Interlacing Families and Bipartite Ramanujan Graphs

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Goals

In this talk I plan to

1. Give a brief introduction to graph expansion
2. Give a brief survey of what is known about Ramanujan families
3. Motivate the approach we took in trying to find Ramanujan families
4. Introduce a technique for showing the existence of combinatorial objects we call “the method of interlacing polynomials”
5. Use this to prove the existence of Ramanujan families of arbitrary degree
6. Discuss some related open questions not necessarily (but mostly) in this order.
Throughout the talk, the following things will hold:

- $G = (V, E)$ will be a $d$-regular graph
- We will assume a fixed ordering on $E = \{e_1, \ldots, e_m\}$.
- We will assume a fixed ordering on $V = \{v_1, \ldots, v_n\}$.
Simplifications

Throughout the talk, the following things will hold:

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- We will assume a fixed ordering on $V = \{v_1, \ldots, v_n\}$.

And please interrupt if there are any questions.
Expander graphs

Expander Graphs are sparse, regular, well-connected graphs that “approximate” random graphs.
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- Sets of vertices have many external (equivalently, few internal) neighbors
- No “small” cuts
- Random walks mix quickly
- Can get from \(a\) to \(b\) using few edges
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Used throughout computer science.

- Error-correcting codes
- Pseudorandom generators
- Computational complexity
  - PCP theorem (Dinur 2007)
  - $SL=L$ (Reingold 2005)
Types of Expansion

There are different definitions of “expansion”

▶ Edge expansion

\[
h_E(G) = \min_{0 < \vert S \vert \leq n/2} \frac{\vert \{e \in E : e \cap S = 1\} \vert}{\vert S \vert}
\]

▶ Vertex expansion

\[
h_V(G) = \min_{0 < \vert S \vert \leq n/2} \frac{\vert \{v \in V(G) : v \sim S\} \vert}{\vert S \vert}
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▶ Spectral Expansion (defined momentarily)
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▶ Spectral Expansion (defined momentarily)

For constant degree graphs, these are interchangable (up to a constant).
Given $G$, the adjacency matrix $A$ is defined as

$$
\begin{array}{cccccc}
  & a & c & d & b & e & f \\
 a & 0 & 0 & 0 & 1 & 1 & 0 \\
 c & 0 & 0 & 0 & 1 & 0 & 1 \\
 d & 0 & 0 & 0 & 0 & 1 & 1 \\
 b & 1 & 1 & 0 & 0 & 0 & 0 \\
 e & 1 & 0 & 1 & 0 & 0 & 0 \\
 f & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
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Adjacency Matrix

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1. $A_{i,j} = 1$ if and only if $\{v_i, v_j\} \in E$
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1. $A_{i,j} = 1$ if and only if $\{v_i, v_j\} \in E$
2. Since the graph is $d$-regular, each row sums to $d$
The spectrum

Since $A$ is symmetric, it has all real eigenvalues

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\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n
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Since the rows sum to $d$, $\mathbf{1}$ is an eigenvector with eigenvalue $d$. This (as well as $-d$ if $G$ is bipartite) is called a trivial eigenvalue.
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$G$ is a good expander (spectrally) if all non-trivial eigenvalues are small (in absolute value).

For example, $K_{d+1}$ has all non-trivial eigenvalues $-1$ and $K_d$, $d$ has all non-trivial eigenvalues $0$.

But in practice, we need big graphs with small degree. In particular, we would like infinite families of such graphs.
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But in practice, we need *big* graphs with *small* degree.

In particular, we would like *infinite families* of such graphs.
Outline

Brief Introduction

Brief History

The Approach

Method of Interlacing Polynomials

Wrap Up
What can we hope for?

Theorem (Alon–Boppana (1986))

For every $\epsilon > 0$, there exists an $N$ such that any $d$-regular graph on $N$ vertices has a non-trivial eigenvalue $\alpha$ with $|\alpha| \geq 2\sqrt{d - 1} - \epsilon$
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A $d$-regular graph that has all non-trivial eigenvalues inside the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ is called Ramanujan.

