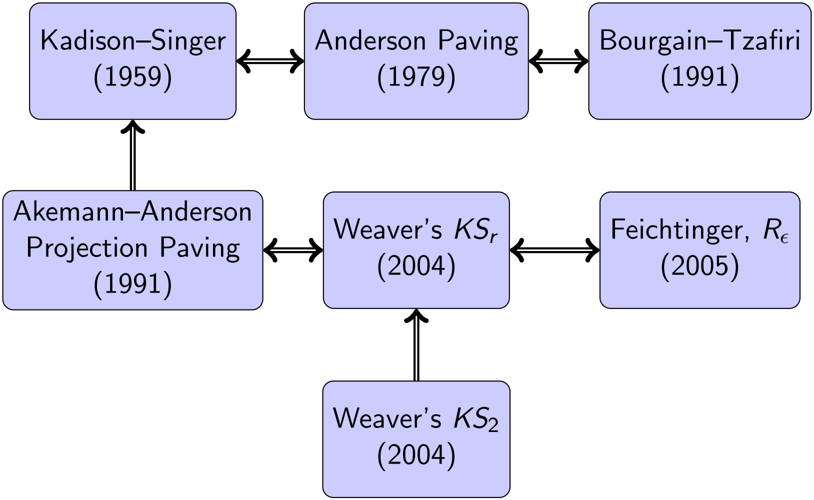

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Examples:

1. Orthonormal bases
2. Regular simplex centered at the origin (properly scaled)
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In general, no (there may be huge vectors). So what if the vectors are all bounded in size?

What if I choose randomly?

Tropp (2011) showed that a uniformly random choice of vectors works with high probability if

$$\|v_i\| \leq \frac{C(\epsilon)}{\log d}$$

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The goal would be to trade the $\log d$ factor in exchange for nonzero (instead of high) probability.

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3. The fundamental question is whether a Parseval frame can be partitioned into two "almost Parseval" frames
4. The fundamental question is true (with high probability) when the vectors have norm $O(1/\log d)$
5. We want to know what happens when the vectors bounded in norm by a *constant*

Outline

Probabilistic Approach

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Let $\widehat{v}_1, \ldots, \widehat{v}_m$ be random vectors that are Parseval in expectation:

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But what if (as before) the vectors were also bounded in norm?

Main Theorem

Our main technical theorem says the following:

Theorem

Let $\epsilon > 0$ and $\widehat{v}_1, \ldots \widehat{v}_m$ be independent random vectors such that

$$\sum_{i=1}^{m} \mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = I$$

and

$$\mathbb{E}\left[\|\widehat{v}_i\|^2\right] \leq \epsilon$$

for all i. Then there exists an assignment $\widehat{v}_i = v_i$ such that

$$\left\| \sum_{i=1}^{m} v_i v_i^T \right\| \leq (1 + \sqrt{\epsilon})^2.$$

In this talk, I will assume $\epsilon < 1/4$ and prove this for $(1 + 3\sqrt{\epsilon})$ (a slight weakening).

Translation and experimentation:

There are two benefits to dealing with vectors (rather than algebras):

1. Translation into the world of polynomials
   Given vectors $v_1, \ldots, v_m$, set $V = \sum_i v_i {v_i}^T$. Then the maximum eigenvalue of V is the largest root of $\chi_V(x)$ (the characteristic polynomial). So we can turn this into a question about the behavior of a special class of polynomials.
2. This formulation is ripe for experimentation.
   We can optimize over collections of vectors that satisfy the given constraints to see what the worst case scenarios are. We can see what the average scenarios are. We can see what the worst case average scenarios are.

Experimental Observations

So that is what we did, and these are the observations we made.

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Observation 2: When we looked at all possible values of $\widehat{V} = \sum_i \widehat{v}_i \widehat{v}_i^T$ (satisfying the hypotheses of the theorem), there always seemed to be one whose characteristic polynomial $\chi_V(x)$ had a smaller largest root than the expected characteristic polynomial $\mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$.

Experimental Observations, cont.

Observation 3: The expected characteristic polynomial $\mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$ seemed to have maximal largest root when $\mathbb{E}\left[\|\widehat{v}_i\|^2\right] = \epsilon$ and $\mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = \frac{\epsilon}{d} I$ for all i.

Experimental Observations, cont.

