Non-uniqueness for the Navier–Stokes equations from critical data

Part 2

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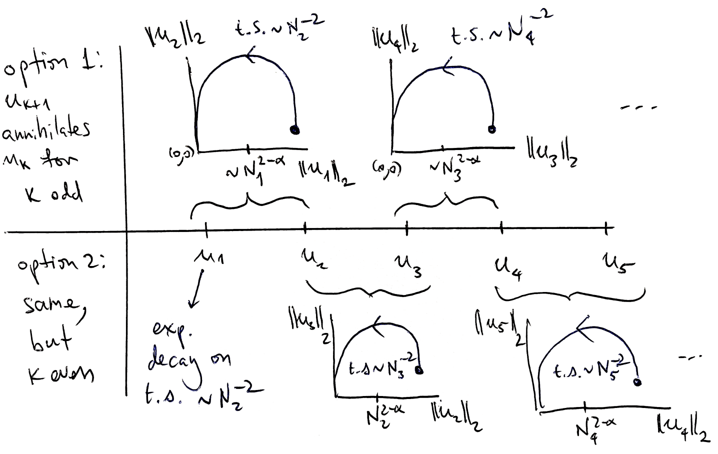
NSF-FRG Summer School on Fluids and Computer Assisted Proofs

Where we left off: non-uniqueness mechanism

- Decompose u into frequency shells:

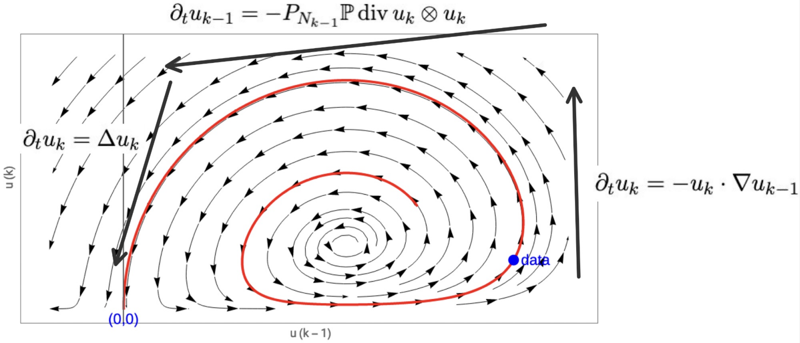
$$u = \sum_{k \geq 0} u_k, \qquad \operatorname{supp} \hat{u}_k \sim N_k, \qquad N_k \ll N_{k+1}$$



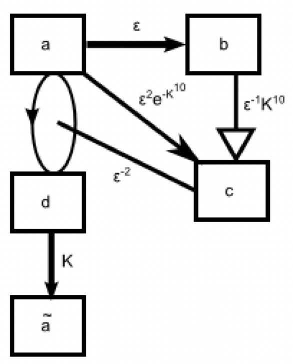
Dynamics of two adjacent modes

The mechanism requires the following terms to dominate:

$$\partial_t u_{k-1} \approx -P_{k-1}\,\mathrm{div}(u_k \otimes u_k)$$
$$\partial_t u_k \approx \Delta u_k - u_k \cdot \nabla u_{k-1}$$



Analogous program: blow-up from a dyadic model



T. Tao, JAMS 2016

Transfer to Navier–Stokes equations

Joint with M. Coiculescu

Define the critical space BMO^{-1} with the norm

$$\|U\|_{BMO^{-1}} := \sup_{R>0} \sup_{x_0 \in \mathbb{R}^3} \left(\int_0^{R^2} \int_{B(x_0,R)} |e^{t\Delta} U|^2 dxdt \right)^{\frac{1}{2}} < \infty$$

This is the *largest* space where one can implement small data global regularity.

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Theorem (Koch and Tataru 2001)

There exists $\epsilon > 0$ such that if U^0 is divergence-free with $\|u\|_{BMO^{-1}} < \epsilon$, then there exists a global-in-time solution that is regular for $t > 0$.

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Essentially, they show that the Navier–Stokes bilinear operator

$$u \mapsto -\int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u(t') \otimes u(t') dt'$$

is bounded on X_{KT}, the path space corresponding to BMO^{-1}.