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$$\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ
\end{bmatrix}
\begin{pmatrix}
-d & -2\sqrt{d-1} & 0 & 2\sqrt{d-1} & d
\end{pmatrix}
$$

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We will call an infinite collection of $d$-regular Ramanujan graphs a *Ramanujan family*. 
Recall that expander graphs have the property “sets of vertices have many external neighbors”.
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The largest eigenvalue of the $d$-regular infinite tree is $2\sqrt{d - 1}$ (the smallest is $-2\sqrt{d - 1}$).
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In this sense, the “ultimate” $d$-regular expander is the $d$-regular infinite tree.

The largest eigenvalue of the $d$-regular infinite tree is $2\sqrt{d - 1}$ (the smallest is $-2\sqrt{d - 1}$).

So being Ramanujan can be seen as being a good (finite) approximation of a $d$-regular infinite tree.
Previous results

Theorem (Margulis, Lubotzky–Phillips–Sarnak (1988))

Ramanujan families exist for \( d = p + 1 \) where \( p \) is a prime number.

Proof depends heavily on algebraic techniques (they show certain collections of Cayley graphs are Ramanujan).
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Question: Are “algebraic graphs” the only families?

Answer: Almost no.

Theorem (Friedman (2008))
A randomly chosen \( d \)-regular graph has its non-trivial eigenvalues in the interval
\[
[-2\sqrt{d-1} - \epsilon, 2\sqrt{d-1} + \epsilon]
\]
with high probability.
Previous results

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with high probability.

So are Ramanujan families “special” or are they just “everywhere”? 
Main result

We prove the existence of Ramanujan families of every degree $d$. 

Two caveats:
1. The families guaranteed in our proof will be families of bipartite graphs.
2. Despite showing that such families are “everywhere”, we are not actually able to construct one. Instead, we use a new technique for showing existence of combinatorial objects we call “the method of interlacing polynomials.”
Main result

We prove the existence of Ramanujan families of every degree \( d \).

Furthermore, the proof technique will show that such families are more “everywhere” than they are “special”.

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Brief History 14/58
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Garbage Collection

What you need to keep in your brain:
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3. Optimal (in a spectral sense) means all non-trivial eigenvalues are in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. 

Brief History
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4. Such families were known to exist but only for certain values of $d$ and only for algebraic reasons.
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4. Such families were known to exist but only for certain values of $d$ and only for algebraic reasons.
5. We will prove the existence of (bipartite) Ramanujan families of every degree.
Outline

Brief Introduction

Brief History

The Approach

Method of Interlacing Polynomials

Wrap Up
General Idea

Start with a good graph.

And make a copy of it.
And perturb it.
Want to find perturbations that cause new graph to be good.
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General Idea

Start with a good graph.

And make a copy of it. And perturb it.

Want to find perturbations that cause new graph to be good.
2-lifts


For each edge \((a, b)\) in the original graph, choose either

\begin{align*}
  &a_1 \quad b_1 \\
  &a_2 \quad b_2
\end{align*}

Positive Edge Lift
Bilu and Linial (2006) studied perturbations called 2-lifts. For each edge \((a, b)\) in the original graph, choose either

\[
\begin{align*}
(a_1, b_1) & \quad \text{or} \quad (a_2, b_2) \\
\text{Positive Edge Lift} & \quad \text{Negative Edge Lift}
\end{align*}
\]

for a total of \(2^{|E|}\) possible 2-lifts.
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\[ a_1 b_1 a_2 b_2 \] \hspace{1cm} \text{Positive Edge Lift}

\[ a_1 b_2 a_2 b_1 \] \hspace{1cm} \text{Negative Edge Lift}

for a total of \(2|E|\) possible 2-lifts.

We will refer to a 2-lift by its \textit{signing} \(s \in \{\pm\}^m\) and refer to the corresponding (lifted) graph as \(G_s\).
Important properties

Let $e_i = (a, b)$ be an edge in $G$. Then (depending on the value of $s_i$) we have either

- Positive Edge Lift
  - $a_1 b_1 \quad a_2 b_2$

- Negative Edge Lift
  - $a_1 b_1 \quad a_2 b_2$

in $G_s$. 