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Observation 4: In the case where $\mathbb{E}\left[\|\widehat{v}_i\|^2\right] = \epsilon$ and $\mathbb{E}\left[\|\widehat{v}_i\widehat{v}_i^T\|^2\right] = \frac{\epsilon}{d}I$ and for all i, the expected characteristic polynomial is an associated Laguerre polynomial (a classical orthogonal polynomial whose roots satisfy the bounds we were hoping for).

Suggests an approach

If we could

1. bound the largest root of $\mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$ over the set of random vectors that satisfy the hypotheses of the theorem, and then
2. show there always exists an assignment $v_1, \ldots, v_m$ such that the largest root of $\chi_V(x)$ is smaller than the largest root of $\mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$

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We start with part (2). Main Idea: define a process by which we pick the assignments one by one and try to understand how the (now conditional) expected characteristic polynomial changes.

Outline

Formalization

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We also define a *partial choice* vector $\sigma' \in \{0,1\}^k$ $(k < m)$. The corresponding polynomial will be the conditional expectation.

$$p_{\sigma'} = \mathbb{E}_{\widehat{v}_{k+1},\ldots,\widehat{v}_d}\left[\chi(\widehat{V})(x) \mid \widehat{v}_i = v_i^{\sigma'_i} \text{ for } 1 \leq i \leq k\right]$$

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Note that $p_\varnothing = \mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$, the expected characteristic polynomial we are interested in.

Sums of polynomials

We have the relation

$$p_{\sigma'}(x) = p_{\sigma',0}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v_{k+1}^{0}\right] + p_{\sigma',1}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v_{k+1}^{1}\right]$$

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Approach: forget this issue and see what we can prove.

A Lemma

Lemma

Let f *and* g *be monic polynomials. Assume there exists a point* $c \in \mathbb{R}$ *such that* f *and* g *each has* exactly *one real root larger than* c *(call these the "extreme roots"). Then the largest real root of* $f + g$ *lies between these extreme roots.*

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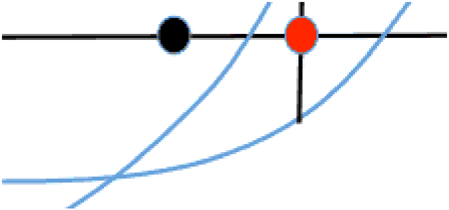
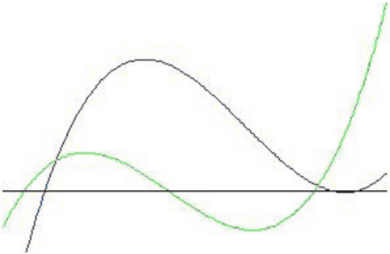
Proof.

By picture



□

Without c to "anchor"



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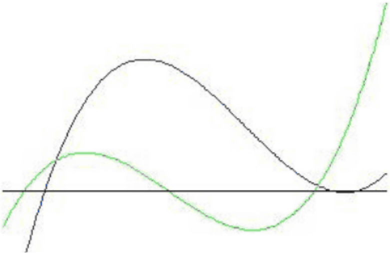

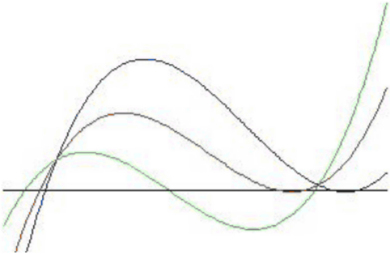

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While our original polynomials (characteristic polynomials of Hermitian matrices) are real-rooted, in general the sums of real-rooted polynomials can be arbitrary.

Example: $p(x) = (x-2)^2 - 1$ (has double root at 1) and $q(x) = (x+2)^2 - 1$ (has double root at -1).

$$p(x) + q(x) = x^2 + 6$$

does not have any real roots (roots are $\pm\sqrt{-6}$).

Equation Revisited

Back to our equation

$$p_{\sigma'}(x) = p_{\sigma',0}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v_{k+1}^{0}\right] + p_{\sigma',1}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v_{k+1}^{1}\right]$$

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The lemma tells us that if

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Let's worry about c for the moment (keeping real-rootedness on the back burner).

Interlacing polynomials

Let p be a real-rooted polynomial of degree n and q a real-rooted polynomial of degree $n-1$

$$p(x) = \prod_{i=1}^{n}(x-\alpha_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{n-1}(x-\beta_i)$$

with $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_{n-1}$

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We say q *interlaces* p if $\alpha_1 \leq \beta_1 \leq \alpha_2 \cdots \leq \alpha_{d-1} \leq \beta_{n-1} \leq \alpha_n$.