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Theorem (Coiculescu–P. 2025)

There exists divergence-free initial data $U^0 \in BMO^{-1}$ *such that the Navier–Stokes initial value problem admits two distinct global solutions:*

$$u^{(1)},\, u^{(2)} \in X_{KT} \cap C^\infty_{t,x}((0,\infty)\times\mathbb{T}^3) \cap C_t([0,\infty); W^{-1,p}(\mathbb{T}^3))$$

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- The solutions are in the Banach space where Koch–Tataru run the fixed point argument:

$$\sup_{t>0} t^{\frac{1}{2}} \|u^{(i)}(t)\|_{L^\infty_x} + \sup_{x_0,\, R>0} \left(R^{-3} \int_0^{R^2} \int_{B(R)} |u^{(i)}|^2 dx dt \right)^{1/2} < \infty$$

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- The data is critical and smooth a.e. (but not L^2).

Idea of the proof

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  - *Heat-dominated flow*: let $v_k(x,t)$ the heat flow of the data:

$$\begin{aligned} \partial_t v_k - \Delta v_k &= l.o.t. \\ v_k|_{t=0} &= V_k^0. \end{aligned}$$

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  - *Inverse-cascade-dominated flow*: let $\overline{v}_k(x,t)$ evolve as:

$$\begin{aligned} \partial_t \overline{v}_k + P_{\sim N_k} \mathbb{P} \operatorname{div} v_{k+1} \otimes v_{k+1} &= 0 \\ \overline{v}_k|_{t=0} &= V_k^0 \end{aligned}$$

    with v_{k+1} as above. By a particular choice of V_{k+1}^0, we can arrange that $\|\overline{v}_k(t)\|_{L^\infty}$ decays exponentially to zero on the time scale N_{k+1}^{-2}.

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- Taking for granted that there exist v_k, $\overline{v}_k$ as above, we have two distinct *approximate* solutions of NSE with data $U^0 = \sum_{k \geq 0} V_k^0$:

$$v^{(1)} := v_0 + \overline{v}_1 + v_2 + \overline{v}_3 + \cdots \,, \quad v^{(2)} := \overline{v}_0 + v_1 + \overline{v}_2 + v_3 + \cdots \,.$$

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- The distinctness is immediate from the fact that v_k and $\overline{v}_k$ have different decay rates.

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- These two events are incompatible because the kth mode vanishes on a time scale (i.e., N^{-2}_{k+1}) much shorter than the time it needs to act on V^0_{k-1} (i.e., N^{-2}_k).

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- These two events are incompatible because the kth mode vanishes on a time scale (i.e., N_{k+1}^{-2}) much shorter than the time it needs to act on V_{k-1}^0 (i.e., N_k^{-2}).
- As a result, the dynamics are only consistent if we alternate $v_0 + \overline{v}_1 + v_2 + \overline{v}_3 + \cdots$.

Constructing the principal parts v_k, $\overline{v}_k$

- The initial data $V_k^0(x)$ is (very roughly) of the form

$$a_k(x)\theta\sin(N_k x\cdot\eta)$$

for some fixed $\eta,\theta\in\mathbb{Z}^3$ with $\eta\cdot\theta=0$, where $a_k(x)$ is a scalar coefficient that is principally supported at frequencies $|\xi|\ll N_k$.

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- Then the heat dominated flow should be

$$v_k(x,t) = a_k(x)\theta \sin(N_k x \cdot \eta) \exp(-N_k^2 |\eta|^2 t)$$

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- Recall we require $\partial_t\overline{v}_k=-P_{\sim N_k}\mathbb{P}\operatorname{div}(v_{k+1}\otimes v_{k+1})$ which becomes

$$\partial_t\overline{v}_k=-P_{\sim N_k}\mathbb{P}\operatorname{div}\left(a_{k+1}^2\theta\otimes\theta\sin^2(N_{k+1}x\cdot\eta)\right)\exp(-2N_{k+1}^2|\eta|^2 t)$$

$$\overline{v}_k|_{t=0}=V_k^0.$$

Constructing v_k, $\overline{v}_k$, continued

- One can directly integrate this and obtain

$$\overline{v}_k(t) = V_k^0 - CN_{k+1}^{-2} P_{\sim N_k} \mathbb{P} \operatorname{div}(a_{k+1}^2 \theta \otimes \theta)(1 - \exp(-2N_{k+1}^2 |\eta|^2 t))$$

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- Replace the simple ansatz $V_k^0 = a_k(x)\theta \sin(N_k x \cdot \eta)$ with one based on a Mikado flow:

$$\sum_i a_{i,k} \theta_i \sin(N_k x \cdot \eta_i)$$

Constructing v_k, $\overline{v}_k$, continued

- Write $V_k^0 = -\mathbb{P}\Delta\psi_k^0$. This is both divergence-free, and in the form $\mathrm{div}(\mathcal{D}\psi_k)$ for $\mathcal{D}\psi_k$ a symmetric tensor.