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The Approach

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Important properties

Let $e_i = (a, b)$ be an edge in $G$. Then (depending on the value of $s_i$) we have either

Positive Edge Lift

or

Negative Edge Lift

in $G_s$. We can observe that

1. $\deg(a) = \deg(a_1) = \deg(a_2)$ for all vertices $a$
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$\begin{align*}
\text{a}_1 & \quad \text{a}_2 \\
\text{b}_1 & \quad \text{b}_2
\end{align*}$

or

**Negative Edge Lift**

$\begin{align*}
\text{a}_1 & \quad \text{a}_2 \\
\text{b}_1 & \quad \text{b}_2
\end{align*}$

in $G_s$. We can observe that

1. $\deg(a) = \deg(a_1) = \deg(a_2)$ for all vertices $a$
2. $x_i \sim y_j$ for some $i, j$ if and only if $x \sim y$
Important properties

Let \( e_i = (a, b) \) be an edge in \( G \). Then (depending on the value of \( s_i \)) we have either

- Positive Edge Lift
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in \( G_s \). We can observe that

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2. \( x_i \sim y_j \) for some \( i, j \) if and only if \( x \sim y \)

So, in particular, if \( G \) is bipartite and \( d \)-regular, then \( G_s \) is bipartite and \( d \)-regular (for all signings \( s \)).
Examples

For example:

\[ (a_1, b_1), (c_1, d_1) \]

\[ (a_2, b_2), (c_2, d_2) \]
Examples

For example:

1. All positive
Examples

For example:

1. All positive
2. All negative
Examples

For example:

1. All positive
2. All negative
3. \((a, b)\) and \(c, f\) negative (rest positive)
Signed Adjacency Matrix

Let $A$ be the adjacency matrix of $G$ and $s$ a signing of $G$.

$$
\begin{array}{c|cccccc}
\text{ } & a & c & d & b & e & f \\
\hline
a & 0 & 0 & 0 & 1 & 1 & 0 \\
c & 0 & 0 & 0 & 1 & 0 & 1 \\
d & 0 & 0 & 0 & 0 & 1 & 1 \\
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$$
Signed Adjacency Matrix

Let $A$ be the adjacency matrix of $G$ and $s$ a signing of $G$.

$$
\begin{array}{cccccc}
  & a & c & d & b & e & f \\
  a & 0 & 0 & 0 & -1 & 1 & 0 \\
  c & 0 & 0 & 0 & 1 & 0 & 1 \\
  d & 0 & 0 & 0 & 0 & 1 & 1 \\
  b & -1 & 1 & 0 & 0 & 0 & 0 \\
  e & 1 & 0 & 1 & 0 & 0 & 0 \\
  f & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
$$

Multiply each value in $A$ by the corresponding sign from $s$.

This is called the *signed adjacency matrix* $A_s$. 
Same Examples

For example:
**Same Examples**

For example:

<table>
<thead>
<tr>
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1. All positive

The Approach
## Same Examples

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Main Eigenvalue lemma

Theorem (Bilu–Linial (2006))

Let $G$ be any graph, $s$ a signing of $G$ and $G_s$ the 2-lift of $G$ corresponding to $s$.
Then the eigenvalues of $A(G_s)$ (the new adjacency matrix) are the union of the eigenvalues of $A$ (the original adjacency matrix) and the eigenvalues of $A_s$ (the signed adjacency matrix).
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Therefore if $G$ was Ramanujan and the eigenvalues of $A_s$ were in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$, then $G_s$ would be Ramanujan.
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Therefore if $G$ was Ramanujan and the eigenvalues of $A_s$ were in the interval $[-2\sqrt{d - 1}, 2\sqrt{d - 1}]$, then $G_s$ would be Ramanujan.