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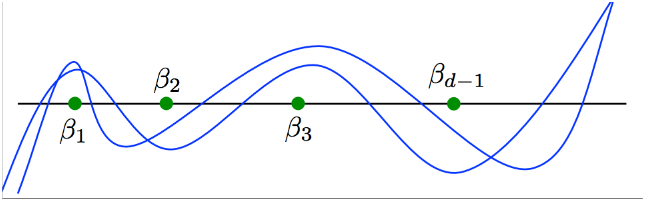
Example 1: $p'(x)$ interlaces $p(x)$
Example 2: If p has no multiple roots (and largest root R), then let $q = p/(x - R)$. Then $q(x + \epsilon)$ interlaces $p(x)$

Common Interlacers

We say that two degree n polynomials p and r have a *common interlacer* if there exists a q such that q interlaces *both* p and r simultaneously.

Think: the roots of q split up $\mathbb{R}$ into n intervals, each of which contains exactly one root of p and one root of r



Common Interlacers

We say that two degree n polynomials p and r have a *common interlacer* if there exists a q such that q interlaces *both* p and r simultaneously.

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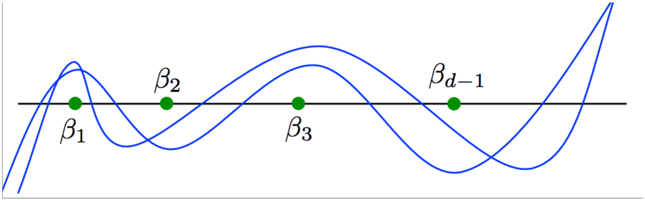

Note, if p and r have a common interlacer (say q), then $c = \beta_{d-1}$ can serve as the anchor from the lemma!

Interlacing families

We say $\{p\}_{\sigma \in \Sigma}$ is an *interlacing family* if for all partial assignments σ' we have that

1. Each polynomial $p_{\sigma'}$ is real-rooted, and
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Corollary

If $\{p\}_{\sigma}$ forms an interlacing family, then there exists an assignment σ_0 such that the largest root of p_{σ_0} is at most the largest root of $p_{\varnothing}$ (the expected polynomial).

Proof.

Start at the expected polynomial and walk backwards. $\square$

Interlacing for free

Fortunately, the interlacing follows directly from a well-known lemma:

Lemma (folklore, Fisk)

Let f, g be polynomials of the same degree such that every $\lambda f + (1 - \lambda) g$ is real-rooted for all $\lambda \in [0, 1]$. Then f and g have a common interlacer.

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Recall (again) our equation

$$p_{\sigma'}(x) = p_{\sigma',0}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v^0_{k+1}\right] + p_{\sigma',1}(x)\mathbb{P}\left[\widehat{v}_{k+1} = v^1_{k+1}\right]$$

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Time to pull real-rootedness from the back burner.

Where to start?

The issue with real-rooted polynomials is that it is hard to see how to get from one to another.

Parking garage phenomenon

The issue with real-rooted polynomials is that it is hard to see how to get from one to another.

Unless you consider them to be a projection of higher dimensional objects.

Real stable polynomials

There have been many recent advances in understanding real-rootedness using theory of *real stable polynomials*, a multivariate extension of real-rooted polynomials.

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A polynomial p is *real stable* if all coefficients are real and $p(z_1, \ldots, z_n) \neq 0$ whenever $\Im(z_i) > 0$ for all i (if $p(z_1, \ldots, z_n) = 0$ then some z_i has $\Im(z_i) \leq 0$).

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Some important properties:

- Univariate polynomials are real-rooted if and only if they are real stable.
- Real stable polynomials are closed under substitution of reals $(z_1, z_2, \ldots, z_n) \to (a, z_2, \ldots, z_n)$ for $a \in \mathbb{R}$.

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Similar to *hyperbolic polynomials*.

Borcea and Brändén

Borcea and Brändén developed numerous techniques for showing real stability. In particular,

Lemma

Let $A_1, \ldots, A_m$ *be Hermitian positive semidefinite matrices and* $x_1 \ldots x_m$ *variables. Then*

$$p(x_1, \ldots, x_m) = \det\left[\sum_{i=1}^{m} x_i A_i\right]$$

is real stable.

Lemma

If $p(x_1, \ldots, x_m)$ *is a real stable polynomial, then*

$$p(x_1, \ldots, x_m) - \frac{\partial p(x_1, \ldots, x_m)}{\partial x_i}$$

is real stable.

Our polynomials

Fortunately, our polynomials have a nice general form.