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- Use the "Nash lemma": any positive definite symmetric tensor can be decomposed as
$$\sum_i a_{i,k+1}^2 \theta_i \otimes \theta_i,$$
so one can solve
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- This creates a recursive dependence between the modes: V_k^0 determines V_{k+1}^0. Roughly,

$$\|V_{k+1}^0\|_{L^\infty} \approx C_1 N_{k+1} \left(\frac{\|V_k^0\|_{L^\infty}}{N_k} \right)^{\frac{1}{2}}.$$

Constructing v_k, $\overline{v}_k$, continued

- The relation becomes

$$\frac{\|V^0_{k+1}\|_{L^\infty}}{N_{k+1}} \sim \left(\frac{\|V^0_k\|_{L^\infty}}{N_k}\right)^{\frac{1}{2}}.$$

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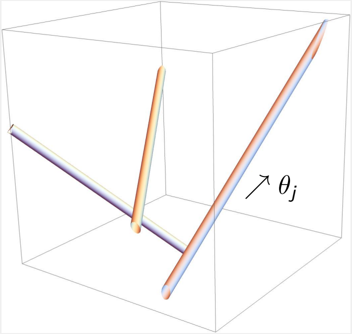
  - The construction *cannot* produce small data.

Building blocks of the construction

- Pick a small $\delta_0 > 0$ to be the pipe width, say $1/1000$.
- Can place six periodic "pipes" of width δ_0 pointing in the directions $\theta_1, \ldots, \theta_6$ in $\mathbb{T}^3$.
- For an even $\varphi \in C_c^\infty([0,1))$, define the pipe profiles

$$\tilde{\varphi}_j(x) = \varphi(\delta_0^{-1} d_{\mathbb{T}^3}(x, \ell_j)).$$



Building blocks of the construction, continued

- Let $\eta_j \perp \theta_j$ be integers.
- Sequence of frequencies: $M_0 \ll N_0 \ll M_1 \ll N_1 \ll M_2 \ll \cdots$. N_k is the oscillation frequency, M_k^{-1} is the period.

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- Then define the Mikado flow potentials:

$$\Psi^0_{j,k}(x) = N_k^{-2}\tilde{\varphi}_j(M_k x)\sin(N_k(x - x_j)\cdot \eta_j)\theta_j$$

as well as the (approximately) heat evolved potential

$$\Psi_{j,k}(x,t) = \Psi^0_{j,k}(x)\exp(-|\eta_j|^2 N_k^2 t).$$

- Then the velocity will be close to

$$-\Delta\Psi^0_{j,k} \approx |\eta_j|^2\tilde{\varphi}_j(M_k x)\sin(N_k(x - x_j)\cdot \eta_j)\theta_j$$

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where

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- $a_{j,k}$ oscillates much more slowly, designed carefully so that

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- $a_{j,k}$ is chosen to be supported only on $\operatorname{supp} v_{k-1}^0$.

Upgrading the regularity

- If $\xi_k \in \mathbb{Z}^3$ is lacunary and $|c_k| \sim |\xi_k|$, then

$$\sum_{k \geq 0} c_k e^{ix\cdot\xi_k} \in B^{-1}_{\infty,\infty} \setminus BMO^{-1}$$

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- But if χ_k are bump functions with $\sum_k |\operatorname{supp} \chi_k| < \infty$,

$$\sum_{k \geq 0} c_k e^{ix \cdot \xi_k} \chi_k(x) \in BMO^{-1}$$

so we just need a small amount of "intermittency"

Upgrading the regularity

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- But if χ_k are bump functions with $\sum_k |\operatorname{supp} \chi_k| < \infty$,

$$\sum_{k\geq 0} c_k e^{ix\cdot\xi_k} \chi_k(x) \in BMO^{-1}$$

  so we just need a small amount of "intermittency"

- By making the pipes (boundedly) narrow and $\operatorname{supp} v_k \subset \operatorname{supp} v_{k-1}$, we obtain

$$|\operatorname{supp} v_k^0| \lesssim 2^{-k}$$

  which is sufficient.