Conjecture (Bilu–Linial (2006))

Every $d$-regular graph contains a signing $s$ for which all of the eigenvalues of $A_s$ (the signed adjacency matrix) lie inside the interval $[-2\sqrt{d - 1}, 2\sqrt{d - 1}]$.
Implications

If the conjecture was true, this would imply the existence of Ramanujan families of degree $d$ (for any $d$).
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If the conjecture was true, this would imply the existence of Ramanujan families of degree $d$ (for any $d$).

1. Start with a $d$-regular Ramanujan graph $G$
2. Find a signing $s$ for which all eigenvalues of $A_s$ lie in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$
3. Perform the 2-lift associated with the signing to get graph $G_s$ (with twice as many vertices).
4. Then $G_s$ is Ramanujan, so iterate

Note: we can always start with $G = K_{d+1}$ or $G = K_{d,d}$. 
Implications

If the conjecture was true, this would imply the existence of Ramanujan families of degree $d$ (for any $d$).

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We will prove the conjecture for every bipartite graph $G$.

Since 2-lifts preserve bipartiteness, the same proof applies.
Bipartite Adjacency Matrices

What is so special about being bipartite?

In this case, the adjacency matrix can be written in block form
\[
\begin{pmatrix}
0 & B \\
B^T & 0
\end{pmatrix}
\]
causing eigenvalues/vectors to come in pairs
\[v_i = [u_i | u_i] \quad \text{and} \quad v_{n-i} = [u_i | -u_i]\]
for \(1 \leq i \leq n/2\) and so the eigenvalues satisfy
\[\lambda_i = -\lambda_{n-i}.
\]

Corollary
A bipartite graph \(G\) has all of its non-trivial eigenvalues in the range
\[\left[-2\sqrt{d-1}, 2\sqrt{d-1}\right]\]
if and only if it has all non-trivial eigenvalues at most
\[2\sqrt{d-1}.
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Intermission

Recall our goal: to show the existence of Ramanujan families of any degree $d$. 
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To do this, we are going to use an idea of Bilu and Linial that involves making a new graph from an old one using 2-lifts (and iterating).

Theorem

Every $d$-regular bipartite graph has a signing $s$ such that the largest eigenvalue of the signed adjacency matrix is at most $2 \sqrt{d-1}$.

Side note: a random signing does not (in general) work: $\|A_s\| \gg 2 \sqrt{d-1}$ as noted by Bilu and Linial.
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To show that the iteration produces a Ramanujan family, it suffices to show the following

**Theorem**

*Every $d$-regular bipartite graph has a signing $s$ such that the largest eigenvalue of the signed adjacency matrix is at most $2\sqrt{d} - 1$.***
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**Theorem**

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Side note: a random signing does not (in general) work:

$$\mathbb{E}_s [\|A_s\|] \gg 2\sqrt{d - 1}$$

as noted by Bilu and Linial.
Step By Step

Our approach will be to build the signing one edge at a time and see what happens to the eigenvalues.

\( \chi_M(x) = \text{det}(xI - M) = \prod_{i}(x - \lambda_i) \)

where the \( \lambda_i \) are the eigenvalues of \( M \).

Given a vector \( t \in \{\pm\}^k \) for \( k \leq m \), define the partial assignment polynomial \( p_t(x) := \sum_{s \in \{\pm\}^m} [\chi_A(s) | s_1 = t_1, \ldots, s_k = t_k] \)
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To “keep track” we will use a generalization of the characteristic polynomial of a matrix

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\[ p_t(x) := \mathbb{E}_{s \in \{\pm\}^m} [\chi_{A_s}(x) \mid s_1 = t_1, \ldots, s_k = t_k] \]
For $s \in \{\pm\}^m$, the partial assignment polynomial is just the characteristic polynomial of the matrix $A_s$

$$p_s(x) = \chi_{A_s}(x)$$
Some notes

For \( s \in \{\pm\}^m \), the partial assignment polynomial is just the characteristic polynomial of the matrix \( A_s \)

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p_s(x) = \chi_{A_s}(x)
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Also,

\[
p_{\emptyset}(x) = \mathbb{E}_{s \in \{\pm\}^m} \chi_{A_s}(x)
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is the expected characteristic polynomial over all possible signed adjacency matrices.
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Also,

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is the expected characteristic polynomial over all possible signed adjacency matrices.