Theorem

Let $\widehat{v}_1, \ldots \widehat{v}_m$ be random vectors such that $\mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = A_i$. Then

$$\mathbb{E}\left[\chi_{\widehat{V}}(x)\right] = \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right) \det\left[xI + \sum_{i=1}^{m} z_i A_i\right]\Bigg|_{z_1=\cdots=z_m=0}$$

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We call this a *mixed characteristic polynomial* and denote it $\mu[A_1, \ldots, A_m]$.

A world of mixed characteristic polynomials

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1. Normal characteristic polynomials (for an assignment $\sigma = v_1, \ldots, v_m$ with $\sum_i v_i v_i^T = V$)
$$p_\sigma(x) = \chi_V(x) = \mu[v_1 v_1^T, \ldots, v_m v_m^T](x)$$
2. The expected characteristic polynomial (with $\mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = A_i$)
$$\mathbb{E}\left[\chi_{\widehat{V}}(x)\right] = \mu[A_1, \ldots, A_m](x)$$
3. The partial assignment polynomials
$$\begin{aligned} p_{\sigma'} &= \mathbb{E}_{v_{k+1},\ldots,v_d}\left[\chi_{\widehat{V}}(x) \mid \widehat{v}_i = v_i^{\sigma'_i} \text{ for } 1 \le i \le k\right] \\ &= \mu[v_1 v_1^T, \ldots, v_k v_k^T, A_{k+1}, \ldots, A_m] \end{aligned}$$

Putting it all together

Theorem

Mixed characteristic polynomials are real-rooted.

Proof.

By the first lemma of Borcea and Brändén,

$$p(z_1, \ldots, z_m) = \det\left[xI + \sum_{i=1}^{m} z_i A_i\right]$$

is real stable and so by the second lemma of Borcea and Brändén,

$$\prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left[xI + \sum_{i=1}^{m} z_i A_i\right]$$

is real stable.

Putting it all together, cont.

Since real stability is preserved under substitution by reals, (setting $z_1 = \cdots = z_m = 0$), we have

$$\mu[A_1, \ldots, A_m](x) = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det \left[xI + \sum_{i=1}^{m} z_i A_i \right] \Bigg|_{z_1 = \cdots = z_m = 0}$$

is univariate and real stable (and therefore real-rooted). $\square$

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Corollary

Our polynomials form an interlacing family.

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So we are left with bounding the largest root of the expected characteristic polynomial.

Outline

"Roots" of multivariate polynomials

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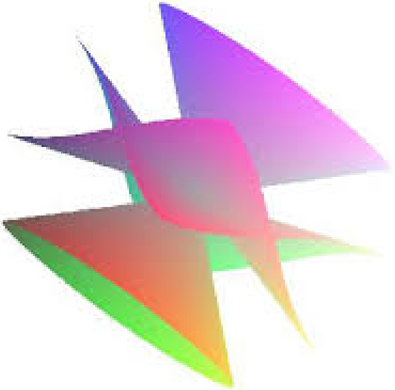
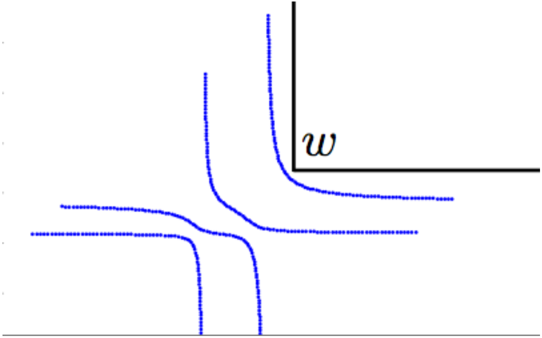


Above the roots

Let $p(x_1, \ldots, x_n)$ be a multivariate real stable polynomial.

We say a point $\vec{w} = (w_1, \ldots, w_n)$ is *above the roots* of p if $p(w_1 + t_1, w_2 + t_2, \ldots, w_n + t_n)$ is nonzero whenever $t_1, \ldots, t_n > 0$.



Diagonalization

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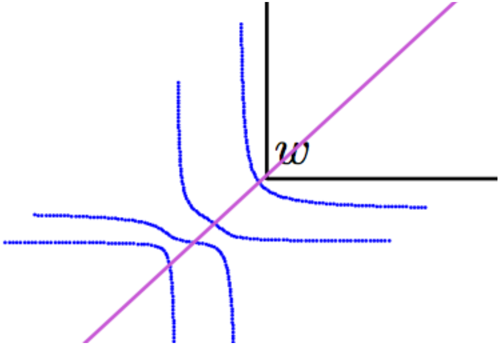
If $t\mathbb{1}$ is above the roots of p, then t is an upper bound on largest root of its diagonalization.