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- the inverse cascade-driven part

$$\overline{v}_k(t,x) = \frac{1}{2} N_{k+1}^{-2} \mathbb{P} \operatorname{div} \sum_j A_{j,k+1} |\eta_j|^2 e^{-2|\eta_j|^2 N_{k+1}^2 t} a_{j,k+1}^2(x) \theta_j \otimes \theta_j.$$

Smallness of the error

- The above construction produces solutions $v^{(1)}$ and $v^{(2)}$ of

$$\partial_t v^{(i)} - \Delta v^{(i)} + \mathbb{P}\operatorname{div} v^{(i)} \otimes v^{(i)} = \operatorname{div} F^{(i)}, \quad v^{(i)}|_{t=0} = U^0.$$

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- Claim: $F^{(i)}$ can be chosen so that it contains terms of size:

$$\sum_k N_{k-1} N_k \exp(-N_k^2 t) + N_k^2 \exp(-N_{k+1}^2 t)$$

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- E.g.: If $N_k = \lambda^{b^k}$, then

$$\sum_k N_{k-1} N_k \exp(-N_k^2 t) \lesssim t^{-\frac{1}{2}(1+1/b)}$$

  which is *sub*critical

Smallness of the error, continued

Define the norm

$$\|a\|_Y := \sup_{t\in(0,1]} \left(t^{1-\alpha}\|a\|_{L^\infty(\mathbb{T}^3)} + t^{\frac{3}{2}-\alpha}\|\nabla a\|_{C^\kappa(\mathbb{T}^3)}\right) < \infty,$$

$\alpha > 0$ a small *subcriticality parameter*.

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Proposition

$F^{(i)}$ can be chosen so that

$$\|F^{(i)}\|_Y \le \epsilon.$$

Construction of the perturbation

- We have produced solutions $v^{(1)}$ and $v^{(2)}$ of

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- The linear terms obstruct a fixed point method. The difficulty is that the drift is large in a very weak space X_{KT}.

Construction of the perturbation (continued)

- To make the fixed point method possible, we use the semigroup for the linearized NSE around $v^{(i)}$:

$$\begin{aligned} \partial_t S^{(i)}(t,t')a - \Delta S^{(i)}(t,t')a + 2\mathbb{P}\,\mathrm{div}(v^{(i)}(t) \odot S^{(i)}(t,t')a) = 0 \\ S^{(i)}(t',t') = \mathbb{P}\,\mathrm{div}\, a(t'). \end{aligned}$$

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- It is not clear the semigroup is well-defined all the way back to $t' = 0$. But it does not need to be because $w^{(i)}|_{t=0} = 0$.

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- By making the separation between N_k large,

$$\exp(\#\{k : N_k^{-2} \in [t', t]\})$$

can be made mild.

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- Combining with $\|F\|_Y \leq \epsilon$ and the estimates on $S^{(i)}$, the fixed point argument closes in a small ball.

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- Explicitly,

$$\begin{aligned}\|T(w)(t)\|_{L^\infty} &\lesssim t^{-\frac{1}{2}+\epsilon}\int_0^t (t')^{-1+\alpha-\epsilon}dt'\|F - w\otimes w\|_Y \\ &\lesssim t^{-\frac{1}{2}+\epsilon}\int_0^t (t')^{-1+\alpha-\epsilon}dt'(\|w\|_X^2 + \|F\|_Y) \\ &\lesssim t^{-\frac{1}{2}+\alpha}(\|w\|_X^2 + \epsilon_0),\end{aligned}$$

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- We also have, $\|w(t)\|_{\mathcal{C}^{-1+\alpha/2}} \lesssim t^{\alpha/4}$.

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- Look at (say) time $t = N_0^{-2}$. All the parts of both solutions have dissipated away except the lowest mode of $v^{(1)}$.
- One can show that $v^{(i)} \to u^0$ in $W^{-1,p}$ ($p < \infty$).

Concluding remarks

- One can compute that the solutions we construct lie in the exact path space of Koch and Tataru. So the smallness assumption in the mild solution theory.

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- The initial energy is $+\infty$ so that solutions are not Leray–Hopf. Nonetheless, L^2 strictly decreases.
- Constructions are also possible for finite energy data where the energy comes in from infinite wavenumber.
- This might be a flexible mechanism with further applications: 2D, other equations, other building blocks, etc.

Thank you!