Miraculously, the expected characteristic polynomial is something we can get our hands on.
Theorem (Godsil–Gutman (1981))

For any graph $G$,

$$\mathbb{E}_{s \in \{\pm\}^m} \chi_A_s(x) = \sum_i x^{n-2i} (-1)^i m_i$$

where $m_i$ is the number of matchings (subsets of $E$ that touch each vertex at most once) in $G$ of size $i$. 
Miracle 1

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Proof.
Expand the determinant as a sum over permutations:
1. Permutations that hit any off-diagonal non-edge are 0
2. Permutations that hit $A_{i,j}$ but not $A_{j,i}$ cancel (in expectation)
3. All that remains are permutations with $n - 2i$ entries on the diagonal and matchings of size $i$
Matching polynomials

For a graph $G$ with $m_i$ matchings of size $i$, the polynomial

$$
\mu_G(x) := \sum_i x^{n-2i}(-1)^i m_i
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is (fittingly) called the matching polynomial.
Matching polynomials

For a graph $G$ with $m_i$ matchings of size $i$, the polynomial

$$\mu_G(x) := \sum_i x^{n-2i}(-1)^i m_i$$

is (fittingly) called the matching polynomial. Some properties:

1. $m_0 = 1$
2. $m_1 = |E|$
3. $m_2$ is the number of pairs of edges that do not share an endpoint
4. $m_{n/2}$ is the number of perfect matchings

In particular, $\mu_G(0)$ is (in general) NP-hard to compute.
The matching polynomial was introduced by Heilmann and Lieb in their study of monomers–dimers. In their paper, they prove the somewhat remarkable theorem:

**Theorem (Heilmann–Lieb (1972))**

Let $G$ be a graph with maximum degree $\Delta$. Then

1. $\mu_G(x)$ is real-rooted
2. $\mu_G(y) > 0$ for all $y > 2\sqrt{\Delta - 1}$
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1. $\mu_G(x)$ is real-rooted
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**Proof.**

Use the identity

$$\mu_G(x) = \mu_{G-e}(x) - \mu_{G\setminus\{u,v\}}(x)$$

for some $e = (u, v) \in E$ and (a clever) induction. \qed
Suggests an approach

We now have a bunch of polynomials.
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Approach: Look for Miracle 3 (some way of relating the roots of individual polynomials to the roots of the average polynomial).

And that is the approach we will take.
Outline

Brief Introduction

Brief History

The Approach

Method of Interlacing Polynomials

Wrap Up
In search of a miracle

Unwinding the definition, we get the recurrence equation

$$p_t(x) = \frac{1}{2} p_{t+}(x) + \frac{1}{2} p_{t-}(x)$$

but now we have reached our first major issue.
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Adding polynomials is a function of the coefficients and we are interested in the roots.

In general, it is easy to get the coefficients from the roots but hard to get the roots from the coefficients.
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In general, it is easy to get the coefficients from the roots but hard to get the roots from the coefficients.

Approach: forget this and see what we can prove.
A Lemma

Lemma

Let $f$ and $g$ be monic polynomials. Assume there exists a point $c \in \mathbb{R}$ such that $f$ and $g$ each has exactly one real root larger than $c$ (call these the “extreme roots”). Then the largest real root of $f + g$ lies between these extreme roots.
Lemma

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Proof.

By picture
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So what?

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While each $p_s(x)$ was real-rooted for $s \in \{\pm\}^m$ (characteristic polynomials of symmetric matrices), in general the sums of real-rooted polynomials can be arbitrary.