Shift

Fortunately, we can transform our target polynomial into a diagonalization:

Lemma

In the case that $\sum_i A_i = I$*, we have*

$$\mu[A_1, \ldots, A_m](x) = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det \left[xI + \sum_{i=1}^{m} z_i A_i \right] \Bigg|_{z_1 = \cdots = z_m = 0}$$
$$= \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial y_i}\right) \det \left[\sum_{i=1}^{m} y_i A_i \right] \Bigg|_{y_1 = \cdots = y_m = x}$$

Proof.

Substitute $y_i = z_i + x$. □

The new framework

Recall we are interested in bounding the roots of $\mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$ in the case that

$$\mathbb{E}\left[\sum_{i=1}^{m} \widehat{v}_i \widehat{v}_i^T\right] = I \quad \text{and} \quad \mathbb{E}\left[\|\widehat{v}_i\|^2\right] \leq \epsilon.$$

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Given what we know about mixed characteristic polynomials, this is equivalent to showing (for some t) that $t\mathbb{1}$ is above the roots of

$$\prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right) \det\left[\sum_{i=1}^{m} z_i A_i\right]$$

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We will apply the operators one by one and see what happens to the roots.

Potential function

We use a *multivariate potential function* to help understand the behavior as the operators are applied.

$$\Phi_p^i(z_1, \ldots, z_m) = \frac{\partial}{\partial z_i} \log p(z_1, \ldots, z_m)$$

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- Blows up whenever a variable x_i gets close to a zero surface of p
- Monotone nonincreasing at any $\vec{w}$ that is above the roots of p
- Convex at any $\vec{w}$ that is above the roots of p
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Generalization of potential function from Batson, Spielman, Srivastava (2008)

$$\Phi_p(x) = \frac{\partial}{\partial x} \log p(x) = \frac{p'(x)}{p(x)} = \sum_i \frac{1}{x - r_i}$$

Testing the water lemma

Lemma

If p is real stable, $\vec{w}$ is above the roots of p, and

$$\Phi_p^i(\vec{w}) < 1$$

then $\vec{w}$ is above the roots of $(1 - \partial_{z_i})p$

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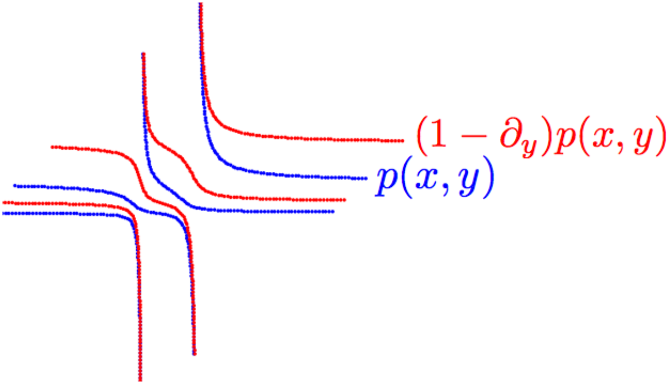
(this is just the definition Φ_p^i). Rearranging, gives

$$\left(1 - \frac{\partial}{\partial z_i}\right) p(\vec{w} + \vec{t}) > 0$$

□

Interpreting the lemma

Applying the operator $(1 - \partial_{z_j})$ causes the roots to get closer.



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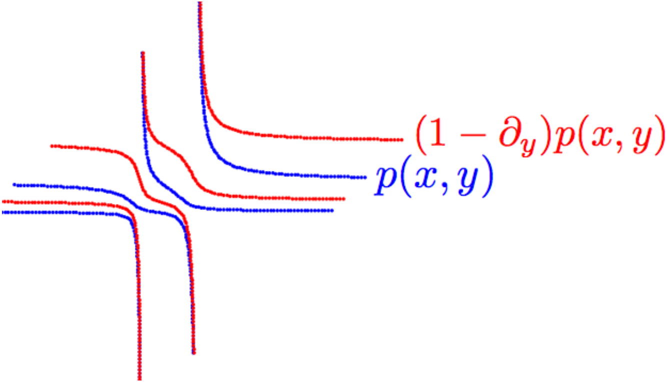

If $\Phi_p^j < 1$ then we are still above the roots after the shift.

