Non-uniqueness for the Navier–Stokes equations from critical data

Part 1

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Navier–Stokes equations

- Incompressible Navier–Stokes equations on $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$

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\partial_t u - \Delta u + \operatorname{div} u \otimes u + \nabla p &= 0 \\
\operatorname{div} u &= 0 \\
u(x,0) &= U_0(x)
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- Unknowns are $u : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}$
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- Given initial data U_0 with $\operatorname{div} U_0 = 0$.
- *Guiding question*: What class of initial data and notion of "solution" guarantees existence, uniqueness, regularity?

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- Sufficiently regular solutions obey the energy equality

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  - Natural examples of critical spaces: $H^{\frac{1}{2}}$, L^3, $L^{3,w}$, $B^{-1+3/p}_{p,\infty}$ $(p > 3)$
  - There is small data global existence from data in all these spaces (Fujita–Kato '64, Kato '84, Cannone '94, Planchon '98)

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  - No control over small scales $\rightarrow$ wild behavior possible.

Navier–Stokes equations—Leray's picture

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Local theory

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- Local well-posedness when data is in a subcritical space (e.g. L^p, $p > 3$)
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Leray weak solutions

- I.e., weak solutions obeying an energy *in*equality
- Global existence for any L^2 data
- Weak-strong uniqueness
- Global regularity?
- Uniqueness?

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  - u is a smooth solution for $t > 0$

Approach #1: non-uniqueness by instability

Idea of Jia-Sverak (2013):

- Write Navier–Stokes in the self-similar coordinates $\xi = x/t^{1/2}$, $\tau = \log t$ with the ansatz

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- Find a steady solution $U(\xi, \tau) = \overline{U}(\xi)$ that is nonlinearly unstable in the self-similar dynamics
- Then one can write down infinitely-many distinct solutions:
  - the stationary (in self-similar coordinates) solution $\overline{U}$
  - solutions on the unstable manifold of $\overline{U}$ (i.e., converging exponentially to $\overline{U}$ as $\tau \to -\infty$)

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$$U(\xi) = \sigma a(\frac{\xi}{|\xi|})|\xi|^{-1} + o(|\xi|^{-1})$$

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  - Conjectured scenario: as σ increases, the solution set bifurcates, an eigenvalue crosses the imaginary axis.

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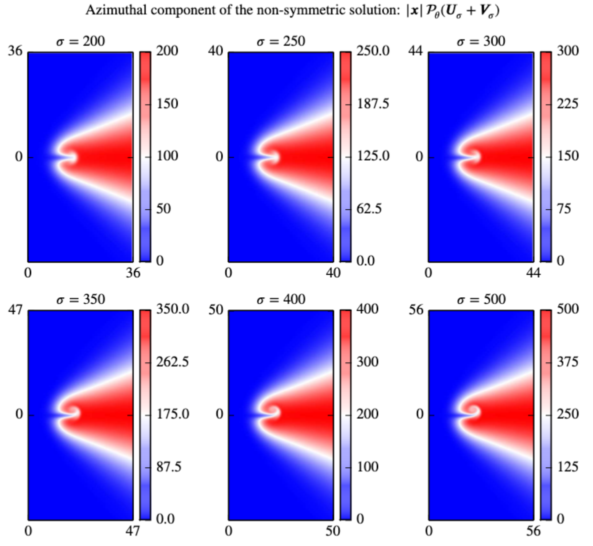
This scenario is supported by numerics: (Guillod-Šverák, JMFM 2023)



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- Albritton-Brué-Colombo (2021): non-unique Leray solutions of *forced* Navier–Stokes from critical initial data

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- For Navier–Stokes: Buckmaster–Vicol 2017, Buckmaster–Colombo–Vicol 2018, Cheskidov–Luo 2020-21, Cheskidov–Zeng–Zhang 2025, etc.
- Current techniques seem very far from reaching the energy space $L^\infty_t L^2_x \cap L^2_t H^1_x$.

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- The Navier-Stokes nonlinearity transfers energy between the modes. Accounting for the nearest-neighbor interactions:

$$\begin{aligned} \dot{X}_k = -N_k^2 X_k &+ c_1 \left(N_{k-1}^\alpha X_{k-1}^2 - N_k^\alpha X_k X_{k+1}\right) \\ &+ c_2 \left(N_{k-1}^\alpha X_{k-1} X_k - N_k^\alpha X_{k+1}^2\right) \end{aligned}$$

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- This model has an analogous theory of Leray solutions which exhibit weak-strong uniqueness.