Example: $p(x) = (x - 2)^2 - 1$ (has double root at 1) and $q(x) = (x + 2)^2 - 1$ (has double root at $-1$).

$$p(x) + q(x) = x^2 + 6$$

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But recall that

1. \( p_s(x) \) is real-rooted for any \( s \in \{\pm\}^m \) (see above)
2. \( p_\emptyset(x) = \mu_G(x) \) is real-rooted (by Heilmann and Lieb)
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Perhaps this is true in more generality?
Equation Revisited

Back to our equation

\[ p_t(x) = \frac{1}{2} p_{t+}(x) + \frac{1}{2} p_{t-}(x) \]

The lemma tells us that if
1. \( p_t(x) \) is real-rooted
2. \( p_{t+}(x) \) is real-rooted
3. \( p_{t-}(x) \) is real-rooted
4. There exists a \( c \) "anchoring" the largest roots of \( p_{t+}(x) \) and \( p_{t-}(x) \)

Then we know the largest root of \( p_t(x) \) lies between the largest root of \( p_{t+}(x) \) and the largest root of \( p_{t-}(x) \).

Let's worry about \( c \) for the moment (keeping real-rootedness on the back burner).
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Let’s worry about \( c \) for the moment (keeping real-rootedness on the back burner).
Interlacing polynomials

Let $p$ be a real-rooted polynomial of degree $n$ and $q$ a real-rooted polynomial of degree $n - 1$

$$p(x) = \prod_{i=1}^{n} (x - \alpha_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{n-1} (x - \beta_i)$$

with $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_{n-1}$
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We say $q$ interlaces $p$ if $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \alpha_{d-1} \leq \beta_{n-1} \leq \alpha_n$.

Think: The roots of $q$ separate the roots of $p$
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Think: The roots of $q$ separate the roots of $p$.

Example 1: $p'(x)$ interlaces $p(x)$

Example 2: If $p$ has no multiple roots (and largest root $R$), then let $q = p/(x - R)$. Then $q(x + \epsilon)$ interlaces $p(x)$.
Common Interlacers

We say that two degree $n$ polynomials $p$ and $r$ have a *common interlacer* if there exists a $q$ such that $q$ interlaces *both* $p$ and $r$ simultaneously.

Think: the roots of $q$ split up $\mathbb{R}$ into $n$ intervals, each of which contains exactly one root of $p$ and one root of $r$. 
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Note, if $p$ and $r$ have a common interlacer (say $q$), then $c = \beta_{d-1}$ can serve as the anchor from the lemma!
Interlacing families

We say \( \{p\}_{s \in \{\pm\}^m} \) is an *interlacing family* if for all partial assignments \( t \) we have that

1. Each polynomial \( p_t \) is real-rooted, and
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2. The polynomials \( p^+_t \) and \( p^-_t \) have a common interlacer

Corollary

If \( \{p\}_s \) forms an interlacing family, then there exists an assignment \( s^* \) such that the largest root of \( p_{s^*} \) is at most the largest root of \( p_{\emptyset} \) (the expected polynomial).

Proof.

Start at the expected polynomial and walk backwards.
Interlacing for free

Fortunately, interlacing follows directly from a well-known lemma:

Lemma (folklore, Fisk)

Let \( f, g \) be polynomials of the same degree such that every \( \lambda f + (1 - \lambda)g \) is real-rooted for all \( \lambda \in [0, 1] \). Then \( f \) and \( g \) have a common interlacer.
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Note this is similar to our recurrence equation:

$$p_t(x) = \frac{1}{2} p_{t+}(x) + \frac{1}{2} p_{t-}(x)$$

with $1/2$ replaced by $\lambda \in [0, 1]$. 
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with 1/2 replaced by \( \lambda \in [0, 1] \).

So if we can prove that our polynomials are real-rooted for all independent distributions, we get interlacing for free!
We defined a collection of *partial assignment* polynomials.
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We defined an interlacing family \( \{p\}_s \) and showed that any such family has a polynomial \( p^*_s \) whose largest root is at most the largest root of \( p_\emptyset \) (the expected polynomial).
We defined a collection of *partial assignment* polynomials.