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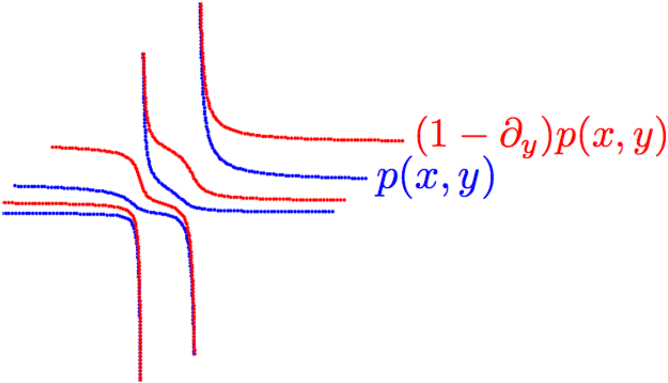

If $\Phi_p^j < 1$ then we are still above the roots after the shift.

But we have messed with the potential functions in the other directions (we decreased the cushion)!

Jumping in lemma

Lemma

If p is real stable, $\vec{w}$ is above the roots of p, and

$$\Phi_p^i(\vec{w}) < 1 - \frac{1}{\delta}$$

then $\vec{w}$ is above the roots of $(1 - \partial_{z_i})p$ and

$$\Phi_{p-p_j}^i(\vec{w} + \delta e_j) \leq \Phi_p^i(\vec{w})$$

for all i (where $p_j = \partial p / \partial z_j$).

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Proof.

Uses convexity mentioned above. □

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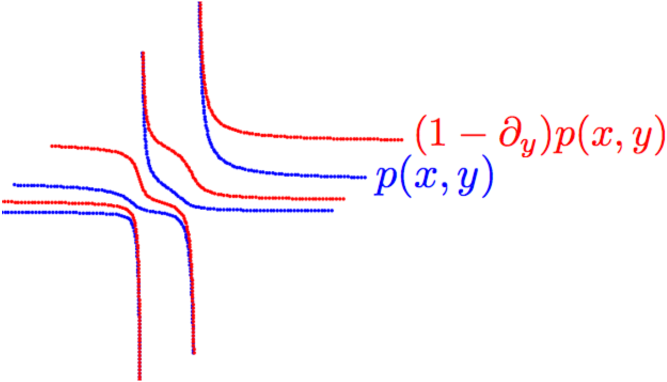

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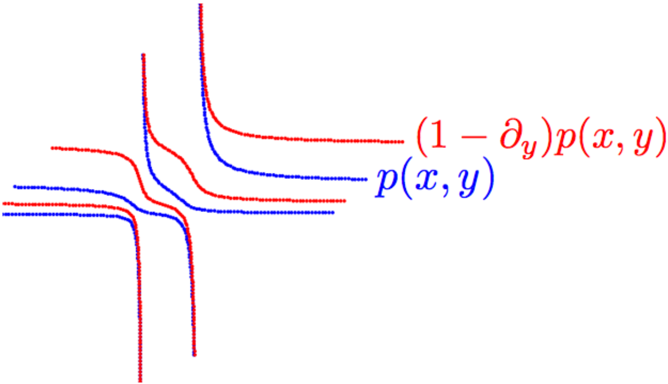

If $\Phi_p^j < 1 - 1/\delta$ then we are still above the roots after the shift *and* if we then move δ in the direction of the shift, we can get back the original cushion we had (in all other directions).

Proof of bound

Theorem

$(1+3\sqrt{\epsilon})\mathbb{1}$ *is above the roots of*

$$\prod_{i=1}^{m}\left(1-\frac{\partial}{\partial z_i}\right)\det\left[\sum_{i=1}^{m} z_i A_i\right]$$

for all $\epsilon < 1/4$.

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Proof.

Start with

$$Q_0(z_1,\ldots,z_m) := \det\left[\sum_{i=1}^{m} z_i A_i\right]$$

so that $t\mathbb{1}$ is above the roots of Q_0 for any $t > 0$.

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We will use $t = \sqrt{\epsilon}$ (we'll need the extra cushion).

Set $\vec{w}_0 = t\mathbb{1}$, so that

$$\Phi^{i}_{Q_0}(\vec{w}_0) = Tr\left[\left(\sum_{j=1}^{m} tA_j\right)^{-1} A_i\right] = \frac{Tr[A_i]}{t} \leq \frac{\epsilon}{\sqrt{\epsilon}} = \sqrt{\epsilon}.$$

This satisfies the cushion lemma for any

$$\delta > \frac{1}{1-\sqrt{\epsilon}}$$

so pick $\delta = 1 + 2\sqrt{\epsilon}$ (here we use $\epsilon < 1/4$).