Breaking a discrete scaling symmetry (cont'd)

- The scenario is captured by the Obukhov model ($c_1 = 0$, $c_2 = 1$):

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- There is an energy identity for regular solutions:

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  - $\alpha < 2$: energy is subcritical (local regularity)
  - $\alpha = 2$: energy critical
  - $\alpha > 2$: energy supercritical (ill-posedness?)

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- It follows that X^0 gives rise to distinct solutions: $X(t)$ and $\lambda^{2-\alpha} LX(\lambda^2 t)$ where L is the left shift operator.
- In practice, the solutions are not *exactly* discretely self-similar, but are nearly so (and converge to exactly SS as $\lambda \to \infty$).

Breaking a discrete scaling symmetry (cont'd)

- Suppose X is now indexed by $k \in \mathbb{Z}$ and:
  - Data X^0 is invariant under scaling by $\lambda > 1$:

$$X_k(0) = \lambda^{(2-\alpha)n} X_{k-n}(0), \quad \forall n \in \mathbb{Z}$$

  - Solution $X(t)$ invariant under scaling by multiples of λ^2 (but not λ^1):

$$X_k(t) = (\lambda^2)^{(2-\alpha)n} X_{k-2n}((\lambda^2)^{2n} t), \quad \forall n \in \mathbb{Z}$$

- It follows that X^0 gives rise to distinct solutions: $X(t)$ and $\lambda^{2-\alpha} L X(\lambda^2 t)$ where L is the left shift operator.
- In practice, the solutions are not *exactly* discretely self-similar, but are nearly so (and converge to exactly SS as $\lambda \to \infty$).
- Additional symmetry breaking is necessary to get a finite energy solution. (Truncation of low frequencies.)

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Remarks:

- The solutions decay exponentially in N_k for $t > 0$ and are positive.
- We have the stronger critical bound $|X_{0,k}| \lesssim N_k^{-\alpha+2}$ corresponding to data in $B_{p,\infty}^{-1+2(\alpha-1)/p}$.

Idea of the construction

$$\dot{X}_k = -N_k^2 X_k + N_{k-1}^{\alpha} X_{k-1} X_k - N_k^{\alpha} X_{k+1}^2, \quad k \geq 1$$

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  - X_{k+1} is exponentially small.

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We have (approximately, on a suitable time scale) the *decoupled* system

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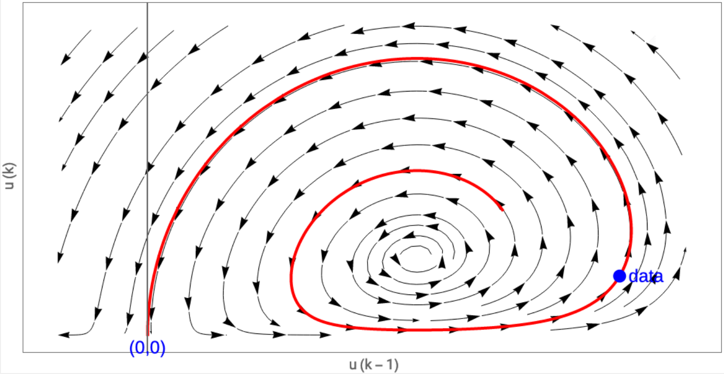

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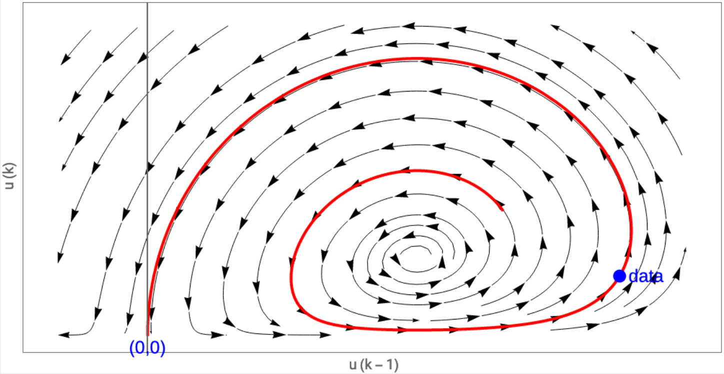

For data on the stable manifold, $(X_{k-1}, X_k) \to (0,0)$ on timescale N_k^{-2}.