We defined an interlacing family \( \{ p \}_s \) and showed that any such family has a polynomial \( p_s^* \) whose largest root is at most the largest root of \( p_\emptyset \) (the expected polynomial).

If we can show the collection of polynomials

\[
P = \left\{ \sum_{s \in \{\pm\}^m} \prod_{s_i = +} \theta_i \prod_{s_i = -} (1 - \theta_i) p_s(x) \right\} \quad \theta_i \in [0, 1]
\]

are all real-rooted, then our polynomials form an interlacing family.
Even more general

In fact, we will prove something more general.
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In fact, we will prove something more general. Note that

\[ p_s(x - d) = \chi_{A_s}(x - d) = \det[x I - (dl + A_s)] \]

and that \( dl + A_s \) can be written

\[
dl + A_s = \sum_{s_i=-} (\delta_i - \delta_j)(\delta_i - \delta_j)^T + \sum_{s_i=} (\delta_i + \delta_j)(\delta_i + \delta_j)^T
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so we can write

\[ \sum_{s \in \{\pm\}^m} \prod_{s_i=} \theta_i \prod_{s_i=-} (1 - \theta_i) p_s(x) = \mathbb{E} \det \left[ xI - \sum_{e \in E} \vec{u}_e \vec{u}_e^T \right] \]

where for \( e_k = \{i, j\} \)

\[ \vec{u}_e = \begin{cases} (\delta_i + \delta_j) & \text{with probability } \theta_k \\ (\delta_i - \delta_j) & \text{with probability } 1 - \theta_k \end{cases} \]
Thus the real-rootedness of \( P \) would follow from the following theorem:

**Theorem**

Let \( \vec{u}_1, \ldots, \vec{u}_m \) be any independent random vectors. Then the expected characteristic polynomial

\[
\mathbb{E} \det \left[ xI - \sum_i \vec{u}_i\vec{u}_i^T \right]
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is real-rooted.
Master Real-rootedness theorem

Thus the real-rootedness of $\mathcal{P}$ would follow from the following theorem:

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is real-rooted.

Time to prove some real-rootedness.
Where to start?

The issue with real-rooted polynomials is that it is hard to see how to get from one to another.
Parking garage phenomenon

The issue with real-rooted polynomials is that it is hard to see how to get from one to another.

Unless you consider them to be a projection of higher dimensional objects.
Real stable polynomials

There have been many recent advances in understanding real-rootedness using theory of real stable polynomials, a multivariate extension of real-rooted polynomials.

A polynomial \( p \) is real stable if all coefficients are real and \( p(z_1,\ldots,z_n) \neq 0 \) whenever \( \Im(z_i) > 0 \) for all \( i \) (if \( p(z_1,\ldots,z_n) = 0 \) then some \( z_i \) has \( \Im(z_i) \leq 0 \)).

Some important properties:

▶ Univariate polynomials are real-rooted if and only if they are real stable.
▶ Real stable polynomials are closed under substitution of reals \((z_1, z_2, \ldots, z_n) \rightarrow (a, z_2, \ldots, z_n)\) for \( a \in \mathbb{R} \).

Similar to hyperbolic polynomials.
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There have been many recent advances in understanding real-rootedness using theory of \textit{real stable polynomials}, a multivariate extension of real-rooted polynomials.

A polynomial $p$ is \textit{real stable} if all coefficients are real and $p(z_1, \ldots, z_n) \neq 0$ whenever $\Re(z_i) > 0$ for all $i$ (if $p(z_1, \ldots, z_n) = 0$ then some $z_i$ has $\Re(z_i) \leq 0$).

Some important properties:

- Univariate polynomials are real-rooted if and only if they are real stable.
- Real stable polynomials are closed under substitution of reals $(z_1, z_2, \ldots, z_n) \rightarrow (a, z_2, \ldots, z_n)$ for $a \in \mathbb{R}$.
Real stable polynomials

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Similar to hyperbolic polynomials.
Borcea and Brändén

Borcea and Brändén developed numerous techniques for showing real stability. In particular,

**Lemma**

Let $A_1, \ldots, A_m$ be Hermitian positive semidefinite matrices and $x_1 \ldots x_m$ variables. Then

$$p(x_1, \ldots, x_m) = \det \left[ \sum_{i=1}^{m} x_i A_i \right]$$

is real stable.