Apply the operator $1 - \partial_{z_1}$ and then move δ in the direction of $\vec{e}_1$. By the lemma, $\vec{w}_1 = \vec{w}_0 + \delta\vec{e}_1$ is above the roots of

$$Q_1 = \left(1 - \frac{\partial}{\partial z_1}\right) Q_0$$

and we still satisfy the cushion lemma with $\delta = 1 + 2\sqrt{\epsilon}$.

Apply the operator $1 - \partial_{z_1}$ and then move δ in the direction of $\vec{e}_1$. By the lemma, $\vec{w}_1 = \vec{w}_0 + \delta\vec{e}_1$ is above the roots of

$$Q_1 = \left(1 - \frac{\partial}{\partial z_1}\right) Q_0$$

and we still satisfy the cushion lemma with $\delta = 1 + 2\sqrt{\epsilon}$.

Do this for $i = 2, \ldots, m$ (using the lemma each time). This shows that

$$\vec{w}_m = \vec{w}_0 + \delta \sum_i \vec{e}_i = (\delta + t)\mathbb{1} = (1 + 3\sqrt{\epsilon})\mathbb{1}$$

is above the roots of

$$Q_m = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) Q_0 = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det\left[\sum_i z_i A_i\right]$$

as required. $\square$

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3. We defined a multivariate barrier function to help us understand the evolution of the zero surfaces of multivariate polynomials
4. We used this to show that (for our polynomials) the largest root of $p_\emptyset$ was at most $1 + 3\sqrt{\epsilon}$.

Outline

Our theorem

We have proved our main technical theorem:

Theorem

Let $0 < \epsilon < 1/4$ *and* $\widehat{v}_1, \ldots \widehat{v}_m$ *be independent random vectors such that*

$$\sum_{i=1}^{m} \mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = I$$

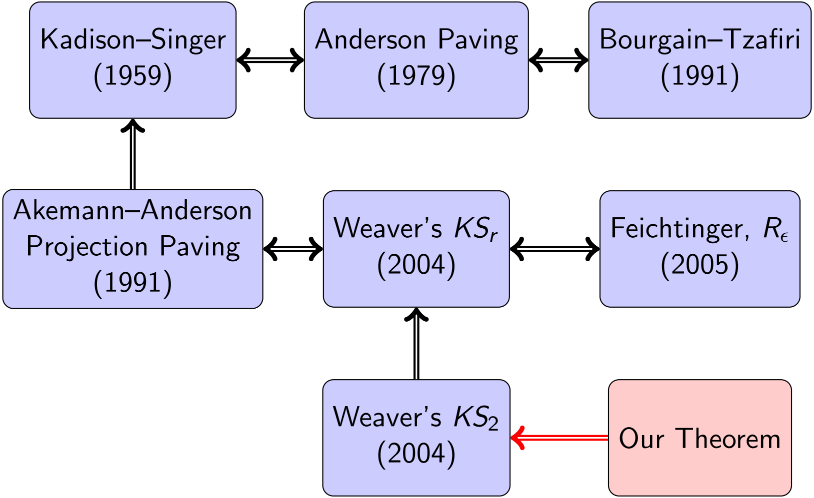
and

$$\mathbb{E}\left[\|\widehat{v}_i\|^2\right] \leq \epsilon$$

for all i*. Then there exists an assignment* $\widehat{v}_i = v_i$ *such that*

$$\left\| \sum_{i=1}^{m} v_i v_i^T \right\| \leq 1 + 3\sqrt{\epsilon}.$$

Claim:



Kadison–Singer (1959)
Anderson Paving (1979)
Bourgain–Tzafiri (1991)
Akemann–Anderson Projection Paving (1991)
Weaver's KS_r (2004)
Feichtinger, R_ϵ (2005)
Weaver's KS_2 (2004)
Our Theorem

Recall what KS_2 says:

Conjecture (KS_2)

There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds: if $w_1, \ldots, w_m \in \mathbb{C}^d$ satisfy $\|w_i\| \leq 1$ for all i and

$$\sum_i |\langle u, w_i \rangle|^2 = \eta$$

for all unit vectors $u \in \mathbb{C}^d$. Then there exists a partition of the vectors into two parts S_0, S_1 so that

$$\sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq \eta - \theta$$

for all unit vectors $u \in \mathbb{C}^d$ and each $j \in \{0, 1\}$.