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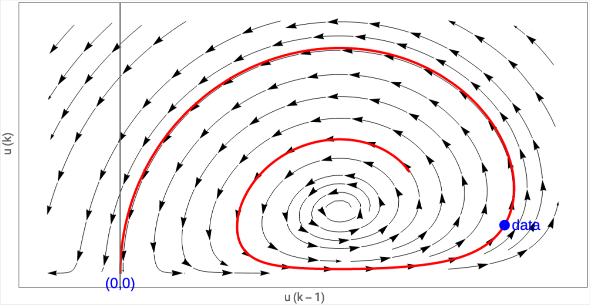

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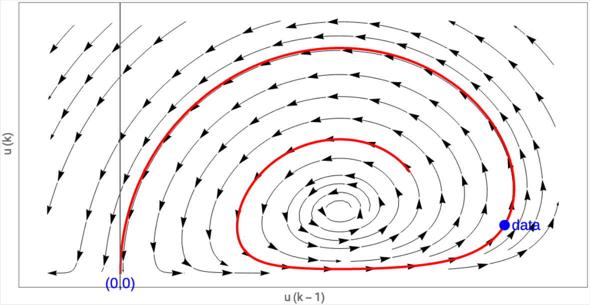

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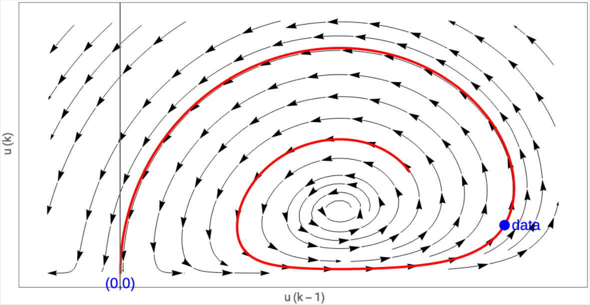

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- In other words, there is a family of data for which (X_{2j-1}, X_{2j}) annihilate, and another for which (X_{2j}, X_{2j+1}) annihilate.
- These families have an intersection, so there exists data for which either behavior can happen.

Transfer to the Navier–Stokes equations?

- Consider the actual NSE nonlinearity at frequency N_k:

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- To achieve the same mechanism, one needs to find data so that dominant interactions in the NSE are the following:

$$\begin{aligned} \partial_t u_{\sim N_{k-1}} &\approx -P_{\sim N_{k-1}} \operatorname{div}(u_{\sim N_k} \otimes u_{\sim N_k}) \\ \partial_t u_{\sim N_k} &\approx \Delta u_{\sim N_k} - u_{\sim N_k} \cdot \nabla u_{\sim N_{k-1}} \end{aligned}$$

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- Bad news: it is not so clear how to capture $-u_{\sim N_k} \cdot \nabla u_{\sim N_{k-1}}$ in the full PDE

Transfer to Navier–Stokes equations

Joint with M. Coiculescu

Define the critical space BMO^{-1} with the norm

$$\|U\|_{BMO^{-1}} := \sup_{R>0} \sup_{x_0 \in \mathbb{R}^3} \left(\int_0^{R^2} \int_{B(x_0,R)} |e^{t\Delta} U|^2 dxdt \right)^{\frac{1}{2}} < \infty$$

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There exists $\epsilon > 0$ such that if U^0 is divergence-free with $\|u\|_{BMO^{-1}} < \epsilon$, then there exists a global-in-time solution that is regular for $t > 0$.

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Essentially, they show that the Navier–Stokes bilinear operator

$$u \mapsto -\int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u(t') \otimes u(t') dt'$$

is bounded on X_{KT}, the path space corresponding to BMO^{-1}.

What about large initial data?

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Theorem (Coiculescu–P. 2025)

There exists divergence-free initial data $U^0 \in BMO^{-1}$ *such that the Navier–Stokes initial value problem admits two distinct global solutions:*

$$u^{(1)},\, u^{(2)} \in X_{KT} \cap C^\infty_{t,x}((0,\infty) \times \mathbb{T}^3) \cap C_t([0,\infty); W^{-1,p}(\mathbb{T}^3))$$

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- The solutions are in the Banach space where Koch–Tataru run the fixed point argument:

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- The data is critical and smooth a.e. (but not L^2).

Thank you!