**Lemma**

If $p(x_1, \ldots, x_m)$ is a real stable polynomial, then

$$p(x_1, \ldots, x_m) - \frac{\partial p(x_1, \ldots, x_m)}{\partial x_i}$$

is real stable.
Master Identity

We can write the polynomial we want using these operations:

**Theorem**

Let $\vec{u}_1, \ldots, \vec{u}_m$ be independent random vectors with $\mathbb{E} \vec{u}_i \vec{u}_i^T := A_i$. Then

\[
\mathbb{E} \det \left[ xI + \sum_i \vec{u}_i \vec{u}_i^T \right] = \prod_{i=1}^{m} \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left[ xI + \sum_{i=1}^{m} z_i A_i \right] \bigg|_{z_1=\cdots=z_m=0}
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In particular, the expected polynomial does not depend on the vectors or the probabilities — only the expected outer products.
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In particular, the expected polynomial does not depend on the vectors or the probabilities — only the expected outer products.

We call this a *mixed characteristic polynomial* and denote it $\mu[A_1, \ldots, A_m]$. 
Putting it all together

Theorem

Mixed characteristic polynomials are real-rooted.
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**Theorem**

*Mixed characteristic polynomials are real-rooted.*

**Proof.**

By the first lemma of Borcea and Brändén,

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is real stable.
Putting it all together, cont.

Since real stability is preserved under substitution by reals, (setting $z_1 = \cdots = z_m = 0$), we have

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\mu[A_1, \ldots, A_m](x) = \prod_{i=1}^{m} \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left[ xI + \sum_{i=1}^{m} z_i A_i \right] \bigg|_{z_1 = \cdots = z_m = 0}
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is univariate and real stable (and therefore real-rooted).
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**Corollary**

*Our polynomials form an interlacing family.*
Quick Review

A quick review:

1. We defined interlacing families and showed that any such family has a polynomial whose largest root is smaller than the largest root of the expected polynomial.

2. We argued that (for our polynomials) the interlacing condition was implied by the real-rootedness of the partial assignment polynomials (and the recurrence equation).

3. We defined mixed characteristic polynomials and showed that our partial assignment polynomials belonged to this class.

4. We showed that mixed characteristic polynomials were real-rooted by using Borcea and Brändén's theory of real stable polynomials.
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Outline

Brief Introduction

Brief History

The Approach

Method of Interlacing Polynomials

Wrap Up
Piecing things together

**Theorem**

*There exist bipartite Ramanujan families of degree $d$ for any $d$.***
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Proof.

Set $G_0 = K_{d,d}$ (which is bipartite, $d$-regular, and Ramanujan for any $d$) and let $G_i$ be a bipartite, $d$-regular, Ramanujan graph (for some $i$).
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Set $G_{i+1}$ to be the 2-lift associated with $s^*$ — this is bipartite, $d$-regular, and (by Bilu and Linial) Ramanujan — and proceed by induction.
Open Problem 1

Show that there exist non-bipartite Ramanujan families of all degrees.

Note that we used “bipartiteness” as a way to bound the largest and smallest eigenvalues simultaneously.
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It is possible to do this in the general case using polynomials like

$$q_s(x) = p_s(x)p_s(-x)$$

but this would require new proofs of all three stages of the “method of interlacing polynomials”.

1. Showing that the \( \{q_s\} \) form an interlacing family
2. Calculating the expected polynomial, and
3. Bounding the largest root of the expected polynomial
Open Problem 2

Find constructions for Ramanujan families of arbitrary degrees.

The families found by Margulis and Lubotzky–Phillips–Sarnak are constructive, but only exist for $d = p + 1$. 
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Remember, these things are essentially “everywhere”!
Thank you to the organizers for providing me the opportunity to speak to you today.
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And thank you for your attention!