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Again, we prove the real case (though the complex case is identical).

Proof.

Given the w_i, let $\widehat{v}_i$ be the random vector (in $\mathbb{R}^{2d}$!) taking values in

$$\left\{ \sqrt{\frac{2}{\eta}} \begin{pmatrix} w_i \\ 0^d \end{pmatrix}, \sqrt{\frac{2}{\eta}} \begin{pmatrix} 0^d \\ w_i \end{pmatrix} \right\}$$

each with probability $1/2$ and set $\epsilon = 2/\eta$.

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Then (this is just a rescaling to fit our main theorem)

$$\sum_{i=1}^{m} \mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = I \quad \text{and} \quad \mathbb{E}\left[\|\widehat{v}_i\|^2\right] \leq \epsilon$$

for all i, so let σ be the assignment guaranteed by our main theorem.

For the given σ, let

$$M_0 = \frac{2}{\eta} \sum_{i:\sigma(i)=0} w_i w_i^T \quad \text{and} \quad M_1 = \frac{2}{\eta} \sum_{i:\sigma(i)=1} w_i w_i^T$$

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Then by the theorem, the matrix

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Set

$$S_0 = \{w_i \mid \sigma_i = 0\} \quad \text{and} \quad S_1 = \{w_i \mid \sigma_i = 1\}$$

Then for all $j \in \{0,1\}$ and $u \in \mathbb{C}^d$

$$\frac{2}{\eta} \sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq 1 + 3\sqrt{\frac{2}{\eta}}$$

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Setting $\eta = 32$ gives

$$\sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq 28$$

proving the theorem for $\eta = 32$ and $\theta = 4$. $\square$

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Using the (stronger) original theorem, we can get $\eta = 18$ and $\theta = 2$.

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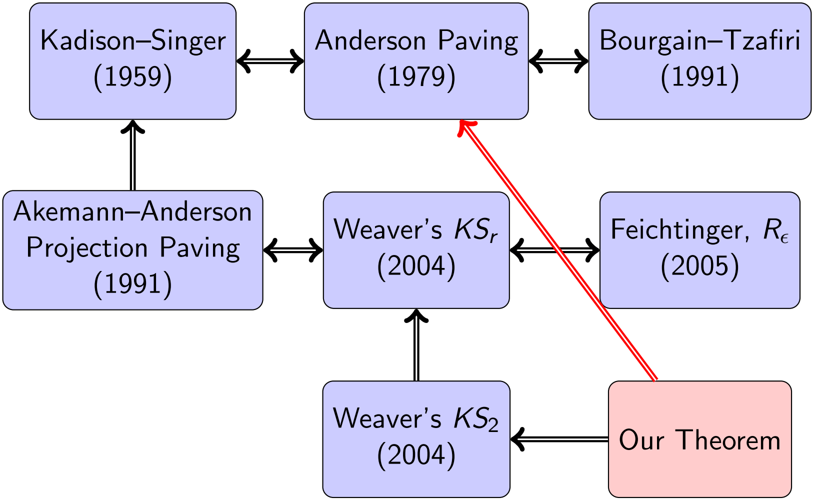
Using the (stronger) original theorem, we can get $\eta = 18$ and $\theta = 2$.

Casazza showed $\eta = 2$ is not possible (optimal answer lies somewhere in between).

Or if you prefer paving



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Conjecture (Casazza)

For all $\epsilon > 0$ *and even integers* $N > 0$*, there exists* $r = r(N, \epsilon)$ *such that for any* $d > 0$ *and any vectors* $v_1, \ldots, v_m \in \mathbb{C}^d$ *satisfying*

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for all j*, there exists a partition* $\{A_1, \ldots, A_r\}$ *of* $[m]$ *such that for all* $i \in [r]$

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Argument similar to proof of KS_2 shows this holds for $r \geq 6N/\epsilon^2$.

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Polynomial problems

Find more interlacing families and use them to solve problems!

Both results here used characteristic polynomials, but any real rooted polynomials will do (for example, Gurvits has interesting results using matching polynomials).

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Can any of the real-rootedness conditions be relaxed?

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What is the worst (in terms of largest root) $\mu[A_1, \ldots, A_m](x)$? We conjecture it is when all A_i are the same (and multiples of the identity).
This would improve all of the constants in this talk.

"The general feeling in the community is that the original question (and therefore all equivalent forms) have a negative solution" (Casazza–Kutinyiok, 2013).

What are the implications of a *positive* solution?

Thanks

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And thank you for your attention!